AUTOMORPHISMS OF DRINFELD HALF-SPACES OVER A FINITE FIELD

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Abstract: We show that the automorphism group of Drinfeld's half-space over a finite field is the projective linear group of the underlying vector space. The proof of this result uses analytic geometry in the sense of Berkovich over the finite field equipped with the trivial valuation. We also take into account extensions of the base field.

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Introduction

In this note we determine the automorphism group of Drinfeld's half-spaces over a finite field. Given a finite-dimensional vector space V over a finite field k, the Drinfeld half-space $\Omega(V)$ is defined as the complement of all k-rational hyperplanes in the projective space $\mathbf{P}(V)$; it is an affine algebraic variety over k. We show that every k-automorphism of $\Omega(V)$ is induced by a k-automorphism of $\mathbf{P}(V)$. Hence the automorphism group of $\Omega(V)$ is equal to PGL(V). More generally, for an arbitrary field extension K of k, we prove that the natural injection of PGL(V) into $\mathrm{Aut}_K(\Omega(V)\otimes_k K)$ is an isomorphism. Our result answers a question of Dat, Orlik and Rapoport [**5**, p. 338] which was motivated by the analogous statement for Drinfeld half-spaces over a non-Archimedean local field (with non-trivial absolute value).

Drinfeld defined his p-adic upper half-spaces in [**6**]. They are the founding examples of the theory of period domains [**12**]. Analogs of period domains over finite fields have been considered by Rapoport in [**11**]. They are open subvarieties of flag varieties characterized by a semi-stability condition. Recently, they have been studied by Rapoport, Orlik and others, see e.g. [**9**], [**10**]. A good introduction is given in the book [**5**].

Over local non-Archimedean fields with non-trivial absolue value, Drinfeld half-spaces are no longer algebraic varieties and must be defined in the context of analytic geometry. In this setting, it was shown by Berkovich that every automorphism is induced by a projective linear transformation [**3**]. This was generalized to products of Drinfeld half-spaces by Alon [**1**], who also pointed out and corrected a discrepancy in Berkovich's proof. Berkovich's strategy is based on the fact that in the case of a local non-Archimedean ground field with non-trivial absolute value, the Bruhat-Tits building of the group PGL(V) is contained in $\Omega(V)$ as the subset of points satisfying a natural maximality condition. This implies that every automorphism of $\Omega(V)$ induces an automorphism of the Bruhat-Tits building, and with some further work (see [**1**]) one can prove the claim.

One could in fact use a similar strategy in order to determine the automorphism group of $\Omega(V)$ over a finite field. Indeed, if we endow the finite ground field with the trivial absolute value and look at the corresponding Berkovich analytic space $\Omega(V)^{an}$, by [**2**] the *vectorial building* associated to the group PGL(V) is contained in $\Omega(V)^{an}$. We believe that one can follow Berkovich's and Alon's arguments to deduce that every automorphism comes from an element of PGL(V).

However, in this note we adopt a slightly different and maybe more natural viewpoint. Thereby, we want to highlight that the true content of this theorem is about *extension* of automorphisms, and that it has in fact very little to do with buildings, see Remark 2.3. Our approach is the following. We consider the space X obtained by blowing up all linear subspaces of the projective space $\mathbf{P}(V)$. Irreducible components of the boundary divisor correspond bijectively to linear subspaces of $\mathbf{P}(V)$. Moreover, a family of components has non-empty intersection if and only if the corresponding linear subspaces form a flag. We use Berkovich analytic geometry to prove in Proposition 2.1 that every automorphism of $\Omega(V)$ preserves the set of discrete valuations on the function field induced by boundary components of X. Hence by Proposition 1.4 it extends to an automorphism of X. By taking a closer look at the Chow ring of X in section 3, we deduce that this automorphism preserves the set of discrete valuations corresponding to hyperplanes, which allows us to conclude that it induces an automorphism of the projective space.

This paper illustrates the usefulness of Berkovich analytic geometry over trivially valued ground fields as a method to solve algebraic problems. We believe that similar analytic tools can be applied to other questions in algebraic geometry.

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1. Automorphisms of Drinfeld's half-spaces

Let k be a finite field and let V be a k-vector space of dimension $n+1 \geqslant 2$. We denote by $\mathbf{P}(V)$ the projective scheme $\mathrm{Proj}\left(\mathrm{Sym}^\bullet V\right)$ and define the k-scheme $\Omega(V)$ as the complement of all (rational) hyperplanes in $\mathbf{P}(V)$:

$$\Omega(V) = \mathbf{P}(V) - \bigcup_{\substack{W \subset V \\ \dim W = 1}} \mathbf{P}(V/W).$$

For every field extension K/k we denote by $V_K = V \otimes_k K$ the induced vector space over K. Then the base change $\Omega(V)_K = \Omega(V) \otimes_k K$ is the complement of all k-rational hyperplanes in $\mathbf{P}(V_K) = \mathbf{P}(V) \otimes_k K$.

The main result of this note is the following.

Theorem 1.1. — *Let* V *be a vector space of dimension* $\geqslant 2$ *over a finite field* k.

(i) *The restriction map*

$$\mathrm{PGL}(V) = \mathrm{Aut}_k\big(\mathbf{P}(V)\big) \to \mathrm{Aut}_k\big(\Omega(V)\big), \quad \varphi \mapsto \varphi_{|\Omega(V)}$$

is an isomorphism. Equivalently, every k-automorphism of $\Omega(V)$ *extends to a k-automorphism of* $\mathbf{P}(V)$.

(ii) *For every field extension* K/k *the natural map*

$$\mathrm{PGL}(V) \longrightarrow \mathrm{Aut}_K\big(\Omega(V)_K\big)$$

is an isomorphism. Equivalently, every K*-automorphism of* $\Omega(V)_K$ *comes by base change from a* K*-automorphism of* $\mathbf{P}(V)$.

The proof combines analytic geometry in the sense of Berkovich with algebraic arguments. As a first step we show that every k-automorphism of $\Omega(V)$ can be extended to an automorphism of the k-scheme X we get by blowing-up all linear subspaces of $\mathbf{P}(V)$. For this step we use Berkovich analytic geometry over the field k endowed with the trivial absolute value. The second step is of an algebraic nature and consists in checking that this automorphism of X is induced by a k-automorphism of $\mathbf{P}(V)$. Here we analyze the geometry of the boundary divisor more closely and use an induction argument.

Given a proper subvector space W of V, applying Proj to the natural map $\mathrm{Sym}^\bullet(V) \twoheadrightarrow \mathrm{Sym}^\bullet(V/W)$ leads to a closed immersion $\mathbf{P}(V/W) \hookrightarrow \mathbf{P}(V)$ whose image L is called a *linear subspace* of $\mathbf{P}(V)$. Such a subscheme is said to be trivial if $L = \varnothing$ or $L = \mathbf{P}(V)$; it is called a *hyperplane* if it is of codimension 1. We denote by $\mathscr{L}^i(V)$ the set of linear subspaces of dimension i in $\mathbf{P}(V)$, and by $\mathscr{L}(V) = \bigcup_{0 \leqslant i \leqslant n-1} \mathscr{L}^i(V)$ the set of non-trivial linear subspaces.

Definition 1.2. — *We denote by* $\pi : \mathrm{X} \to \mathbf{P}(\mathrm{V})$ *the blow-up of* $\mathbf{P}(\mathrm{V})$ *along the full hyperplane arrangement. To be precise, X is defined as*

$$\mathrm{X} = \mathrm{X}_{n-1} \xrightarrow{\pi_{n-1}} \mathrm{X}_{n-2} \longrightarrow \cdots \longrightarrow \mathrm{X}_1 \xrightarrow{\pi_1} \mathrm{X}_0 \xrightarrow{\pi_0} \mathrm{X}_{-1} = \mathbf{P}(\mathrm{V})$$

with

$$\pi = \pi_0 \circ \pi_1 \circ \ldots \circ \pi_{n-1},$$

where π_i *denotes the blow-up of* X_{i-1} *along the strict transforms of linear subspaces of* $\mathbf{P}(\mathrm{V})$ *of dimension* i.

The scheme X is projective and smooth over k. It contains $\Omega(\mathrm{V})$ as an open dense subscheme since each π_i induces an isomorphism over $\Omega(\mathrm{V})$. We write $\mathrm{D} = \mathrm{X} - \Omega(\mathrm{V})$ for the complement.

Note that π_{n-1} is an isomorphism and that the strict transforms of two distinct linear subspaces $\mathrm{L}, \mathrm{L}' \subset \mathbf{P}(\mathrm{V})$ of dimension i in X_{i-1} are disjoint since (the strict transform of) $\mathrm{L} \cap \mathrm{L}'$ has been previously blown-up.

Each non-trivial linear subspace $\mathrm{L} \subset \mathbf{P}(\mathrm{V})$ defines a smooth and irreducible hypersurface E_L in X as follows. If L has dimension i, its strict transform by $\pi_0 \circ \pi_1 \circ \ldots \circ \pi_{i-1}$ in X_{i-1} (by convention L itself if it is a point) is blown-up under the map $\pi_i : \mathrm{X}_i \to \mathrm{X}_{i-1}$ to give rise to a hypersurface $\widetilde{\mathrm{L}}$ in X_i. The (codimension 1) subscheme E_L of X is then the strict transform of $\widetilde{\mathrm{L}}$ by $\pi_{i+1} \circ \ldots \circ \pi_{n-1}$. The induced map $\mathrm{E}_\mathrm{L} \to \widetilde{\mathrm{L}}$ coincides with the blow-up of $\widetilde{\mathrm{L}}$ along the hypersurface arrangement induced by hyperplanes of $\mathbf{P}(\mathrm{V})$ containing L. We have an alternative description of E_L as the closure

$$\overline{\pi^{-1}\Big(\mathrm{L} - \bigcup_{\substack{\mathrm{L}' \in \mathscr{L}(\mathrm{V}) \\ \mathrm{L}' \subsetneq \mathrm{L}}} \mathrm{L}'\Big)}$$

taken in X.

It follows from the construction of X that the boundary divisor D is the union of all hypersurfaces E_L, i.e. we have

$$\mathrm{D} = \pi^{-1}\Big(\bigcup_{\substack{\mathrm{W} \subset \mathrm{V} \\ \dim \mathrm{W} = 1}} \mathbf{P}(\mathrm{V}/\mathrm{W})\Big) = \bigcup_{\mathrm{L}} \mathrm{E}_\mathrm{L}.$$

Two components $\mathrm{E}_\mathrm{L}, \mathrm{E}_{\mathrm{L}'}$ have non-empty intersection if and only if $\mathrm{L} \subset \mathrm{L}'$ or $\mathrm{L}' \subset \mathrm{L}$. Indeed, if none of the inclusions holds, then L and L' intersect along a smaller linear subspace, say of dimension i, and the strict transforms of L and L' in X_i are disjoint. It follows that a family of components has non-empty intersection if and only if the indexing linear spaces lie in a flag.

Lemma 1.3. — *The divisor* D *has simple normal crossings and the strata occurring in its stratification by iterated regular loci are in one-to-one correspondence with flags of non-trivial linear subspaces of* $\mathbf{P}(\mathrm{V})$. *Moreover, if* Z *is the stratum corresponding to the flag* $\mathscr{F}$, *then*

$$\mathrm{U}_\mathrm{Z} = \mathrm{X} - \bigcup_{\mathrm{L} \notin \mathscr{F}} \mathrm{E}_\mathrm{L}$$

is an affine open subset of X *containing* Z *as a closed subset.*

Proof — Let $\mathscr{F} = (\mathrm{L}_0 \subset \ldots \subset \mathrm{L}_{n-1})$ be a complete flag of non-trivial linear subspaces of $\mathbf{P}(\mathrm{V})$. We consider the blow-up of $\mathbf{P}(\mathrm{V})$ along $\mathscr{F}$, i.e.

$$p : \mathrm{Y} = \mathrm{Y}_{n-1} \xrightarrow{p_{n-1}} \mathrm{Y}_{n-2} \longrightarrow \cdots \longrightarrow \mathrm{Y}_1 \xrightarrow{p_1} \mathrm{Y}_0 \xrightarrow{p_0} \mathrm{Y}_{-1} = \mathbf{P}(\mathrm{V})$$

where p_i denotes the blow-up of Y_{i-1} along the strict transform of L_i. By the universal property of blow-up, there exists a (unique) morphism of towers $f_\bullet : \mathrm{X}_\bullet \to \mathrm{Y}_\bullet$.

For every $i \in \{-1, \ldots, n-2\}$, we define two open subsets $\mathrm{U}_i \subset \mathrm{X}_i$ and $\mathrm{W}_i \subset \mathrm{Y}_i$ as follows:

- $\mathrm{U}_{-1} = \mathrm{W}_{-1}$ is the complement in $\mathbf{P}(\mathrm{V})$ of all 0-dimensional linear subspaces distinct from L_0;
- if $0 \leqslant i \leqslant n-2$, then U_i (resp. W_i) is the complement in $\pi_i^{-1}(\mathrm{U}_{i-1})$ (resp. in $p_i^{-1}(\mathrm{W}_{i-1})$) of the strict transforms of all $(i+1)$-dimensional linear subspaces $\mathrm{L} \subset \mathbf{P}(\mathrm{V})$ not in $\mathscr{F}$;

- $U_{n-1} = \pi_{n-1}^{-1}(U_{n-2})$ and $W_{n-1} = p_{n-1}^{-1}(W_{n-2})$.

Arguing by induction on i, we see that $U_i = f_i^{-1}(W_i)$, and that f_i induces an isomorphism between U_i and W_i respecting the restrictions of exceptional divisors. It is clear that

$$U_{n-1} = X - \bigcup_{L \notin \mathscr{F}} E_L.$$

On the other side, we claim that W_{n-1} coincides with the complement W'_{n-1} in Y of the strict transforms of all hyperplanes of $\mathbf{P}(V)$ which do not belong to $\mathscr{F}$ (i.e., distinct from L_{n-1}). The inclusion $W_{n-1} \subset W'_{n-1}$ is obvious. For every point $y \in Y - W_{n-1}$ there exists an index $i \in \{-1, \ldots, n-2\}$ such that the image y_i of y in Y_i lies in the strict transform of a $(i+1)$-dimensional linear subspace $L \subset \mathbf{P}(V)$ distinct from L_{i+1}. Let us consider a hyperplane H which contains L but not L_{i+1}, and let $\widetilde{H}$ denote its strict transform in Y_i; by construction, $y_i \in \widetilde{H}$. Since $L_j \not\subset H$ for $j \in \{i+1, \ldots, n-1\}$, the strict transform of $\widetilde{H}$ in Y_j is transverse to the center of p_j and thus coincides with the inverse image of $\widetilde{H}$ in Y_j. It follows that y belongs to the strict transform of $\widetilde{H}$ in Y, hence to the strict transform of H in Y, and thus $y \in Y - W'_{n-1}$. This proves the converse inclusion $W'_{n-1} \subset W_{n-1}$.

Given a basis $(e_0, e_1, \ldots, e_n)$ of V such that $L_i = Z(e_{i+1}, \ldots, e_n)$ for every $i \in \{0, \ldots, n-1\}$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k[t_1, \ldots, t_n]) & \xhookrightarrow{\ j\ } & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Spec}(k[x_1, \ldots, x_n]) & \hookrightarrow & \mathbf{P}(V) \end{array}$$

where the horizontal arrows are open immersions identifying $t_1, \ldots, t_n$ (resp. $x_1, \ldots, x_n$) with the rational functions $e_1/e_0, \ldots, e_n/e_{n-1}$ (resp. $e_1/e_0, \ldots, e_n/e_0$) and where q is the morphism defined by $q^*(x_i) = \prod_{j \leqslant i} t_j$. Via j, the open subscheme W_{n-1} of Y is isomorphic to the principal open subset $D(f)$ of Spec $(k[t_1, \ldots, t_n])$, where

$$f = \prod_{i=1}^{n} \prod_{(a_i, \ldots, a_n) \in k^{n-i+1}} (1 + a_i t_i + a_{i+1} t_i t_{i+1} + \ldots + a_n t_i \ldots t_n).$$

In particular, W_{n-1} is affine. Moreover, the intersection of the divisor D with the open affine set U_{n-1} has simple normal crossings. Since the sets U_{n-1} for all choices of complete flags form an open affine covering of X, the divisor D has simple normal crossings on X.

We now claim that the intersection Σ of any family of d irreducible components of D is either empty or irreducible. Indeed, assume that Σ is non-empty and reducible. Non-emptiness amounts to saying that these components are indexed by linear spaces in some flag $\mathscr{F}$. Pick a complete flag $\mathscr{F}'$ containing $\mathscr{F}$. As $U_{n-1} \cap \Sigma$ is irreductible there must be a component Σ_0 of Σ which lies in $X - U_{n-1}$. Since $X - U_{n-1}$ is by construction the union of some irreducible components of D, we see that Σ_0 must be contained in a $(d+1)$-th irreducible component of D. But this contradicts the normal crossing property of D. In view of the discussion before Lemma 1.3, this shows that the strata of D are in one-to-one correspondence with flags of linear subspaces.

Finally, if we start with a stratum Z corresponding to a partial flag $\mathscr{F}$, the set $U_Z = X - \bigcup_{L \notin \mathscr{F}} E_L$ is the intersection of all $U_{Z'}$ for strata Z' corresponding to complete flags containing $\mathscr{F}$. Hence it is open affine as a finite intersection of open affines in a separated k-scheme. □

In order to extend an automorphism of $\Omega(V)$ to first X and then to $\mathbf{P}(V)$, we look at its action on the discrete valuations associated to the components of D. For each $L \in \mathscr{L}(V)$, the local ring at the generic point of the hypersurface E_L is a discrete valuation ring in the function field $\kappa(V)$ of X. We denote by ord_L the corresponding discrete valuation on $\kappa(V)$, and we write

$$\Gamma(V) = \{\mathrm{ord}_L : L \in \mathscr{L}(V)\}$$

for the set of all these valuations. Note that $\kappa(V)$ is the function field of both $\mathbf{P}(V)$ and $\Omega(V)$. If L is a *hyperplane* in $\mathbf{P}(V)$, then the valuation ord_L is the one given by the local ring of $\mathbf{P}(V)$ at the generic point of L.

The sets $\mathscr{L}(\mathrm{V})$ and $\Gamma(\mathrm{V})$ come with a natural simplicial structure, for which the q-simplices correspond to flags of linear subspaces of length $q-1$.

Proposition 1.4. — *Let φ be a k-automorphism of $\Omega(\mathrm{V})$ and let φ^* be the induced automorphism of the set of valuations on the function field $\kappa(\mathrm{V})$.*

(i) *The birational map φ extends to a k-automorphism of* X *if and only if φ^* preserves the set $\Gamma(\mathrm{V})$ and its simplicial structure.*

(ii) *The birational map extends to a k-automorphism of $\mathbf{P}(\mathrm{V})$ if and only if φ^* preserves the subset of $\Gamma(\mathrm{V})$ defined by hyperplanes.*

Proof — (i) The condition is necessary because the simplicial set $\Gamma(\mathrm{V})$ describes the incidence relations between irreducible components of D (Lemma 1.3). To see that it is sufficient, we use the covering of X by the open affine subsets

$$\mathrm{U}_{\mathrm{Z}} = \mathrm{X} - \bigcup_{\mathrm{L}\notin\mathscr{F}} \mathrm{E}_{\mathrm{L}}$$

where Z denotes a stratum of D and $\mathscr{F}$ is the corresponding flag of linear subspaces of $\mathbf{P}(\mathrm{V})$. If φ preserves $\Gamma(\mathrm{V})$ with its simplicial structure, then there exists for every stratum Z another stratum Z' such that the rational map

$$\mathrm{U}_{\mathrm{Z}'} \dashrightarrow \mathrm{U}_{\mathrm{Z}}$$

induced by φ is defined at each point of height 1. Since U_{Z} is affine and $\mathrm{U}_{\mathrm{Z}'}$ is noetherian and normal, this rational map is everywhere defined on $\mathrm{U}_{\mathrm{Z}'}$ [**7**, 20.4.12] and therefore φ extends to an automorphism from X to X (apply this argument to φ^{-1}).

(ii) If the morphism $\varphi : \Omega(\mathrm{V}) \to \Omega(\mathrm{V})$ preserves all valuations $\mathrm{ord}_{\mathrm{L}}$ coming from hyperplanes, then for every hyperplane L in $\mathbf{P}(\mathrm{V})$ there exists a hyperplane L' such that the rational map

$$\mathbf{P}(\mathrm{V}) - \mathrm{L}' \dashrightarrow \mathbf{P}(\mathrm{V}) - \mathrm{L}$$

induced by φ is defined at every point of height 1, and the conclusion follows as for (i). □

2. Step 1 – Valuations and analytic geometry

This section is devoted to the first step toward the theorem, namely the fact that every k-automorphism of $\Omega(\mathrm{V})$ extends to a k-automorphism of X.

Proposition 2.1. — *Let* $\mathrm{Aut}_k(\mathrm{X},\mathrm{D})$ *denote the group of k-automorphisms of* X *which preserve* D. *The canonical map*

$$\mathrm{Aut}_k(\mathrm{X},\mathrm{D}) \to \mathrm{Aut}_k\big(\Omega(\mathrm{V})\big), \quad \varphi \mapsto \varphi_{|\Omega(\mathrm{V})}$$

is an isomorphism. Equivalently, every k-automorphism of $\Omega(\mathrm{V})$ extends to a k-automorphism of X.

We can study this problem from a nice geometric viewpoint in the framework of Berkovich spaces. Endowed with the trivial absolute value, k becomes a complete non-Archimedean field and there is a well-defined category of k-analytic spaces, together with an analytification functor $\mathrm{Z} \rightsquigarrow \mathrm{Z}^{\mathrm{an}}$ from the category of k-schemes locally of finite type. If Z is affine, then the topological space underlying Z^{an} is the set of multiplicative k-seminorms on $\mathscr{O}(\mathrm{Z})$ with the topology generated by evaluation maps $x \mapsto |f(x)| := x(f)$, where $f \in \mathscr{O}(\mathrm{Z})$. We refer to [**2**, Section 3.5] and [**13**, Section 1] for a detailed account.

Working in the analytic category over k allows us to realize $\Gamma(\mathrm{V})$ as a set of rays in $\Omega(\mathrm{V})^{\mathrm{an}}$: for each $\mathrm{L} \in \mathscr{L}(\mathrm{V})$, the map

$$\varepsilon_{\mathrm{L}} : (0,1] \to \Omega(\mathrm{V})^{\mathrm{an}}, \quad r \mapsto r^{\mathrm{ord}_{\mathrm{L}}(\cdot)}$$

is an embedding and $\varepsilon_{\mathrm{L}}(1)$ is the canonical point of $\Omega(\mathrm{V})^{\mathrm{an}}$, namely the point corresponding to the trivial absolute value on $\kappa(\mathrm{V})$. Now, the proposition will follow from the fact that this collection of rays is the 1-skeleton of a *conical complex* $\mathfrak{S}(\mathrm{V})$ in $\Omega(\mathrm{V})^{\mathrm{an}}$ which is preserved by every k-automorphism of $\Omega(\mathrm{V})$.

This conical complex $\mathfrak{S}(V)$ is the *fan* $\mathfrak{S}_0(X,D)$ of the toroidal embedding $\Omega(V) \hookrightarrow X$ described in [**13**, Section 3.1 and Proposition 4.7], following [**4**]. The lemma below shows how to see this conical complex inside the analytic space $\Omega(V)^{an}$. Note that "sp" below denotes the *specialization map* $X^{an} \to X$ (denoted by r in [**13**]).

Lemma 2.2. — *The following properties hold for any normal crossing divisor* D *on a proper smooth (connected) scheme* X*, with* $\Omega = X - D$.

(i) *The subset* $\mathfrak{S}_0(X,D)$ *is closed in* $\Omega^{an} = X^{an} - D^{an}$ *and there is a retraction* $\tau : \Omega^{an} \to \mathfrak{S}_0(X,D)$ *such that* $\tau^{-1}(x)$ *is a k-affinoid domain with Shilov boundary* $\{x\}$ *for every* $x \in \mathfrak{S}_0(X,D)$.

(ii) *For every stratum* Z *of* D *with generic point* η_Z*, the closed subset*

$$C_Z = \mathfrak{S}_0(X,D) \cap \overline{\mathrm{sp}^{-1}(\eta_Z)} = \mathfrak{S}_0(X,D) \cap \bigcap_{\eta_Z \in U \text{ open}} \mathrm{sp}^{-1}(U)$$

is a cone with an integral affine structure: if we let Λ_Z *denote the group of germs of invertible functions at* η_Z *on* $X - D$*, then the natural map*

$$C_Z \to \mathrm{Hom}_{\mathbf{Ab}}(\Lambda_Z, \mathbf{R}_{>0}), \quad x \mapsto (f \mapsto |f(x)|)$$

is an embedding whose image is a rational polyhedral cone.

In our particular situation, we have the following additional property:

(iii) *the map*

$$\iota : \mathfrak{S}(V) \to \mathrm{Hom}_{\mathbf{Ab}}\Big(\mathscr{O}\big(\Omega(V)\big)^{\times}, \mathbf{R}_{>0}\Big), \; x \mapsto (f \mapsto |f(x)|)$$

is a closed embedding inducing the integral affine structure on each cone. Moreover, (the image of) distinct cones span distinct linear spaces.

Proof. — First of all, note that X^{an} coincides with the space $X^{\beth}$ considered in [**13**] because X is proper [**loc.cit.**, Proposition 1.10]. Part (i) and (ii) of the statement are proved in [**13**, Section 3.1]. The assertion on the Shilov boundary of $\tau^{-1}(x)$ follows from the corresponding statement in the toric case [**loc.cit.**, Proposition 2.8]. Note also that the integral affine structure on the cone C_Z, corresponding to a stratum Z of D with generic point η_Z, is spanned by the $r = \mathrm{codim}\, Z$ functions $|t_i| : C_Z \to \mathbf{R}_{>0}$, where $(t_1, \dots, t_r)$ is a regular system of parameters on X at η_Z which defines D; compare [**loc.cit**, Definition 3.9 and Proposition 3.11].

Roughly speaking, point (iii) means that there are enough invertible functions on $\Omega(V)$. Consider a stratum Z of D corresponding to a flag $\mathscr{F}$ of non-trivial linear subspaces of $\mathbf{P}(V)$ and pick a basis $(e_0, \dots, e_n)$ of V such that $\mathscr{F}$ is a subflag of

$$Z(e_1, \dots, e_n) \subset Z(e_2, \dots, e_n) \subset \dots \subset Z(e_n).$$

The explicit description of X given at the end of the proof of Lemma 1.3 shows that $(e_1/e_0, e_2/e_1, \dots, e_n/e_{n-1})$ is a tuple of elements of $\mathscr{O}_{X,\eta_Z}$ which contains a regular system of parameters defining D at η_Z. Therefore, the map ι induces an integral affine embedding of the cone C_Z.

Furthermore, we claim that the following fact is true: *given two distinct cones* C, C'*, there exists* $f \in \mathscr{O}\big(\Omega(V)\big)^{\times}$ *such that* $|f| = 1$ *on one of them and* $|f| < 1$ *on the interior of the other.* Injectivity of the map ι and the last statement of (iii) follow immediately.

We finish the proof by establishing the claim. Given two non-zero vectors $v, v' \in V$ and a non-trivial linear subspace $L \subset \mathbf{P}(V)$, the function v/v' is either a unit, a uniformizer or the inverse of a uniformizer at the generic point of E_L. It follows that

(a) $|v/v'| < 1$ on $\varepsilon_L(0,1)$, if $L \subset Z(v)$ and $L \not\subset Z(v')$
(b) $|v/v'| > 1$ on $\varepsilon_L(0,1)$, if $L \subset Z(v')$ and $L \not\subset Z(v)$
(c) $|v/v'| = 1$ on $\varepsilon_L(0,1]$, if the hyperplanes $Z(v)$ and $Z(v')$ are in the same position with respect to L.

Consider two distinct strata Z, Z′ of D, corresponding to distinct flags $\mathscr{F}, \mathscr{F}'$ of non-trivial linear subspaces. Pick a linear space L occurring in only one of them, say $\mathscr{F}$, and set $i = \dim L$. We embed $\mathscr{F}'$

into a complete flag $(L_0 \subset L_1 \subset \ldots \subset L_{n-1})$ such that $L_i \neq L$. This assumption guarantees the existence of two hyperplanes H, H' such that

- $L \subset H$ and $L_i \not\subset H$
- $L_i \cap H = L_i \cap H'$ and $L \not\subset H'$.

In particular, H and H′ are in the same position with respect to $L_0, \ldots, L_{n-1}$. Given any equations $v, v' \in V$ of H and H′ respectively, we thus obtain $|v/v'| = 1$ on $C_{Z'}$. On the other hand, we have $|v/v'| < 1$ on the interior of $\varepsilon_L(0,1)$, hence also on the interior of C_Z since $\varepsilon_L(0,1] \subset C_Z$. □

Proof of proposition 2.1. — First, we observe that $\mathfrak{S}(V)$ coincides with the set $\Omega(V)^{an}_{max}$ of maximal points of $\Omega(V)^{an}$ for the following ordering:

$$x \preccurlyeq y \iff \forall f \in \mathcal{O}(\Omega(V)^{an}),\ |f(x)| \leqslant |f(y)|.$$

The inclusion $\Omega(V)^{an}_{max} \subset \mathfrak{S}(V)$ follows from (i) since we have $x \preccurlyeq \tau(x)$ for every point $x \in \Omega(V)^{an}$. We apply (iii) to get the converse inclusion: for any two distinct points x, y in $\mathfrak{S}(V)$, there exists $f \in \mathcal{O}(\Omega(V)^{an})^\times$ such that $|f(x)| \neq |f(y)|$, hence such that $|f(x)| < |f(y)|$ and $|(\frac{1}{f})(x)| > |(\frac{1}{f})(y)|$ or vice versa, and therefore x and y are incomparable.

The above characterization of $\mathfrak{S}(V)$ as a closed subset of $\Omega(V)^{an}$ implies that it is preserved by any k-automorphism φ of $\Omega(V)$. It remains to check that the homeomorphism of $\mathfrak{S}(V)$ induced by φ also preserves the conical structure. Let Φ denote the linear automorphism of $\mathrm{Hom}_{\mathbf{Ab}}(\mathcal{O}(\Omega(V))^\times, \mathbf{R}_{>0})$ deduced from φ. Given an n-dimensional cone $C \subset \mathfrak{S}(V)$, the image of its interior is disjoint from the $(n-1)$-skeleton of $\mathfrak{S}(V)$; otherwise it would meet the interiors of two distinct n-dimensional cones C', C'' (note that x lies in the interior of $\mathfrak{S}(V)$, hence $\langle \iota\, C' \rangle = \Phi(\langle \iota\, C \rangle) = \langle \iota\, C'' \rangle$ contradicting (iii). It follows that if $\varphi(C)$ is contained in some n-dimensional cone C', and thus $\varphi(C) = C'$ by considering φ^{-1}. The assertion for lower dimension cones follows at once by considering faces.

In particular, we see that φ preserves the 1-skeleton of $\mathfrak{S}(V)$, hence the set $\Gamma(V)$ of discrete valuations on $\kappa(V)$ associated with irreducible components of $D = X - \Omega(V)$, together with the simplicial structure reflecting the incidence relations between these components. By Proposition 1.4 (i), this implies that φ extends to a k-automorphism of X. □

Remark 2.3. —

1. Let D be a simple normal crossing divisor on a smooth and proper (connected) scheme X over k. Even if $\Omega(V) = X - D$ is affine, condition (iii) and its consequences may fail. For example, consider the case $X = \mathbf{P}^n_k$. If D is a hyperplane, then $\mathfrak{S}_0(X, D)$ is a 1-dimensional cone whereas $\Omega(V)^{an}_{max}$ is empty. If D is the union of the coordinate hyperplanes, then $\Omega(V) = \mathbf{G}^n_m$ and $\mathfrak{S}_0(X, D) = \Omega(V)^{an}_{max}$ is the toric fan, but the map ι is bijective, hence all maximal cones span the same linear space. In fact, the inversion $(t_1, \ldots, t_n) \mapsto (t_1^{-1}, \ldots, t_n^{-1})$ on $\mathbf{G}^n_m$ transforms the fan $\mathfrak{S}_0(X, D)$ into its opposite, hence does not preserve the conical structure. This reflects the fact that this automorphism of $\mathbf{G}^n_m$ does not extend to $\mathbf{P}^n$.
2. The conical complex $\mathfrak{S}(V)$ is also the *vectorial building* of PGL(V), but this is somehow fortuitous and irrelevent from the viewpoint of automorphisms. In general, there exists for any connected and split semi-simple k-group G a canonical embedding of the vectorial building $\mathcal{V}(G, k)$ of $G(k)$ into the analytification of an open affine subscheme Ω in any flag variety Y of G [**2**, Section 5.5]. However, this observation does not lead to a generalization of Theorem 1.1, at least along the lines of the present proof. Indeed, while we made crucial use of the fact that $\mathfrak{S}(V)$ is the fan of a normal crossing divisor, we doubt that $\mathcal{V}(G, k)$ can be realized as the fan of a toroidal compactification of $\Omega(V)$ if $(G', Y) \neq \big(\mathrm{PGL}(V), \mathbf{P}(V)\big), \big(\mathrm{PGL}(V), \mathbf{P}(V^\vee)\big)$.
3. It may be interesting to try to extend our method, based on the study of toroidal compactifications, to determine the automorphism groups of other period domains.
4. Whether the above proposition can be proved without analytic geometry is not clear.

3. Step 2 – Geometry of the blow-up

The second step in the proof of the theorem relies on elementary intersection theory on X, which we review in this section. The standard reference is [**8**].

The Chow ring CH^* is a contravariant functor from the category of smooth k-schemes to the category of graded commutative rings. For any smooth k-scheme X, the abelian group underlying $\mathrm{CH}^*(\mathrm{X})$ is the free abelian group on integral subschemes of X modulo rational equivalence, and it is graded by codimension. Multiplication comes from the intersection product. We write [Z] for the class of a closed subscheme Z of X.

We are going to use the following two basic facts.

(a) Let Y be a regularly embedded closed subscheme of X and let $\pi : \widetilde{\mathrm{X}} \to \mathrm{X}$ be the blow-up of X along Y, with exceptional divisor $\widetilde{\mathrm{Y}}$. The canonical map

$$\mathrm{CH}^1(\mathrm{X}) \oplus \mathbf{Z}[\widetilde{\mathrm{Y}}] \to \mathrm{CH}^1(\widetilde{\mathrm{X}}), \quad (z, n[\widetilde{\mathrm{Y}}]) \mapsto \pi^*(z) + n[\widetilde{\mathrm{Y}}]$$

is an isomorphism [**8**, Proposition 6.7].

(b) In the situation of (a), let V be an integral subscheme of X with strict transform $\widetilde{\mathrm{V}}$. If $\operatorname{codim}(\mathrm{Y},\mathrm{X}) \leqslant \operatorname{codim}(\mathrm{V}\cap\mathrm{Y},\mathrm{V})$, then

$$\pi^*[\mathrm{V}] = [\widetilde{\mathrm{V}}]$$

in $\mathrm{CH}^*(\widetilde{\mathrm{X}})$ [**8**, Corollary 6.7.2].

Now we focus on the particular case where $\pi : \mathrm{X} \to \mathbf{P}(\mathrm{V})$ is the blow-up along the full hyperplane arrangement, with exceptional divisor D.

Lemma 3.1. — *We have*

$$\mathrm{CH}^1(\mathrm{X}) = \mathbf{Z}h \oplus \bigoplus_{\mathrm{L}} \mathbf{Z}[\mathrm{E}_{\mathrm{L}}],$$

where $h = \pi^*[\mathrm{H}]$ *denotes the pull-back of the hyperplane class* [H] *on* $\mathbf{P}(\mathrm{V})$ *and* L *runs over the set of non-trivial linear subspaces of* $\mathbf{P}(\mathrm{V})$ *of codimension at least* 2.

Proof. — For any non-trivial linear subspace L of $\mathbf{P}(\mathrm{V})$ of dimension $i \in \{0,\ldots,n-1\}$, let $\widetilde{\mathrm{L}} \subset \mathrm{X}_i$ denote the blow-up of its strict transform in X_{i-1}; this is a smooth irreducible hypersurface. Recall that we have $\pi = \pi_0 \circ \pi_1 \circ \ldots \circ \pi_{n-1}$, where π_{n-1} is an isomorphism. Applying (a) iteratively to each blow-up $\pi_0,\ldots,\pi_{n-2}$, we obtain that $\mathrm{CH}^1(\mathrm{X})$ is the free abelian group on $\pi^* h$ and the classes $(\pi_{i+1} \circ \ldots \circ \pi_{n-1})^*[\widetilde{\mathrm{L}}]$, where $i \in \{0,\ldots,n-2\}$ and L runs over the set of i-dimensional linear subspaces of $\mathbf{P}(\mathrm{V})$.

The conclusion follows from the additional fact that we have an equality

$$(\pi_{i+1} \circ \ldots \circ \pi_{n-1})^*[\widetilde{\mathrm{L}}] = [\mathrm{E}_{\mathrm{L}}]$$

in $\mathrm{CH}^1(\mathrm{X})$ for any linear subspace L of dimension $i \in \{0,\ldots,n-2\}$. This is an immediate consequence of (b), since the center of each blow-up π_j, with $j \in \{i+1,\ldots,n-1\}$, is transverse to the strict transform of $\widetilde{\mathrm{L}}$ in X_{j-1}. □

For each integer $d \geqslant 1$, we define

$$\lambda(d) = \#\left\{ \begin{array}{c} \text{non-trivial linear subspaces} \\ \text{of codimension } \geqslant 2 \quad \text{in } \mathbf{P}^d_k \end{array} \right\}.$$

Lemma 3.2. — *Let* $\mathrm{L} \subset \mathbf{P}(\mathrm{V})$ *be a non-trivial linear subspace of dimension* d*; note that* $d \in \{0,\ldots,n-1\}$.

(i) *We have*

$$\operatorname{rk}\mathrm{CH}^1(\mathrm{E}_{\mathrm{L}}) = \lambda(d) + \lambda(n-1-d) + \varepsilon(d),$$

where $\varepsilon(d) = 1$ *if* $d \in \{0, n-1\}$ *and* $\varepsilon(d) = 2$ *otherwise.*

(ii) *For every linear subspace* $\mathrm{L}' \subset \mathbf{P}(\mathrm{V})$ *of dimension* d' *satisfying* $d < d' < n-1-d$*, the following inequality holds*

$$\operatorname{rk}\mathrm{CH}^1(\mathrm{E}_{\mathrm{L}}) > \operatorname{rk}\mathrm{CH}^1(\mathrm{E}_{\mathrm{L}'}).$$

Proof. — (i) Let L_{d-1} (resp. $\widetilde{L}$) denote the strict transform of L in X_{d-1} (resp. in X_d). The scheme E_L is the blow-up of $\widetilde{L}$ along the hypersurface arrangement induced by hyperplanes of $\mathbf{P}(V)$ containing L. Applying (a), we obtain

$$\begin{aligned} \operatorname{rk} CH^1(E_L) &= \operatorname{rk} CH^1(\widetilde{L}) + \#\left\{ \begin{array}{c} \text{linear spaces of codim } \geqslant 2 \\ \text{strictly containing } L \end{array} \right\} \\ &= \operatorname{rk} CH^1(\widetilde{L}) + \lambda(n-d-1). \end{aligned}$$

Since $\widetilde{L} = \mathbf{P}(\mathscr{N})$, where $\mathscr{N}$ is the conormal sheaf to L_{d-1} in X_{d-1}, of rank $n-d$, it follows from [**8**, Theorem 3.3, (b)] that

$$\operatorname{rk} CH^1(\widetilde{L}) = \operatorname{rk} CH^0(L_{d-1}) + \operatorname{rk} CH^1(L_{d-1}) = 1 + \operatorname{rk} CH^1(L_{d-1})$$

if $0 \leqslant d < n-1$, and

$$\operatorname{rk} CH^1(\widetilde{L}) = \operatorname{rk} CH^1(L_{d-1})$$

if $d = n-1$.

Finally, since L_{d-1} is the blow-up of L along the full hyperplane arrangement,

$$\operatorname{rk} CH^1(L_{d-1}) = \operatorname{rk} CH^1(L) + \#\left\{ \begin{array}{c} \text{non-trivial linear subspaces} \\ \text{of codimension } \geqslant 2 \text{ in } L \end{array} \right\},$$

hence

$$\operatorname{rk} CH^1(L_{d-1}) = \begin{cases} 1+\lambda(d) & \text{if } 0 < d \leqslant n-1 \\ 0 & \text{if } d = 0. \end{cases}$$

(ii) In view of (i), it is enough to prove the inequality

$$\lambda(n-1-d) - \lambda(n-1-d') > \lambda(d') - \lambda(d) + 1$$

for any $d, d' \in \{0, \dots, n-1\}$ such that $d < d' < n-1-d$. Since λ is increasing and $0 < d' < n-1-d$, this would follow from the coarser inequality

$$\lambda(t) - \lambda(t-1) > \lambda(t-1) + 1$$

for every integer $t \geqslant 2$. If we fix a hyperplane H and count non-trivial linear subspaces of codimension $\geqslant 2$ in $\mathbf{P}^t_k$ taking into account their position with respect to H (transverse to H, or of codimension $\geqslant 2$ or $= 1$ in H), we obtain for $t \geqslant 2$

$$\lambda(t) = \nu(t) + \lambda(t-1) + \#\,\mathbf{P}^{t-1}(k) > \nu(t) + \lambda(t-1) + 1,$$

where $\nu(t)$ denotes the number of non-trivial linear subspaces of codimension at least 2 in $\mathbf{P}^t_k$ which are not contained in H. Hence, it is enough to prove the inequality

$$\nu(t) \geqslant \lambda(t-1)$$

for every integer $t \geqslant 2$. But this is obvious: given a hyperplane $\mathbf{P}^{t-1}_k \subset \mathbf{P}^t_k$ and a rational point p in the complement of $\mathbf{P}^{t-1}(k)$, the map $L \mapsto \langle L, p\rangle$ embeds the set of codimension d linear subspaces of $\mathbf{P}^{t-1}_k$ into the set of codimension d linear subspaces of $\mathbf{P}^t_k$ which are not contained in $\mathbf{P}^{t-1}_k$. □

4. Conclusion

We can now prove Theorem 1.1.

Let us first show part (i). Every k-automorphism φ of $\Omega(V)$ extends to a k-automorphism $\widetilde{\varphi}$ of the blow-up X by Proposition 2.1. Hence it induces a permutation $\hat{\varphi}$ of non-trivial linear subspaces of $\mathbf{P}(V)$ defined by $\widetilde{\varphi}(E_L) = E_{\hat{\varphi}(L)}$. By Proposition 1.4 (ii) it suffices to prove that this permutation preserves the subset of all *hyperplanes*.

We argue by induction on $n = \dim V - 1 \geqslant 1$. For $n = 1$, the result is obvious. For $n = 2$, it is enough to compare self-intersections of components of D to conclude: for a point p and a line ℓ,

$$\deg\,[E_p]^2 = -1 \quad \text{and} \quad \deg\,[E_\ell]^2 = \deg\left(h - \sum_{p\in\ell(k)} [E_\ell]\right)^2 = 1 - \#\,\ell(k) = -(\#\,k),$$

thus $\hat{\varphi}$ maps a line to a line.

In general, for any rational hyperplane H of $\mathbf{P}(V)$, it follows from Lemma 3.2 that $\hat{\varphi}(H)$ is either a hyperplane or a rational point. Let us now assume that n is at least 3 and that the theorem has been proved in lower dimension. If $\hat{\varphi}(H)$ is a rational point p, then $\widetilde{\varphi}$ induces a k-isomorphism $\bar{\varphi}$ between E_H and E_p which maps the divisor $D_H = \bigcup_{L\neq H} E_H \cap E_L$ onto the divisor $D_p = \bigcup_{L\neq\{p\}} E_p \cap E_L$.

Since E_H (resp. E_p) is the blow-up of H (resp. $\mathbf{P}(T_p^\vee)$, where T_p denotes the tangent space of $\mathbf{P}(V)$ at p) along the full hyperplane arrangement, with exceptional divisor D_H (resp. D_p), the theorem in dimension $n-1$ implies that $\bar{\varphi}$ is induced by a k-isomorphism between H and $\mathbf{P}(T_p^\vee)$, hence maps the components of D_H defined by rational points of H to components of D_p defined by rational points of $\mathbf{P}(T_p^\vee)$, which is to say by (rational) lines in $\mathbf{P}(V)$ containing p.

Let q be a rational point of H and let ℓ denote the line in $\mathbf{P}(V)$ such that

$$\widetilde{\varphi}(E_H \cap E_q) = E_p \cap E_\ell.$$

The two hypersurfaces E_ℓ and $\widetilde{\varphi}(E_q)$ have the same non-empty intersection with $\widetilde{\varphi}(E_H) = E_p$, so

$$\widetilde{\varphi}(E_q) = E_\ell$$

since D is a normal crossing divisor. By Lemma 3.2, this implies $n = 2$ while we assumed $n \geqslant 3$.

Therefore, $\hat{\varphi}$ preserves the set of hyperplanes, and our claim follows from Proposition 1.4 (ii).

We now indicate how to prove the second part of Theorem 1.1. For every field extension K/k, the base change $\Omega(V)_K$ of $\Omega(V)$ coincides with the complement in $\mathbf{P}(V)_K$ of all *k-rational* hyperplanes. Since blow-ups commute with base change, the K-scheme $X_K = X \otimes_k K$ can be obtained by blowing up $\mathbf{P}(V)_K$ along the arrangement of all k-rational hyperplanes. Moreover, every irreducible components E_L of D is geometrically irreducible, and its base change $(E_L)_K$ is the irreducible component of $X_K - \Omega(V)_K$ corresponding to the k-rational linear subspace L_K of $\mathbf{P}(V)_K$.

Let us consider a K-automorphism φ of $\Omega(V)_K$. One proves exactly as in Proposition 1.4 that φ extends to a K-automorphism of X_K (resp. of $\mathbf{P}(V)_K$) if and only if φ preserves the simplicial set $\Gamma(V_K)$ of discrete valuations on $\kappa(V_K)$ coming from irreducible components of D_K (resp. preserves the subset of $\Gamma(V_K)$ corresponding to hyperplanes). Once again, this condition is established via analytic geometry over the field K endowed with the trivial absolute value. The key point is Lemma 2.2, which holds for the fan $\mathfrak{S}(V_K)$ of the normal crossing divisor D_K on X_K: this is clear for the first two assertions, and the proof of the third one works verbatim (one could also argue that $\mathfrak{S}(V_K)$ coincides with the inverse image of $\mathfrak{S}(V)$ under the projection map $p: X_K^{an} \to X^{an}$, so (iii) holds for $\mathfrak{S}(V_K)$ since it holds for $\mathfrak{S}(V)$.) We then prove as above that φ extends to a K-automorphim of X_K.

Lemma 3.1 and 3.2 also hold for $\mathbf{P}(V)_K$, when we replace *linear subspaces* by *k-rational subspaces*. The argument of part (i) shows that the permutation of k-rational linear subspaces induced by $\widetilde{\varphi}$ perserves the hyperplanes, hence φ induces a K-automorphism of $\mathbf{P}(V)_K$. Since this automorphism preserves the set of k-rational hyperplanes, it is induced by a k-automorphism of $\mathbf{P}(V)$.

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