Arithmétique

3.1.

$$3^2 = 9 \equiv 2\ [7] \Rightarrow 3^6 = (3^2)^3 \equiv 2^3 = 1\ [7]$$

donc $\forall k \in \mathbb{Z},\ 3^{6k} = 1^k = 1\ [7]$

Donc $3^{512} = 3^{510} \cdot 3^2 \equiv 2\ [7]$ car $6 \mid 510$.

$$2^3 = 1\ [7] \Rightarrow \forall k \in \mathbb{Z},\ 2^{3k} = 1\ [7]$$

Donc $2^{256} = 2^{255+1} = 2\ [7]$ car $3 \mid 255$

Conclusion $3^{512} - 2^{256} \equiv 2-2\ [7] = 0\ [7]$.

c-à-d $7 \mid 3^{512} - 2^{256}$

$$2^{70} + 3^{70} = (2^2)^{35} + (3^2)^{35} = 4^{35} - (-9)^{35}$$

car $a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \dots + (-b)^{n-1})$ si n impair.

$$= \underbrace{(4+9)}_{=13}(4^{34} - 4^{33} \times 9 + \dots + 9^{34})$$

donc $13 \mid 2^{70} + 3^{70}$.

3.2. Le dernier chiffre en base b (≥ 2) d'un entier n est le reste de la division euclidienne de n par b.

$$2012 = 2\ [3] \Rightarrow 2012^{2001} = 2^{2001}\ [3] = (-1)^{2001}\ [3]$$

$= -1\ [3] = 2\ [3]$.

Réponse : 2.

$2012 = 7 \times 287 + 3 \Rightarrow 2012 = 3\ [7]$.

Or $3^3 = 27 = -1\ [7] \Rightarrow 3^6 = (3^3)^2 = 1\ [7]$

donc $3^{2001} = 3^3\ [7]$ car $2001 = 3\ [6]$

$= -1\ [7] = 6\ [7]$.

Réponse : 6

$2012 = 2\ [10]$ donc $2012^{2001} = 2^{2001}\ [10]$

Or $2^2 = 4\ [10]$, $2^3 = 8\ [10]$, $2^4 = 6\ [10]$, $2^5 = 2\ [10]$,

etc... : $\forall n \in \mathbb{N}$, $2^{4n} = 6\ [10]$, $2^{4n+1} = 2\ [10]$, $2^{4n+2} = 4\ [10]$

$2^{4n+3} = 8\ [10]$

Comme $2001 = 1\ [4]$, $2012^{2001} = 2^{2001}\ [10]$

$= 2\ [10]$

Réponse : 2

$u_0 = 7$, $u_1 = 7^7$, $u_2 = 7^{(7^7)}$ $\left(\neq (7^7)^7\right)$

$u_n = 7^{\underbrace{7^{7^{\cdot^{\cdot^{\cdot^7}}}}}_{n \text{ fois}}}$

Modulo 3 : $u_0 = 7 = 1\ [3]$, $u_1 = 7^7 = 1^7\ [3] = 1\ [3]$

$\forall n$, $u_n = 7^{u_{n-1}} = 1^{u_{n-1}} = 1\ [3]$

donc $u_{7777} = 1\ [3]$

Modulo 7,

$$\forall n, \quad u_n = 7^{u_{n-1}} \equiv 0 \, [7]$$

$$\text{donc } u_{7777} \equiv 0 \, [7]$$

Modulo 10 : $\quad u_n = 7^{u_{n-1}} \equiv \, ?$

Comme $7^2 = 49 \equiv -1 \, [10]$, $7^4 \equiv 1 \, [10]$

et $7^m \, [10]$ ne dépend que de la classe de m modulo 4 :

$$7^{4n} \equiv 1 \quad [10] \quad (\forall n \in \mathbb{N}^*)$$
$$7^{4n+1} \equiv 7 \, [10] \quad (\forall n \in \mathbb{N})$$
$$7^{4n+2} \equiv 9 \, [10] \quad (\forall n \in \mathbb{N})$$
$$7^{4n+3} \equiv 63 \, [10] \equiv 3 \, [10] \quad (\forall n \in \mathbb{N})$$

Or $u_0 = 7 \equiv 3 \, [4]$, $u_1 = 7^7 \equiv 3^7 \, [4] \equiv (-1)^7 \, [4] \equiv -1 \, [4] \equiv 3 \, [4]$.

$$u_n = 7^{u_{n-1}} \equiv 3^{u_{n-1}} \, [4] \equiv (-1)^{u_{n-1}} \, [4] \equiv -1 \, [4]$$

car $\forall n \in \mathbb{N}$, u_n impair

donc $\forall n \in \mathbb{N}$, $u_n \equiv 3 \, [4] \Rightarrow \forall n \in \mathbb{N}, u_{n+1} = 7^{u_n} \equiv 3 \, [10]$

donc $\boxed{u_{7777} \equiv 3 \, [10]}$.

3.3

$$400\,000\,000\,000\,000\,000\,081 = 4 \times 10^{20} + 81$$

Or $10 \equiv -3 \, [13] \Rightarrow 10^3 \equiv -3^3 \, [13] \equiv -1 \, [13]$

$$\Rightarrow \quad 10^{20} = 10^{18} \cdot 10^{2} = (-1)^{6} \cdot (-3)^{2} \ [13]$$

$$= 9 \ [13] = -4 \ [13]$$

$$\Rightarrow \quad 4 \cdot 10^{20} + 81 = -16 + 81 \ [13] = 65 \ [13] = 0 \ [13]$$

Donc $13 \mid 4 \cdot 10^{20} + 81$. $\square$