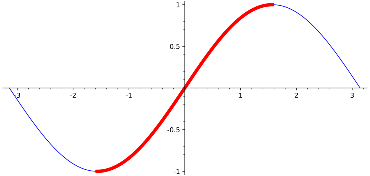
Chapitre II. FONCTIONS TRIGONOMÉTRIQUES RÉCIPROQUES

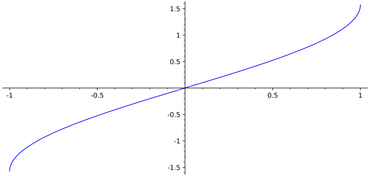
Rappels :

- si $-1 \leq y \leq 1$, alors $x = \arcsin y \Leftrightarrow \begin{cases} -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ \sin x = y \end{cases}$
- si $-1 \leq y \leq 1$, alors $x = \arccos y \Leftrightarrow \begin{cases} 0 \leq x \leq \pi \\ \cos x = y \end{cases}$

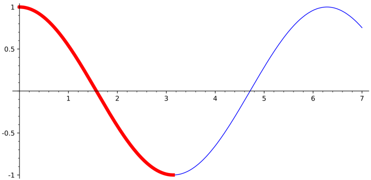
$$y = \sin x$$



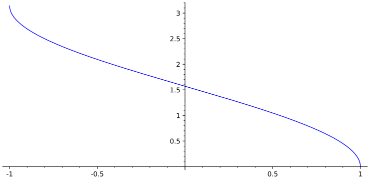
$$y = \arcsin x$$



$$y = \cos x$$



$$y = \arccos x$$



Valeurs à connître

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Valeurs à connaître

y	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\arccos y$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	0
$\arcsin y$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

$$\forall -1 \leq y \leq 1,\ \cos(\arccos(y)) = y$$
$$\forall -1 \leq y \leq 1,\ \sin(\arcsin(y)) = y$$

Pour tout entier $k \in \mathbb{Z}$,

$$^{\forall}x \in \mathbb{R},\ \arccos(\cos(x)) = \begin{cases} 2k\pi - x & \text{si } (2k-1)\pi \leq x \leq 2k\pi \\ x - 2k\pi & \text{si } 2k\pi \leq x \leq (2k+1)\pi \end{cases}$$

Pour tout entier $k \in \mathbb{Z}$,

$$^\forall x \in \mathbb{R},\ \arcsin(\sin(x)) =$$

$$\begin{cases} x - 2k\pi & \text{si } \frac{(4k-1)\pi}{2} \leq x \leq \frac{(4k+1)\pi}{2} \\ (2k+1)\pi - x & \text{si } \frac{(4k+1)\pi}{2} \leq x \leq \frac{(4k+3)\pi}{2} \end{cases}$$

$$\forall -1 \leq y \leq 1,\ \arccos(-y) = \pi - \arccos(y)$$
$$\forall -1 \leq y \leq 1,\ \arcsin(-y) = -\arcsin(y)$$
$$\forall -1 \leq x \leq 1,\ \arccos(x) + \arcsin(x) = \frac{\pi}{2}$$

$$\forall -1 \leq x \leq 1,\ \sin(\arccos x) = \sqrt{1 - x^2}$$

$$\forall -1 \leq x \leq 1,\ \cos(\arcsin x) = \sqrt{1 - x^2}$$

$$\forall -1 < x < 1,\ \arcsin' x = \frac{1}{\sqrt{1-x^2}}$$

$$\forall -1 < x < 1,\ \arccos' x = -\frac{1}{\sqrt{1-x^2}}$$

Théorème-définition. Pour tout $n \in \mathbb{N}$, il existe un polynôme $T_n(X) \in \mathbb{R}[X]$, de degré n, tel que

$$\forall -1 \leq x \leq 1,\ T_n(x) = \cos(n \arccos(x)) \ .$$

Exemples.

$T_0(X) = 1,\ T_1(X) = X,\ T_2(X) = 2X^2 - 1,\ T_3(X) = 4x^3 - 3x$
$T_4(x) = 8x^4 - 8x^2 + 1,\ T_5(x) = 16x^5 - 20x^3 + 5x,\ T_6(x) =$
$32x^6 - 48x^4 + 18x^2 - 1$

Démonstration. Par récurrence sur $n \in \mathbb{N}$.
$\cos((n+1)\arccos(x)) + \cos((n-1)\arccos(x)) = 2\cos(n\arccos(x))\cos(\arccos(x))$ D'où :
$T_{n+1}(x) + T_{n-1}(x) = 2XT_n(x)$.

Exercice. $T_n(x) = \begin{vmatrix} x & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2x & 1 & \ddots & & \vdots \\ 0 & 1 & 2x & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 2x \end{vmatrix}$

Exercice.

$$\arccos\left(\frac{4}{5}\right)+\arccos\left(\frac{12}{13}\right)= \arccos\left(\frac{33}{65}\right)$$

$$\forall x, y, x + y \geq 0 \Rightarrow$$
$$\arccos(x) + \arccos(y) = \arccos\left(xy - \sqrt{1 - x^2}\sqrt{1 - y^2}\right)$$

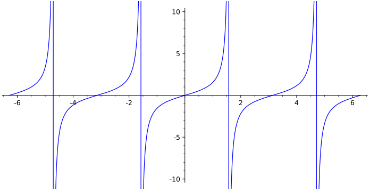
Exercice. $\forall 0 \leq x \leq 1,\ \arcsin(x) = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + ...$
$= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!}\frac{x^{2n+1}}{2n+1}$

$$\Rightarrow {}^{\forall}0 \leq t \leq \frac{\pi}{2},\ t = \sin(t) + + \frac{1}{2}\frac{\sin^3(t)}{3} + \frac{1.3}{2.4}\frac{\sin^5(t)}{5} + \frac{1.3.5}{2.4.6}\frac{\sin^7(t)}{7} + ...$$

$$\Rightarrow 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ... = \frac{\pi^2}{8}$$

La fonction tangente : $\tan x = \frac{\sin x}{\cos x}$ .Le graphe :

$$y = \tan x$$



- $^{\forall}x \neq \frac{\pi}{2} \mod \pi$, $\tan(x+\pi) = \tan x$;
- $^{\forall}x \neq \frac{\pi}{2} \mod \pi$, $\tan(-x) = -\tan x$;

Valeurs à connaître

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$+\infty$

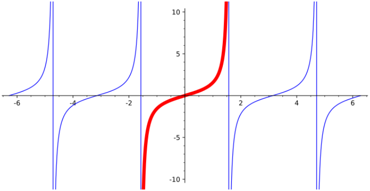
Exercices

$$\tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$$

$$\tan\left(\frac{\pi}{5}\right) = \sqrt{5 - 2\sqrt{5}}$$

Proposition. La fonction tan $:]-\frac{\pi}{2},\frac{\pi}{2}[\to \mathbb{R}$ est strictement croissante et bijective.

$$y = \tan x$$



Démonstration :

$$^{\forall}x \in \mathbb{R},\ \tan' x = \frac{\sin' x \cos x - \cos' x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = 1+\tan^2 x > 0$$

$$\lim_{\substack{x \to \frac{\pi}{2} \\ x < \frac{\pi}{2}}} \tan x = +\infty$$

$$\lim_{\substack{x \to -\frac{\pi}{2} \\ x > \frac{\pi}{2}}} \tan x = -\infty$$

q.e.d.

Définition

$$Si\ y \in \mathbb{R},\ x = \arctan y \Leftrightarrow \begin{cases} -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \tan x = y \end{cases}$$

Quelques valeurs

$$\arctan(0) = 0$$

$$\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$$

$$\arctan(1) = \frac{\pi}{4}$$

$$\arctan(\sqrt{3}) = \frac{\pi}{3}$$

$$\arctan(2) = \arccos\left(\frac{1}{\sqrt{5}}\right)$$

$$\forall -\frac{\pi}{2} < y < -\frac{\pi}{2},\ \tan(\arctan(y)) = y$$

Pour tout entier $k \in \mathbb{Z}$, ${}^{\forall}x \in \mathbb{R}$,

$$\arctan(\tan(x)) = x - k\pi \text{ si } k\pi - \frac{\pi}{2} < x < k\pi + \frac{\pi}{2} \ .$$

Si $z = a + ib$, $a \geq 0$, $b \in \mathbb{R}$, alors $z = \rho e^{i\theta}$ où

$$\rho = \sqrt{a^2 + b^2} \text{ et } \theta = \text{ arctan}\left(\frac{b}{a}\right)$$

Exercice. $^\forall x \in \mathbb{R},\ \tan(n \arctan(x)) = \frac{p_n(x)}{q_n(x)}$
où $p_n(x) = \mathrm{Im}((1+ix)^n)$, $q_n(x) = \mathrm{Re}((1+ix)^n)$ sont des polynômes à coefficients entiers premiers entre eux.

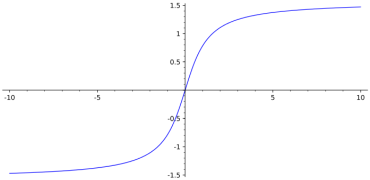
Indication : commencer par le cas où $n = 2$.

Propriétés

- impaire ;
- strictement croissante sur $\mathbb{R}$;
- $\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$, $\lim_{x\to+\infty} \arctan x = \frac{\pi}{2}$;
- ${}^{\forall}x \in \mathbb{R}$, $\arctan' x = \frac{1}{1+x^2}$.

Graphe de arctan

$$y = \arctan x$$



Formules.

$$\forall -1 \leq x \leq 1,\ \arcsin(x) = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\forall x \in \mathbb{R},\ \arctan(x) = \arcsin\left(\frac{x}{\sqrt{1+x^2}}\right)$$

Rappels.

$$\forall x, y \in \mathbb{R},\ \tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$

$$^{\forall}x, y, z \in \mathbb{R},\ \tan(x+y+z) =$$

$$\frac{\tan(x)+\tan(y)+\tan z-\tan(x)\tan(y)\tan(z)}{1-\tan(x)\tan(y)-\tan(y)\tan(z)-\tan(x)\tan(z)}$$

Exercice. Si $x + y + z = 0$, alors
$\tan(x) + \tan(y) + \tan(z) = \tan(x)\tan(y)\tan(z)$.

Quelques formules concernant **arctan**

a) $\arctan 1 + \arctan 2 + \arctan 3 = \pi$;

b) $\arctan(1/2) + \arctan(1/5) + \arctan(1/8) = \pi/4$;

c) $4\arctan(1/5) - \arctan(1/239) = \pi/4$;

d) $2\arctan(1/3) + \arctan(1/7) = \pi/4$.

Une série

$$\forall 0 \leq x \leq 1, \operatorname{arctan}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

En particulier :

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$$