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The spherical p-spin interaction spin glass model: the statics

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The static properties of the spherical p-spin interaction spin glass model are calculated using the replica method. It is shown that within the Parisi scheme the most general solution is the one-step replica symmetry breaking. The transition from the replica symmetric solution to the replica replica symmetry broken one is either continuous or discontinuous in q_1-q_0 depending on the strength of the external magnetic field. The model can be solved explicitly for any p at any temperature and magnetic field. Below the transition we find an infinite number of metastable states.

1. Introduction

In the last years much effort has been spent to understand the low temperature phase of spin systems with random quenched couplings, namely spin glasses (SG). The main feature of these systems is that, due to frustration and to the randomness of the couplings, the free energy landscape presents many minimum states, separated by very high barriers, not related by any symmetry one to another. As a consequence, the equilibrium state of the system at low temperatures may depend on its initial state, even in the thermodynamic limit, and the ergodicity is broken. In this scenario it is clear that even a mean field theory, which usually is the first step towards the understanding of the phases, can be highly non-trivial. Indeed the infinte range Sherrington and Kirkpatrick (SK) model [1], for which the mean field solution should be correct, already exhibits the main problems of such systems.

It is now generally believed that the SK model can be solved by using the replica trick. In this method one considers n non-interacting replicas of the system, which allows for replacing the quenched averages by the $n \to 0$ limit of annealed averages. However, the breaking of the ergodicity in the low-temperature phase implies that the permutation symmetry between replicas is broken [2]. The general form of this breaking is, however, not known. Parisi proposed a specific form based on a hierarchical replica symmetry breaking [3] which, when the number of breaking is sent to infinity, produces a stable mean field theory [4]. This *ansatz* has a natural interpretation in an ultra-metric organisation of the pure states of the system [5]. These results rely, however, on the particular replica symmetry breaking scheme.

Some years ago Gross and Mézard [6] showed that an infinte range spin glass model with p-spin interaction (pSG) is solved exactly by the first step of the Parisi replica symmetry breaking scheme (1 RSB) in the limit $p \to \infty$. In a subsequent work, Gardner found that there is a range of temperature where the 1 RSB gives the correct solution for any $p > 2$ [7]. This range shrinks to zero as $p \to 2$, whereas the lower transition temperature goes to zero for $p \to \infty$. Thus this model is a good candidate for a better understanding of the replica symmetry breaking.

In these works the spins were Ising spins, and this makes it difficult to solve the model completely. It would then be useful to have a model which can be solved exactly, but which retains the main features of the Ising spin pSG. To this end, in this paper we consider a spherical version of the model which presents the required properties, namely it can be solved for any p and it has a 1 RSB phase. A soft-spin version of the pSG for p close to 2 was previously considered by Kirkpatrick and Thirumalai [8] in a context of a dynamical study. In the limit of hard spins model leads to static properties similar to those found by Gardner.

In this paper we consider only the static properties of the spherical pSG. The dynamics will be discussed in a separate paper [9]. The paper is organised as follows. In Sect. 2 we introduce the model. In Sect. 3 the model is solved using the replica trick, but no assumption on the structure of the overlaps $q_{\alpha\beta}$ is done. The replica symmetric (RS) solution is discussed in Sect. 4, together with

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its stability. In Sect. 5 the Parisi one-step replica symmetry breaking solution is introduced. This is shown to be the most general solution of the model in the Parisi scheme. Details of the proof are in Appendix 2. In this section we also discuss the transitions from the RS to the 1 RSB solution. These are of two kinds. For low magnetic fields the transition takes place discontinuously with a jump in q_1-q_0. Above a critical field the transition becomes continuous. The behaviour of some thermodynamic quantities at the transition is discussed in Sect. 6. Finally Sect. 7 is devoted to some conclusions and discussion. In Appendix 3 the interested reader can find some details on the calculation of the eigenvalues of the Gaussian fluctuations about the mean field theory for the 1 RSB.

2. The model

The p-spin interaction spin glass spherical model describes a system of N continuous spins σ_i with random quenched infinite range p-spin interactions. It is defined by the Hamiltonian

$$H=-\sum_{1\leq i_1<\cdots<i_p\leq N} J_{i_1\ldots i_p}\sigma_{i_1}\cdots\sigma_{i_p}-h\sum_{i=1}^{N}\sigma_i. \tag{2.1}$$

The interaction strengths $J_{i_1\ldots i_p}$ are independent random variables which, for simplicity, can be taken to be Gaussian with zero mean and variance

$$\overline{(J_{i_1\ldots i_p})^2}=\frac{J^2 p!}{2N^{p-1}};\qquad 1\leqq i_1<\cdots<i_p\leqq N. \tag{2.2}$$

This scaling with N ensures an extensive free energy, i.e. proportional to N. The spin variables σ_i are continuous real variables which can range from $-\infty$ to $+\infty$, thus to make the model well defined, the global constraint on their magnitude (spherical constraint)

$$\sum_{i=1}^{N}\sigma_i^2=N \tag{2.3}$$

is added. The trace on the spins is then defined as

$$\mathrm{Tr}_\sigma(\cdots)\equiv 2\sqrt{N}\int_{-\infty}^{+\infty}\prod_{i=1}^{N}\mathrm{d}\sigma_i\,\delta\left(N-\sum_{i=1}^{N}\sigma_i^2\right)(\cdots). \tag{2.4}$$

In the case $p=2$, the model reduces to the spherical model introduced by Kosterlitz et al. [10].

Note that $\mathrm{Tr}_\sigma(1)$ is equal to the surface of the N dimensional sphere with radius $N^{1/2}$. Its logarithm is the entropy S of the model at infinite temperature T. For large N it is given by:

$$\mathrm{Tr}_\sigma(1)=\mathrm{e}^{S(\infty)} = \mathrm{e}^{N[1+\ln(2\pi)]/2}\pi^{-1/2}(1+O(1/N)) \tag{2.5}$$

3. The replica method

The static properties of the spherical pSG can be obtained by using the formalism introduced by Gross and Mézard for the Ising spin case [6].

For any fixed realization $\mathbf{J}$ of the couplings, the partition function of the system is given by

$$Z[\mathbf{J}]=\mathrm{Tr}_\sigma\exp[-\beta H[\mathbf{J}]] \tag{3.1}$$

and the quenched free energy

$$\beta F=-\overline{\ln Z}. \tag{3.2}$$

The analytic computation of this average is a very difficult problem, even in simple cases as nearest neighbour one-dimensional models [11]. However, since the integer moments of the partition function are easier to compute, the standard method for performing the averages over the quenched couplings is to introduce n non-interacting replicas of the system, calculate annealed averges and then take the limit to $n\to 0$. In this approach the average free energy is obtained as

$$\beta F=-\lim_{n\to 0}\frac{1}{n}(\overline{Z^n}-1) \tag{3.3}$$

where

$$\begin{aligned}\overline{Z^n}&=\overline{\mathrm{Tr}_\sigma\exp\left[-\beta\sum_{\alpha=1}^{n}H_\alpha\right]}\\ &=\mathrm{Tr}_\sigma\exp\left[\frac{(\beta J)^2}{4}N\sum_{\alpha\beta}\frac{p!}{N^p}\right.\\ &\quad\times\sum_{1\leqq i_1<\cdots<i_p\leqq N}\sigma_{i_1\alpha}\sigma_{i_1\beta}\cdots\sigma_{i_p\alpha}\sigma_{i_p\beta}\\ &\quad\left.+\beta h\sum_{i\alpha}\sigma_{i\alpha}\right].\end{aligned} \tag{3.4}$$

Since we are eventually interested in the $n\to 0$ limit, and in the thermodynamic limit $N\to\infty$ we can calculate the above partition function only up to the first leading terms. For this we replace the restricted sum over $i_1<\cdots<i_p$ by

$$p!\sum_{i_1<\cdots<i_p}=\sum_{i_1,\ldots,i_p}-\frac{p(p-1)}{2}\sum_{i_1,i_1\neq i_3,\ldots,i_p}$$

which is correct to order $O(1/N^2)$.

As in the Ising case, once the average over the couplings is performed, the effective Hamiltonian depends on the overlap function of the replicas

$$q_{\alpha\beta}=\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\alpha}\sigma_{i\beta}. \tag{3.5}$$

For any finite n the matrix $\mathbf{q}$ is symmetric and positive semidefinite, $\mathbf{q}\geqq 0$. Note also that the spherical constraint (2.3) implies $q_{\alpha\alpha}=1$ for the diagonal elements. This, together with the semipositiveness of $\mathbf{q}$, leads to

$$-1\leqq q_{\alpha\beta}\leqq 1\qquad\forall\alpha,\beta. \tag{3.6}$$

The spin trace in (3.4) can be performed by introducing the constraints (2.3) and (3.5) by the conjugate variables $\lambda_{\alpha\alpha}$ and $\lambda_{\alpha\beta}(=\lambda_{\beta\alpha})$ using the relations

$$1=\int_{\mathbf{q}>0}\prod_{\alpha<\beta}\mathrm{d}q_{\alpha\beta}\int_{-i\infty}^{+i\infty}\prod_{\alpha<\beta}\frac{N}{2\pi i}\mathrm{d}\lambda_{\alpha\beta}\times\exp\left[-\tfrac{1}{2}\sum_{\alpha\neq\beta}\lambda_{\alpha\beta}\left(Nq_{\alpha\beta}-\sum_{i=1}^{N}\sigma_{i\alpha}\sigma_{i\beta}\right)\right]$$

and

$$\prod_{\alpha}\delta\left(N-\sum_{i=1}^{N}\sigma_{i\alpha}^2\right)=\int_{-i\infty}^{+i\infty}\prod_{\alpha}\frac{\mathrm{d}\lambda_{\alpha\alpha}}{4\pi i}\exp\left[-\tfrac{1}{2}\sum_{\alpha}\lambda_{\alpha\alpha}\left(Nq_{\alpha\alpha}-\sum_{i=1}^{N}\sigma_{i\alpha}^2\right)\right]$$

where use has been made of $q_{\alpha\alpha}=1$. One then gets

$$\overline{Z^n}=\int_{\mathbf{q}>0}\prod_{\alpha<\beta}\mathrm{d}q_{\alpha\beta}\int_{-i\infty}^{+i\infty}\prod_{\alpha<\beta}\frac{N}{2\pi i}\times\mathrm{d}\lambda_{\alpha\beta}\int_{-i\infty}^{+i\infty}\prod_{\alpha}\frac{\sqrt{N}}{2\pi i}\mathrm{d}\lambda_{\alpha\alpha}\,\mathrm{e}^{-NG} \tag{3.7}$$

with

$$G[\mathbf{q},\boldsymbol{\lambda}]=-\frac{(\beta J)^2}{4}\sum_{\alpha\beta}q_{\alpha\beta}^p+\tfrac{1}{2}\sum_{\alpha\beta}\lambda_{\alpha\beta}q_{\alpha\beta}+\frac{(\beta J)^2}{8N}p(p-1)\times\sum_{\alpha\beta}\langle\sigma_\alpha^2\sigma_\beta^2\rangle q_{\alpha\beta}^{p-2}-\ln\int_{-\infty}^{+\infty}\prod_{\alpha}\mathrm{d}\sigma_\alpha\times\exp\left[\tfrac{1}{2}\sum_{\alpha\beta}\lambda_{\alpha\beta}\sigma_\alpha\sigma_\beta+\beta h\sum_{\alpha}\sigma_\alpha\right] \tag{3.8}$$

where $\langle\cdots\rangle$ means spin average with the Hamiltonian

$$\tfrac{1}{2}\sum_{\alpha\beta}\lambda_{\alpha\beta}\sigma_\alpha\sigma_\beta+\beta h\sum_{\alpha}\sigma_\alpha.$$

The $q_{\alpha\beta}$ integration can be restricted to the region where $\mathbf{q}$ is positive definite.

The integral in the logarithm is Gaussian and, by shifting the $\boldsymbol{\lambda}$ integration so that $\boldsymbol{\lambda}$ gets sufficiently negative real part, it can be made well defined. The integral is easily performed to give

$$2G[\mathbf{q},\boldsymbol{\lambda}]=-n\ln(2\pi)-\frac{(\beta J)^2}{2}\sum_{\alpha\beta}q_{\alpha\beta}^p+\sum_{\alpha\beta}\lambda_{\alpha\beta}q_{\alpha\beta}+\ln\det(-\boldsymbol{\lambda})+(\beta h)^2\sum_{\alpha\beta}(\boldsymbol{\lambda}^{-1})_{\alpha\beta}+\frac{(\beta J)^2}{4N}p(p-1)\sum_{\alpha\beta}\langle\sigma_\alpha^2\sigma_\beta^2\rangle q_{\alpha\beta}^{p-2}. \tag{3.9}$$

The correlation function $\langle\sigma_\alpha^2\sigma_\beta^2\rangle$ can be calculated from the Wick's theorem and expressed in terms of $\langle\sigma_\alpha\sigma_\beta\rangle$ and $\langle\sigma_\alpha\rangle$.

3.1. The $\boldsymbol{\lambda}$-integration

In the limit of large N the integrals in (3.7) can be performed by saddle point method. Namely, $\overline{Z^n}$ is given by the dominant extremum of G and the average free energy is

$$\beta F/N=\lim_{n\to 0}\frac{1}{n}G \tag{3.10}$$

where G, evaluated at its dominant extremum, may include the contributions of the Gaussian fluctuations to take into account the $O(1/N)$ corrections.

Due to the restriction $\mathbf{q}>0$ it is possible to integrate out the auxiliary variables $\lambda_{\alpha\beta}$ parallel to the imaginary axis yielding a real saddle point. In this way $\overline{Z^n}$ is reduced to an integral over the overlaps $q_{\alpha\beta}$. The relevant part for the $\lambda_{\alpha\beta}$ integration is

$$\tilde{G}[\boldsymbol{\lambda}]=\Phi[\boldsymbol{\lambda}]+\frac{\mu}{4N}(p-1)\sum_{\alpha\beta}\langle\sigma_{i\alpha}^2\sigma_{i\beta}^2\rangle q_{\alpha\beta}^{p-2} \tag{3.11}$$

with

$$2\Phi[\boldsymbol{\lambda}]=\sum_{\alpha\beta}\lambda_{\alpha\beta}q_{\alpha\beta}+\ln\det(-\lambda_{\alpha\beta})+b^2\sum_{\alpha\beta}(\boldsymbol{\lambda}^{-1})_{\alpha\beta}=\sum_{\alpha\beta}\lambda_{\alpha\beta}q_{\alpha\beta}+\ln\det(-\lambda_{\alpha\beta}-b^2)+\frac{b^2}{2}\left(\sum_{\alpha\beta}(\boldsymbol{\lambda}^{-1})_{\alpha\beta}\right)^2+O(n^3) \tag{3.12}$$

where we have introduced the parameters

$$\mu=\frac{(\beta J)^2}{2}p\quad\text{and}\quad b=\beta h.$$

In (3.12) we have explicitly retained the terms of order n^2 since, even if they do not contribute to the saddle point, they may be relevant for the $O(1/N)$ corrections if there are eigenvalues of the Gaussian fluctuations with finite - in the $n\to 0$ limit - degeneracy.

The last term in (3.11) is of relative order $1/N$ and gives the $O(1/N)$ corrections to the $N\to\infty$ saddle point value of $\boldsymbol{\lambda}$. However, due to the stationarity of the saddle point, these will lead only to $O(1/N^2)$ corrections to G and, therefore, this term can be neglected in evaluating the saddle point and the Gaussian fluctuations.

The stationary point of $\tilde{G}[\boldsymbol{\lambda}]$ is given by

$$\langle\sigma_\alpha\sigma_\beta\rangle=q_{\alpha\beta} \tag{3.13}$$

which to lowest order in n reads

$$\lambda_{\alpha\beta}+b^2+(\mathbf{q}^{-1})_{\alpha\beta}=O(n). \tag{3.14}$$

The integration of the Gaussian $\boldsymbol{\lambda}$-fluctuations at the saddle point can be done by diagonalising $\mathbf{q}$ and gives the factor

$$\left(\frac{N}{2\pi}\right)^{n(n-1)/4} n^{-n/2}(\det(\mathbf{q}))^{-(1+n)/2}. \tag{3.15}$$

The details are in the Appendix 1. These fluctuations are stable due to the restriction $\mathbf{q}>0$. Note that from the approximations made in (3.12) and (3.14) this result gives the correct $1/N$ contribution to G only up to order $O(n)$.

By inserting these results back into (3.7) and (3.9) one gets

$$\overline{Z^n} = e^{nS(\infty)} \int_{\mathbf{q}>0} \prod_{\alpha<\beta} \sqrt{\frac{N}{2\pi}}\, dq_{\alpha\beta} \times \exp[-NG_0[\mathbf{q}] - G_1[\mathbf{q}] + O(1/N)] \tag{3.16}$$

with

$$2G_0[\mathbf{q}] = -\frac{\mu}{p}\sum_{\alpha\beta} q_{\alpha\beta}^p - b^2 \sum_{\alpha\beta} q_{\alpha\beta} - \ln\det(\mathbf{q}) + \frac{b^4}{2}\left(\sum_{\alpha\beta} q_{\alpha\beta}\right)^2 \tag{3.17}$$

$$2G_1[\mathbf{q}] = \frac{\mu}{2}(p-1)\sum_{\alpha\beta}\langle\sigma_\alpha^2\sigma_\beta^2\rangle q_{\alpha\beta}^{p-2} + \ln\det(\mathbf{q}) \tag{3.18}$$

and

$$\langle\sigma_\alpha^2\sigma_\beta^2\rangle = 1 + 2q_{\alpha\beta}^2 - 2b^4\left(\sum_{\gamma\delta} q_{\alpha\gamma} q_{\beta\delta}\right)^2. \tag{3.19}$$

The last term in (3.17), of order n^2, has been retained since it may be relevant for the $1/N$ corrections. The above expressions give the correct result for the terms of order n, n^2 and n/N.

3.2. The q-integration

The remaining $\mathbf{q}$-integration can be done also by saddle point method. The terms of relative order $1/N$ can again be neglected, and replaced by their value at the stationary point.

The saddle point equation for $q_{\alpha\beta}$ derived from G_0 reads, in the $n\to 0$ limit,

$$\mu q_{\alpha\beta}^{p-1} + b^2 + (\mathbf{q}^{-1})_{\alpha\beta} = 0, \quad \alpha \neq \beta. \tag{3.20}$$

The diagonal terms are fixed to 1 by the spherical constraint.

The second order variation of G_0 with respect to $q_{\alpha\beta}$ gives the Gaussian $\mathbf{q}$-fluctuations

$$2\delta^2 G_0 = -\mu(p-1)\sum_{\alpha\beta} q_{\alpha\beta}^{q-2}(\delta q_{\alpha\beta})^2 + \mathrm{Tr}(\mathbf{q}^{-1}\delta\mathbf{q})^2 + b^4\left(\sum_{\alpha\beta}\delta q_{\alpha\beta}\right)^2 \tag{3.21}$$

where $(\delta\mathbf{q})_{\alpha\beta} = \delta q_{\alpha\beta}\,(=\delta q_{\beta\alpha})$ is the fluctuation of $q_{\alpha\beta}$ from the saddle point value (3.20). Since $q_{\alpha\alpha}=1$ by the spherical constraint, $\delta q_{\alpha\alpha}\equiv 0$.

Note that for finite $n(>1)$ $G_0[\mathbf{q}]$ has to be minimal at the stationary point, i.e. the eigenvalues of the fluctuations have to be positive. However, like in the SK model the dimension of the $\delta\mathbf{q}$-space, $n(n-1)/2$, becomes negative in the limit $n\to 0^+$. In this situation the role of negative and positive eigenvalues is switched and stability requires that $G_0[\mathbf{q}]$ should be maximised [3, 12].

The integration of the Gaussian $\mathbf{q}$-fluctuations can be done by diagonalising the quadratic form (3.21). If we denote by Λ_ν the eigenvalues of (3.21) and by n_ν their degeneracies, then the result of the integration is

$$\left(\prod_\nu \Lambda_\nu^{n_\nu}\right)^{-1/2}. \tag{3.22}$$

Collecting all the terms the average free energy per spin $f=F/N$ in the $N\to\infty$ limit finally reads

$$\beta f = -s(\infty) + \frac{1}{n}G_0[\mathbf{q}] + \frac{1}{Nn}\left(G_1[\mathbf{q}] + \tfrac{1}{2}\sum_\nu n_\nu \ln\Lambda_\nu\right) + O(1/N^2) \tag{3.23}$$

where the $n\to 0$ limit is intended, and $s=S/N$.

4. Replica symmetric solution

In order to evaluate G explicitly we have to impose some *ansatz* on the structure of $\mathbf{q}$. In the high temperature and/or high magnetic field regime we expect only one pure state. Thus it is reasonable to take a replica symmetric (RS) *ansatz*

$$q_{\alpha\beta} = (1-q)\delta_{\alpha\beta} + q. \tag{4.1}$$

For later use, it is useful to introduce also the inverse matrix $\mathbf{q}^{-1}$

$$(\mathbf{q}^{-1})_{\alpha\beta} = A\delta_{\alpha\beta} + B \tag{4.2}$$

with

$$A = (1-q)^{-1}$$

$$B = -\frac{q}{(1-q)[1+(n-1)q]} \underset{n\to 0}{=} -\frac{q}{(1-q)^2}. \tag{4.3}$$

Inserting (4.1) into (3.17) yields for the extensive part of the free energy in the $n\to 0$ limit

$$\frac{2}{n}G_0(q) = -\frac{\mu}{p}(1-q^p) - b^2(1-q) - \ln(1-q) - \frac{q}{1-q}. \tag{4.4}$$

The saddle point equation, derived either from the stationary point of (4.4) or inserting (4.1) and (4.2) into (3.20), is

$$\mu q^{p-1} + b^2 - \frac{q}{(1-q)^2} = 0. \tag{4.5}$$

4.1. *Stability of the* RS *solution*

In the replica symmetric ansatz $\delta^2 G_0$ becomes [see (3.21)]

$$2\delta^2 G_0 = \Lambda_1 \sum_{\alpha\beta} (\delta q_{\alpha\beta})^2 + 2AB \sum_{\alpha\beta\gamma} \delta q_{\alpha\gamma}\,\delta q_{\gamma\beta} + (B^2+b^4)\left(\sum_{\alpha\beta} \delta q_{\alpha\beta}\right)^2 \tag{4.6}$$

where

$$\Lambda_1 = -\mu(1-p)\,q^{p-1} + A^2. \tag{4.7}$$

The eigenvalues of this quadratic form are solutions of the eigenvalue equation

$$\Lambda_1 \delta q_{\alpha\beta} + AB \sum_{\gamma} (\delta q_{\alpha\gamma} + \delta q_{\gamma\beta}) + (B^2+b^4) \sum_{\gamma\delta} \delta q_{\gamma\delta} = \Lambda \delta q_{\alpha\beta}, \quad \alpha \neq \beta. \tag{4.8}$$

For finite n there are three different eigenvalues.

1. The first is

$$\Lambda_1 = -\mu(p-1)\,q^{p-2} + (1-q)^{-2}, \qquad n_1 = n(n-3)/2 \tag{4.9}$$

and corresponds to eigenvectors for which

$$\sum_{\beta} \delta q_{\alpha\beta} = 0 \quad \forall \alpha.$$

2. The second is

$$\Lambda_2 = \Lambda_1 + (n-2)AB, \qquad n_2 = n-1 \tag{4.10}$$

corresponding to eigenvectors

$$\sum_{\alpha\beta} \delta q_{\alpha\beta} = 0 \quad \text{but} \quad \sum_{\beta} \delta q_{\alpha\beta} \neq 0.$$

3. Finally the third is

$$\Lambda_3 = \Lambda_2 + nAB + n(n-1)(B^2+b^4), \qquad n_3 = 1 \tag{4.11}$$

and corresponds to eigenvectors such that

$$\sum_{\alpha\beta} \delta q_{\alpha\beta} \neq 0.$$

Note that in the $n \to 0$ limit $\Lambda_2 = \Lambda_3$. However, since the degeneracy of Λ_3 is 1, its $O(n)$ terms will contribute to the $1/N$ corrections to the free energy f.

The relevant eigenvalue for the stability of the RS solution depends on the sign of q. It is Λ_1 if $q > 0$, whereas it is Λ_2 if $q < 0$. Note that Λ_1 corresponds to the critical eigenvalue found by de Almeida and Thouless for the SK model [13].

4.2. *Solution of the* RS *saddle point equation*

Let us consider first the zero magnetic field case $b = 0$. In this case the saddle point equation (4.5) has always the solution

$$q = 0 \tag{4.12}$$

which is stable at any temperature, for

$$\Lambda_1 = \Lambda_2 = 1, \quad (p > 2). \tag{4.13}$$

In the case of $b \neq 0$ (4.5) has a solution

$$0 < q < 1 \quad \text{for all } T. \tag{4.14}$$

For even p an additional negative solution may also exist. This $q < 0$ solution is, however, always unstable since both Λ_1 and Λ_2 are negative.

The relevant eigenvalue for the stability of the $q > 0$ solution is Λ_1. Using the saddle point equation (4.5) and (4.9) the instability curve $\Lambda_1 = 0$ of the RS solution in the (T, h) plane has the parametric representation

$$\begin{cases} (T/J)^2 = \dfrac{p(p-1)}{2}(1-q)^2 q^{p-2} \\ (h/J)^2 = \dfrac{p(p-2)}{2} q^{p-1}. \end{cases} \quad 0 \leqq q \leqq 1 \tag{4.15}$$

This curve is shown in Fig. 1 for $p = 3$ and Fig. 2 for $p = 10$ (full line). The qualitative shape remains unchanged for all $p > 2$. In particular for any p this line exhibits a critical point $\mathrm{d}h/\mathrm{d}T = 0$ for

$$q_c = 1 - 2/p. \tag{4.16}$$

At low temperature and field the saddle point equation may have other solutions which appear discontinuously when the curve $[q, q/(1-q)^2]$ is tangent to the curve $[q, \mu q^{p-1} + b^2]$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}q}\left[\frac{q}{(1-q)^2} - \mu q^{p-1} - b^2\right] = \Lambda_2 = 0. \tag{4.17}$$

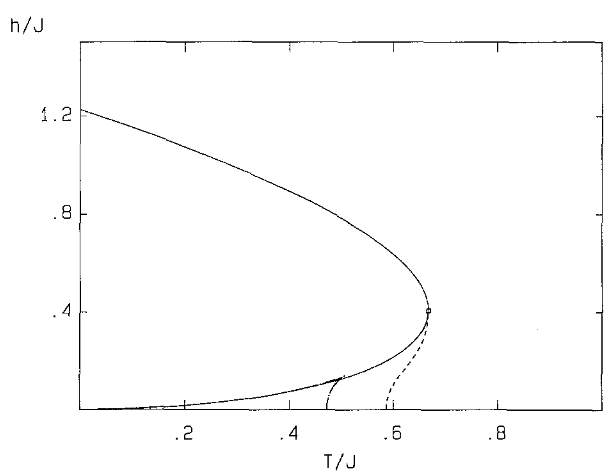

Fig. 1. The lines $\Lambda_1 = 0$ (full line) and $\Lambda_2 = 0$ (dash-dotted line) for $p = 3$. The square marks the critical point. The dash line is the $m = 1$ line where the discontinuous transition to the 1 RSB solution takes place

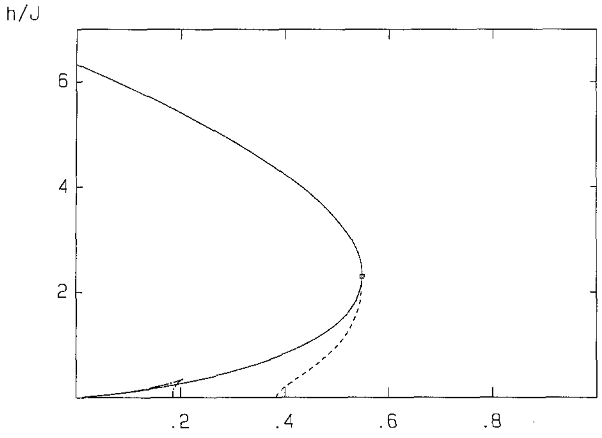

Fig. 2. The lines $\Lambda_1=0$ (full line) and $\Lambda_2=0$ (dash-dotted line) for $p=10$. The square marks the critical point. The dash line is the $m=1$ line where the discontinuous transition to the 1 RSB solution takes place

For $b=0$ the above equation is solved at $T\neq0$ by (4.16), and by $q=0$ at $T=0$. If $h\neq0$ the value of q lies in between 0 and q_c. In the (T,h) plane the line $\Lambda_2=0$ has the parametric form

$$\begin{cases}(T/J)^2=\dfrac{p(p-1)}{2}\,q^{p-2}\,\dfrac{(1-q)^3}{1+q} \\ (h/J)^2=\dfrac{p}{2}\,q^{p-1}\,\dfrac{p-2-pq}{1+q}\,.\end{cases}\quad 0\leqq q\leqq 1-2/p \tag{4.18}$$

This curve is shown in Fig. 1 for $p=3$ and Fig. 2 for $p=10$ (dash-dotted line). It is interesting to note that the curve exhibits a cusp at

$$q=-2/p+\sqrt{(2/p)^2+1-2/p}$$

since for this value both $\mathrm{d}T/\mathrm{d}q$ and $\mathrm{d}h/\mathrm{d}q$ are zero. The tangents to the two branches of the curve are equal at the cusp. Even if the qualitative behaviour does not change with p, when the latter increases the full curve moves towards low temperatures.

Inside the region bounded by the $\Lambda_2=0$ curve each solution of (4.17) splits in two distinct solutions, one with $\Lambda_2>0$ and one with $\Lambda_2<0$. All these solutions are, nevertheless, unstable since $\Lambda_1<0$, as can be easily seen on the border line $\Lambda_2=0$, for $\Lambda_1<\Lambda_2$ for any $q>0$. For $q\to0$ both curves $\Lambda_1=0$ and $\Lambda_2=0$ are tangent to the T-axis at the origin.

The presence of unstable solutions in the region where the RS solution is stable is an indication that another stable solution different from the RS may exist.

5. Replica symmetry broken solution

The analysis of the last section leads to the conclusion that a replica symmetric ansatz cannot give a stable, and hence physical, solution of the model in the full (T,h) plane. Moreover, the existence of (unstable) solutions outside the RS instability region suggests that a replica symmetry broken (RSB) solution may lead to a larger free energy even outside the region $\Lambda_1<0$.

In order to obtain the full solution we must break the replica symmetry, allowing $\mathbf{q}$ to depend, in general, on an infinite number of parameters. The most general form of such a $\mathbf{q}$ is however not known. For the SK model Parisi [3] proposed a particular ansatz of hierarchical breaking of replica symmetry which seems to give the correct solution of the SK model when the number of breaking is sent to infinity. If we use the same ansatz for this model, it is possible to show [see Appendix 2] that within the Parisi scheme the most general RSB solution is the Parisi one-step replica symmetry breaking (1 RSB).

The Parisi 1 RSB is obtained by dividing the $n\times n\,\mathbf{q}$ matrix in $(n/m)\times(n/m)$ blocks of dimension $m\times m$. If $\alpha\neq\beta$ belong to one of the (n/m) diagonal blocks then $q_{\alpha\beta}=q_1$, otherwise $q_{\alpha\beta}=q_0\leqq q_1$. This corresponds to group the n replicas into n/m clusters of m replicas. Any two replicas $\alpha\neq\beta$ within the same cluster have overlap q_1, whereas replicas in different clusters have overlap $q_0\leqq q_1$.

Introducing the matrix ε defined as

$$\varepsilon_{\alpha\beta}=\begin{cases}1 & \text{if } \alpha \text{ and } \beta \text{ are in a diagonal block}\\ 0 & \text{otherwise}\end{cases} \tag{5.1}$$

$q_{\alpha\beta}$ can conveniently be written as

$$q_{\alpha\beta}=(1-q_1)\,\delta_{\alpha\beta}+(q_1-q_0)\,\varepsilon_{\alpha\beta}+q_0\,. \tag{5.2}$$

It is also useful to introduce the eigenvalues of $\mathbf{q}$ (with degeneracies)

$$\begin{aligned}&\eta_0=1-q_1\,, && \text{deg.}=n(m-1)/m\\ &\eta_1=1-(1-m)\,q_1-mq_0\,, && \text{deg.}=n/m-1\\ &\eta_2=1-(1-m)\,q_1-(m-n)\,q_0 && \text{deg.}=1\end{aligned} \tag{5.3}$$

and the inverse matrix $\mathbf{q}^{-1}$,

$$(\mathbf{q}^{-1})_{\alpha\beta}=A\delta_{\alpha\beta}+B\varepsilon_{\alpha\beta}+C \tag{5.4}$$

with

$$\begin{aligned}A&=\eta_0^{-1}\\ B&=-\frac{q_1-q_0}{\eta_0\eta_2}\underset{n\to0}{=}-\frac{q_1-q_0}{\eta_0\eta_1}\\ C&=-\frac{q_0}{\eta_1\eta_2}\underset{n\to0}{=}-\frac{q_0}{\eta_1^2}\,.\end{aligned} \tag{5.5}$$

Note that in the $n\to0$ limit $\eta_1=\eta_2$, and m ranges between 0 and 1.

Substituting the above expressions into (3.17) yields for the extensive part of the free energy in the $n\to0$ limit

$$\frac{2}{n}\,G_0[\mathbf{q}]=-\frac{\mu}{p}\,[1-(1-m)\,q_1^p-mq_0^p]-b^2\eta_1-\frac{q_0}{\eta_1}-\frac{1}{m}\ln(\eta_1)-\frac{m-1}{m}\ln(\eta_0)\,. \tag{5.6}$$

The saddle point equations for q_0 and q_1, obtained either for the stationary point of (5.6) or from (3.20), read for $n \to 0$

$$\mu q_0^{p-1} + b^2 + C = 0 \tag{5.7}$$

$$\mu q_1^{p-1} + b^2 + B + C = 0. \tag{5.8}$$

Each $m(0 \leqq m \leqq 1)$ corresponds to a thermodynamic phase. For any T and h we take the one which maximises G_0 [12]. This leads to the additional equation

$$-\frac{\mu}{p}(q_1^p - q_0^p) - \left(b^2 + C + \frac{1}{m\eta_1}\right)(q_1 - q_0) - \frac{1}{m^2}\ln\left(\frac{\eta_0}{\eta_1}\right) = 0 \tag{5.9}$$

which has to be solved for $0 \leqq m \leqq 1$.

The quadratic form of the Gaussian **q**-fluctuations has for finite n nine eigenvalues, but the relevant ones are only two which, for $n \to 0$, read [see Appendix 3]

$$\Lambda_1^{(1)} = -\mu(p-1)q_1^{p-2} + A^2 \tag{5.10}$$

$$\Lambda_0^{(3)} = -\mu(p-1)q_0^{p-2} + (A + mB)^2. \tag{5.11}$$

The first eigenvalue, $\Lambda_1^{(1)}$, corresponds to fluctuations inside a cluster, while the second, $\Lambda_0^{(3)}$, between different clusters.

5.1. The transition lines

For any value of m the RS solution $q_0 = q_1 = q$ is a solution of the 1 RSB equations, as can be checked by direct substitution. However, even if 1 RSB equations contain the RS solution, we have seen that this is not the correct one in the whole plane (T, h), and hence in general (5.7)–(5.9) will admit a different solution. The transition from the RS to the 1 RSB solution can be either continuous, in the sense that $q_1 - q_0$ goes continuously to zero at the transition, or discontinuous, i.e. $q_1 - q_0$ has a jump. The points where such transitions take place give the critical lines in the (T, h) plane. These yield the equivalent of the de Almeida and Thouless line for the SK model [13].

The condition for a continuous transition can be found by solving the 1 RSB equations in the limit of small $q_1 - q_0$. A simple way of obtaining the equation of the continuous transition critical line is subtracting (5.7) from (5.8) and expanding the result for fixed temperature, field and m, about q_0 in powers of $q_1 - q_0$. One then gets to the lowest order

$$q_1 - q_0 = -2 \times \frac{\mu(p-1)q_0^{p-2} - (1-q_0)^{-2}}{\mu(p-1)(p-2)q_0^{p-3} + 2(m-2)(1-q_0)^{-3}}. \tag{5.12}$$

At the transition $q_1 \to q_0 \to q$ and the right hand side of (5.12) must vanish. This leads to the critical line

$$\mu(p-1)q^{p-2} - \frac{1}{(1-q)^2} = 0 \tag{5.13}$$

where q is solution of the RS equation (4.5). This is the line where the RS solution becomes unstable, for it is nothing but $\Lambda_1 = 0$ [cf. (4.9) and (4.15)].

We have, however, not yet found the value of m. The 1 RSB solution has a well defined m. Therefore, if the critical line is approached in the (T, h) plane from the 1 RSB side along a m-line of constant m, the solution at the transition will be given by $q_1 = q_0 = q$ with that value of m. This implies that m and q are related and may, and indeed does, discard a part of the line (5.13).

Solving the full 1 RSB equations in the small $q_1 - q_0$ limit leads, in addition to (5.13), to the relation

$$m = \frac{p-2}{2q}(1-q) \tag{5.14}$$

where q is solution of (4.5) on the critical line. Since $0 \leqq m \leqq 1$ the above equation implies that the continuous transition takes place only on that part of the curve (5.13) which corresponds to

$$1 - 2/p \leqq q \leqq 1 \tag{5.15}$$

i.e. the upper branch. This is the continuous line in Fig. 3.

We then have the following scenario. When the critical line is approached from high temperatures and fields the RS solution becomes unstable and eventually the 1 RSB solution is continuously set up. The difference $q_1 - q_0$ is zero at the transition, but m has a well defined value fixed by q at the transition through (5.14). Both eigenvalues $\Lambda_1^{(1)}$ and $\Lambda_0^{(3)}$ are zero on the critical line, and positive below the transition.

The continuous transition critical line ends at the critical point $q_c = 1 - 2/p$ where $m = 1$. Below this point the 1 RSB can only be reached discontinuously, i.e. with

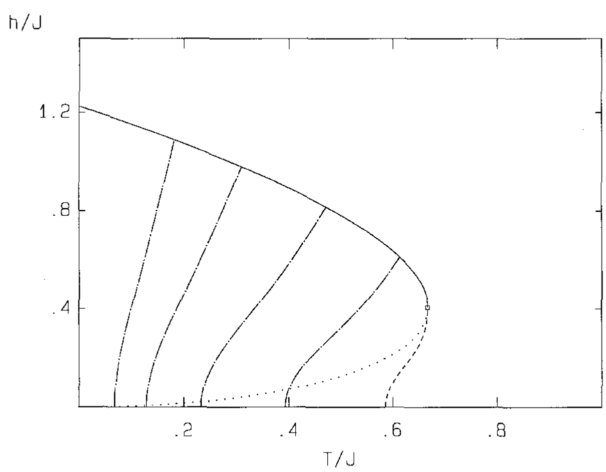

Fig. 3. The continuous transition critical line (full line) and the discontinuous transition critical line (dashed line) for $p = 3$. The square marks the critical point. The dotted line is the lower branch of the $\Lambda_1 = 0$ line. The dash-dotted lines are 1 RSB solutions with $m = 1/2$, 1/4, 1/8 and 1/16

a jump in q_1-q_0. Since $m=1$ at q_c, the discontinuous transition critical line is the m-line with $m=1$. Even if the order parameter is discontinuous, the free energy remains continuous at the transition, for $m=1$. However, on the left side of the critical line, i.e. low temperatures, the 1 RSB solution leads to an higher free energy. In Fig. 3 we show both the continuous transition line (full line) and the discontinuous transition line (dashed line) for $p=3$. The region bounded by these curves is where the 1 RSB gives the maximum of the free energy. For comparision in Fig. 2 one can see these lines for $p=10$. We stress that the qualitative shape of this curve does not depend on $p\,(>2)$. However, the scale on the h-axis increases with p. The temperature T, on the other side, is not too sensitive to p.

The nature of the two transitions is very different. Indeed, while at the continuous transition critical line the RS solution becomes unstable, along the $m=1$ line both the RS and the 1 RSB solutions are stable and give the same free energy. However, the 1 RSB leads to a larger value for the free energy below the transition. Thus we can expect the presence of metastable RS solutions. They appear on the $m=1$ line and last till the lower branch of the curve (5.13) is reached, where the RS solution becomes unstable. These states have a lower free energy.

We conclude by noting that on the $m=1$ line both eigenvalues $\Lambda_1^{(1)}$ and $\Lambda_0^{(3)}$ are positive with $\Lambda_1^{(1)}>\Lambda_0^{(3)}$. The latter, in particular, is equal to the Λ_1 eigenvalue of the RS solution evaluated on this line.

5.2. The $0 \leqq m \leqq 1$ solution

The 1 RSB equations (5.7)–(5.9) can be easily solved for any p, at least with the help of a computer, by rewriting them in a more convenient way. We now briefly outline how this can be achieved. The first step is to substract (5.7) from (5.8). This leads to an equation in which the explicit dependence on the magnetic field has disappeared

$$\mu\,(q_1^{p-1}-q_0^{p-1})=\frac{q_1-q_0}{\eta_0\eta_1}. \tag{5.16}$$

A second equation without the magnetic field is obtained by inserting (5.7) into (5.9),

$$\frac{\mu}{p}\,(q_1^p-q_0^p)-\mu q_0^{p-1}(q_1-q_0)$$
$$=-\frac{q_1-q_0}{m\eta_1}-\frac{1}{m^2}\ln\left(\frac{\eta_0}{\eta_1}\right). \tag{5.17}$$

Next we note that the temperature T enters in (5.16) and (5.17) only through μ. Therefore by dividing (5.17) by (5.16) we can obtain an equation containing only q_0, q_1 and m. Dividing again the result by q_1-q_0, to have the same power of q_0 and q_1 at the numerator and denominator, one finally ends up with

$$\frac{(q_1^p-q_0^p)-p(q_1-q_0)q_0^{p-1}}{p(q_1-q_0)(q_1^{p-1}-q_0^{p-1})}=-\frac{\eta_0}{m(q_1-q_0)}$$
$$-\left[\frac{\eta_0}{m(q_1-q_0)}\right]^2\frac{\eta_1}{\eta_0}\ln\left(\frac{\eta_0}{\eta_1}\right). \tag{5.18}$$

The left hand side of (5.18) depends only on the ratio q_0/q_1 (and p), while the right hand side is a function of η_1/η_0. We can therefore introduce the variables x and y as

$$q_0=xq_1, \quad 0\leqq x\leqq 1 \tag{5.19}$$

$$\eta_0=y\eta_1, \quad 0\leqq y\leqq 1 \tag{5.20}$$

and write (5.18) as

$$\begin{cases} z=\dfrac{2}{p}\,\dfrac{1-x^p-px^{p-1}(1-x)}{(1-x)(1-x^{p-1})} \\ z=-2y\,\dfrac{1-y+\ln y}{(1-y)^2}. \end{cases} \quad 0\leqq x,y,z\leqq 1 \tag{5.21}$$

The problem is then reduced to solve these coupled equations for $0\leqq x\leqq 1$ and a given p. The value of q_0 and q_1 are then obtained from (5.19) with

$$q_1=\frac{1-y}{1-y[1-m(1-x)]}. \tag{5.22}$$

We note that while the first equation (5.21) depends on p, the second one as a function of z does not depend on any parameter. It is the same for any choice of the parameters of the model. Once the function $y(z)$ is known the 1 RSB solution can be easily found for any p. The plot of $y(z)$ for $0\leqq z\leqq 1$ is reported in Fig. 4. For example the m-lines in the (T,h) plane can be drawn for any p by fixing m and varying x from 0 ($h=0$) to 1 (continuous transition). For each value of x the corresponding value of z is obtained from (5.21) and from this y. The values of T and h can now be found from the 1 RSB equation (5.7) and (5.8),

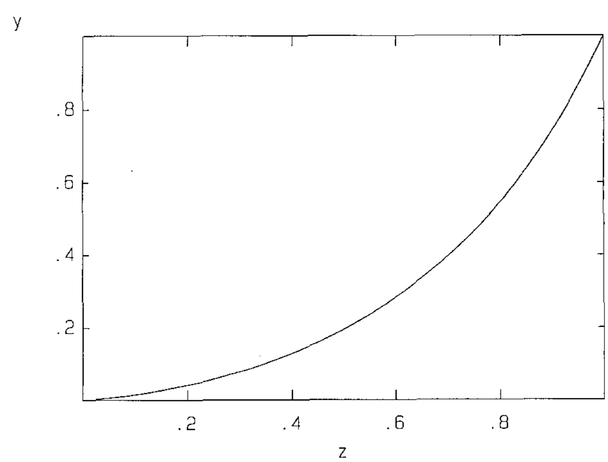

Fig. 4. The function $y(z)$ for $0\leqq z\leqq 1$

$$(T/J)^2 = \frac{p}{2}\,\frac{(1-q_1)[1-(1-m)q_1-mq_0]}{q_1-q_0} \times (q_1^{p-1}-q_0^{p-1}) \qquad (5.23)$$

$$(h/J)^2 = \frac{p}{2}\,\frac{(1-q_1)(q_1^{p-1}-q_0^{p-1})}{(q_1-q_0)[1-(1-m)q_1-mq_0]} - \frac{p}{2}\,q_0^{p-1}. \qquad (5.24)$$

In this way we have obtained the m-lines shown in the Fig. 3 (dash-dotted lines) and the $n=1$ critical line of Figs. 1–3.

The above reformulation of the 1 RSB equation is also useful from an analytical point of view. For example an expansion near $x=1$ would give the solution near the continuous transition critical line. In this way one can find, e.g., (5.14). Similarly $x=0$ for $h=0$, and hence an expansion for small x would give the solution for small field.

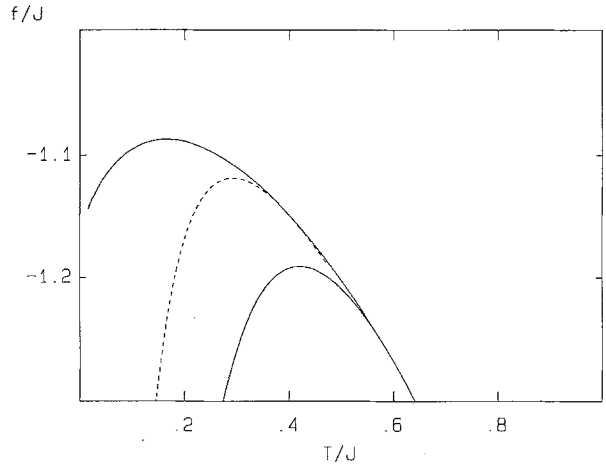

Fig. 5. The free energy f as a function of T for $h=0$ and $p=3$. The upper line is the 1 RSB solution, while the lower line ths RS. The dashed line is the free energy of the metastable state with $m=1/2$

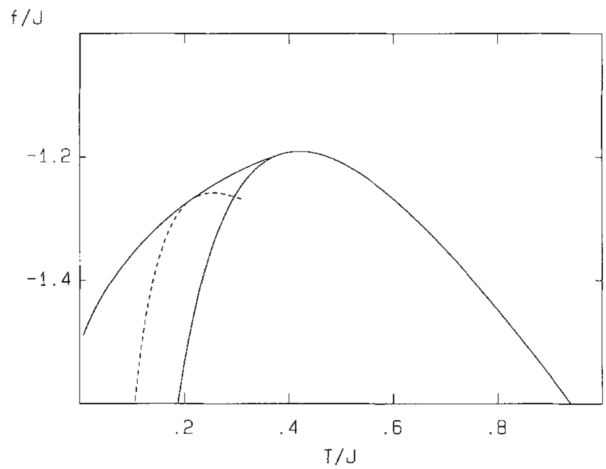

Fig. 6. The free energy f as a function of T for $h=0$ and $p=10$. The upper line is the 1 RSB solution, while the lower line ths RS. The dashed line is the free energy of the metastable state with $m=1/2$

6. Thermodynamic properties

In this section we shall discuss some thermodynamic properties of the spherical pSG. We shall consider first the case of zero external field, since in this case the equations simplify and this help for a better understanding of the discontinuous transition. The case of non-zero external field does not present new qualitative features, as far as the discontinuous transition is concerned.

6.1. Zero external field

In this case the RS solution is $q=0$, and the free energy in the high temperature phase is [see (4.4) with $J=1$]

$$f(T) = -\frac{1}{4T} - T s(\infty) \qquad (6.1)$$

Interestingly, this is of the same form of the free energy in the high temperature phase of the Ising spin case [6]. One has just to replace $s(\infty)$ with the corresponding infinite temperature entropy which for the Ising spins is ln 2.

From (6.1) a simple calculation leads to the entropy per spin

$$s(T) = -\frac{1}{4T^2} + s(\infty) \qquad (6.2)$$

so that $s(T)$ becomes negative, as expected from the similarity with the Ising case, for $T < [4s(\infty)]^{-1/2}$. Nevertheless here this does not imply an instability in the model since for continuous variables the entropy is not strictly positive.

At the critical temperature $T_c(0)$ the 1 RSB solution appears with $m=1$, $q_1-q_0 \neq 0, q_0=0$ and the same free energy of the RS solution. When the temperature is further decreased, m becomes smaller than one and the 1 RSB solution leads to a larger free energy. In Figs. 5 and 6 we show the free energy as a function of T at $h=0$ for $p=3$ and $p=10$, respectively. The upper branch (full line) is the 1 RSB solution, while the lower branch (full line) is the RS solution.

From these figures one sees that in this model the entropy is always negative at low temperatures and diverges as $T \to 0$. Solving the 1 RSB equations in this limit shows that indeed the entropy diverges logarithmically as $T \to 0$. The RS solution gives a stronger, T^{-2}, singularity, and hence the RS entropy is lower than the 1 RSB one. This indicates the presence of freezing.

The spin glass transition is of the first order, as far as the order parameter is concerned. However, due to the maximisation of the free energy, a genuine spin glass first order transition would have a negative latent heat [7]. Moreover, the order parameter is a function and the discontinuity appears on a set of zero measure. These considerations lead to the conclusion that the transition must

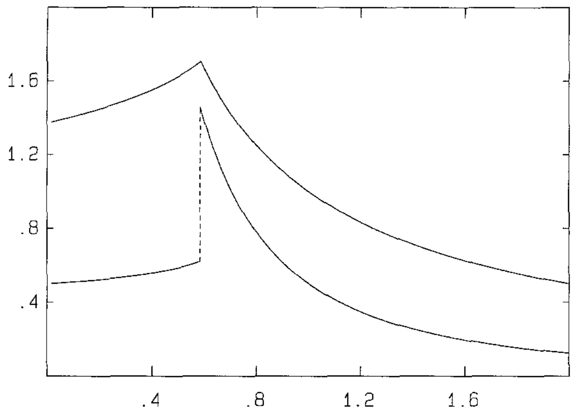

Fig. 7. The specific heat (lower line) and the susceptibility (upper line) as a function of T for $h=0$ and $p=3$

be of the second order in the thermodynamic sense [6, 7], i.e., in the sense that discontinuities or singularities can be seen only from the second derivatives of the free energy. In Fig. 7 we report the susceptibility and the specific heat as a function of T for $p=3$ and $h=0$.

An interesting feature is the presence of metastable states, i.e. of states which are locally stable, but have smaller free energy. Consider for example the RS solution. For $h=0$ it is stable for all temperatures [see (4.13)]. However it leads to a smaller free energy (see Figs. 5 and 6). Other metastable states are obtained by solving the 1 RSB equation (5.8) for $b=0=q_0$ and m fixed. For any m this equation has two solutions which appear for $\Lambda_2^{(3)}=0$, where they coincide, but are unstable since $\Lambda_1^{(1)}<0$. When the temperature is further decreased one of the two solutions becomes stable leading to a metastable state. In Figs. 5 and 6 we show the free energy of one of these metastable states ($m=1/2$) as function of T for $p=3$ and $p=10$ (dashed line). The line starts from the point where the solution becomes stable, i.e. $\Lambda_1^{(1)}=0$. The free energy of these m-states is always lower than of the 1 RSB, but equals the latter at the point where that particular value of m is solution of (5.9) (see Figs. 5 and 6). We note that due to this fact, the free energy of the 1 RSB solution can be seen as the envelope of the free energies of all these metastables m-states.

From Figs. 5 and 6 we see that there exists a particular value of p between 3 and 10 for which the critical temperature $T_c(0)$ equals the temperature where the entropy becomes zero. For larger value of p the latter is always given by the RS value $[4s(\infty)]^{-1/2}$. Apart for this property, this special p has no other special features.

6.2. *Finite external field*

If the field is not zero we have to distinguish two cases: $h<h_c$ and $h>h_c$, where h_c is the value of the field at the critical point $q_c=1-2/p$. Its value for any p can be found by inserting q_c into (4.15). For $T=0$ the critical field h_0 takes the simple form

$$(h_0/J)^2=\frac{p(p-2)}{2}. \qquad (6.3)$$

For $h<h_c$ the transition is discontinuous, and qualitatively similar to the $h=0$ case. The only difference is that the strength of the discontinuities at the transition, e.g. in the order parameter and specific heat, decreases as the field is increased and eventually vanishes at the critical point. The metastable states, which for $h=0$ exist for any temperature below the transition, may now disappear at a temperature which depends on both h and m. This temperature can be found by the vanishing of the most relevant eigenvalue. For example the RS metastable state becomes unstable at the lower branch of the $\Lambda_1=0$ line, see e.g. Figs. 1–3.

For fields larger than h_c the transition becomes continuous. The RS solution is unstable at the transition and the 1 RSB state is continuously set up. The first discontinuities appear only on the third derivatives of the free energy, and hence the transition is thermodynamically of the third order. For comparision we report in Fig. 8 the internal energy near the transitions as a function of T for $p=3$. The upper line refers to the discontinuous transition ($h<h_c$) while the lower line to the continuous transition ($h>h_c$). The square marks the point of the tran-

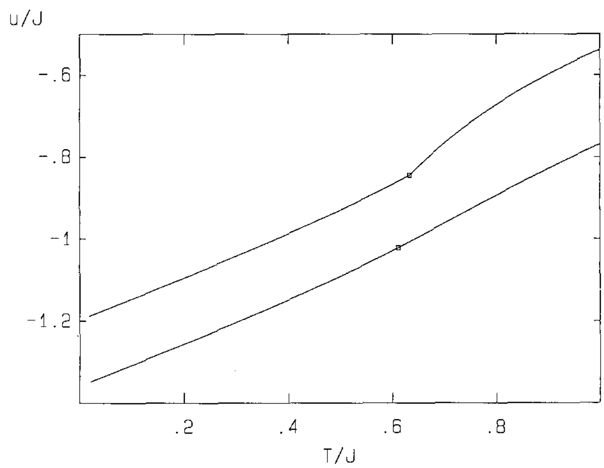

Fig. 8. The internal energy as a function of T for $h<h_c$ (upper line) and $h>h_c$ for $p=3$. The squares mark the transition

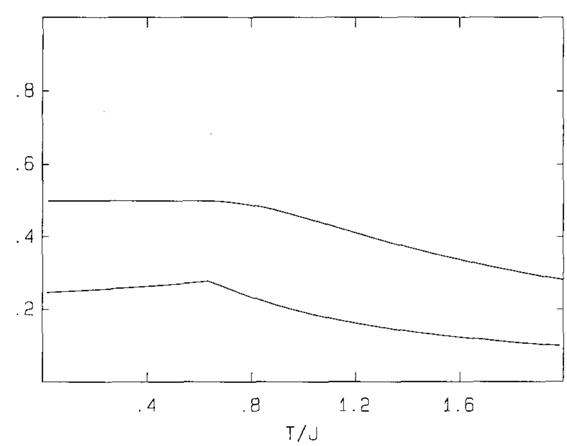

Fig. 9. The magnetisation as a function of T for $p=3$. The upper line refers to $h>h_c$, while the lower line to $h<h_c$

sitions. The difference is evident. In Fig. 9 it is shown the magnetization as a function of T for $p=3$. The freezing is clearly visible.

The metastable states are present also below the continuous transition. These are obtained solving the 1 RSB equation for a fixed value of m, i.e. without requiring the maximisation of the free energy with respect to m. It is clear that not all values of m will give locally stable states. This depends on the field, temperature and p, and is determined by the most relevant eigenvalue. We have verified numerically that these states do exist, even for fields larger than h_c.

7. Conclusions

In this paper we have studied the static properties of the spherical pSG. Using the replica trick the model can be solved exactly for any temperature and field. As the spherical SK model, $p=2$ in our case, the full replica symmetry broken phase is absent. However, for any $p>2$ the model presents at low temperatures and fields a replica symmetry broken phase which, in the Parisi scheme, corresponds to one-step replica symmetry breaking (1 RSB). In this scheme this is the most general solution of the model. The analysis of the eigenvalues of the Gaussian fluctuations about the 1 RSB saddle point shows that the 1 RSB solution is stable in the full low temperature and field phase. In the limit $p \to 2$ this region shrinks to zero and the usual SK spherical model is recovered. This phase corresponds to the 1 RSB phase found in the Ising p-spin model for any $p>2$ [6, 7]. However, the non-linearity of the spherical model is not strong enough to produce a full replica symmetry breaking for lower temperatures [7].

The transition from the RS state to the 1 RSB takes places discontinuously or continuously, depending on the value of the external field. For fields weaker than a critical value h_c, which depends on p, the transition is discontinuous, as far as the order parameter is concerned. At the critical temperature a 1 RSB solution appears with $m=1$ and $q_1-q_0 \neq 0$. This solution has the same free energy of the RS. However when the temperature is lowered and m becomes smaller than 1, the 1 RSB leads to an higher free energy. At the transition the RS solution remains stable. It becomes unstable at a lower temperature where the relevant eigenvalue of the RS Gaussian fluctuations becomes zero. The value of this temperature depends on p and on the field h. In particular for $h=0$ the RS solution is stable for all temperature. Even if the transition is discontinuous in $q(x)$, it turns out to be thermodynamically of the second order, as can be seen from the thermodynamic quantities. Some of them have been discussed in Sect. 6.

For fields larger than h_c the transition is continuous in q_1-q_0. At the transition the RS solution becomes unstable and a 1 RSB state is reached. The 1 RSB solution appears with $q_1-q_0=0$, but with a well defined value of m determined by the RS q at the transition. This transition is thermodynamically of the third order, as can be seen studying the thermodynamic quantities.

At the critical point, joining the continuous and the discontinuous transitions, the RS q is equal to $q_c = 1-2/p$. Therefore in the limit $p \to 2$ ($p \to \infty$) the discontinuous (continuous) transition disappears, in agreement to what is found in the Ising case [6, 7].

An interesting feature of this model is the presence of metastable states, i.e. states which are locally stable but nevertheless have lower free energy. These are 1 RSB states with a given value of m. For $h<h_c$ one of these metastable states is the RS solution. The region of existence of each of these states can be obtained from the most relevant eigenvalue of the Gaussian fluctuations, and in general depends on both h and p. For $h=0$ all these states are stable for all temperatures. These states exist in the full phase of low temperatures and low fields.

The metastable states are not relevant for the static properties of the model. These are indeed obtained from the state with the largest free energy. The exchange of minimum with maximum is due to the $n \to 0$ limit of the replica trick. Nevertheless they may have some relevance for the dynamics of the model, which takes place in the space of random couplings searching for the free energy with maximal probability [12].

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Appendix 1. The Gaussian λ-fluctuations

In this appendix we derive (3.15). As discussed in Sect. 3 the relevant part for the Gaussian $\boldsymbol{\lambda}$-fluctuations at the saddle point is given by (3.12). This can be further simplified by noting that the last term is of $O(n^2)$ and hence will give, after integration, only a contribution of order $O(n^2/N)$, which can be neglected. Thus we are left with

$$2\Phi_0[\boldsymbol{\lambda}] = \sum_{\alpha\beta} \lambda_{\alpha\beta} q_{\alpha\beta} + \ln\det(-\lambda_{\alpha\beta} - b^2). \qquad (\mathrm{A}1.1)$$

The first variation with respect to $\boldsymbol{\lambda}$ leads to the saddle point equation (3.14). The second variation evaluated at the saddle point,

$$\delta^2 \Phi_0 = -\tfrac{1}{2}\mathrm{Tr}(\mathbf{q}\,\delta\boldsymbol{\lambda}\,\mathbf{q}\,\delta\boldsymbol{\lambda}) \qquad (\mathrm{A}1.2)$$

gives the Gaussian fluctuations about the saddle point. They contribute to $\overline{Z^n}$ with the factor [cf. (3.7)]

$$\int_{-i\infty}^{+i\infty} \prod_{\alpha<\beta} \frac{N}{2\pi i}\, d\lambda_{\alpha\beta} \int_{-i\infty}^{+i\infty} \prod_{\alpha} \frac{\sqrt{N}}{2\pi i}\, d\lambda_{\alpha\alpha} \times \exp\left[\frac{N}{4}\mathrm{Tr}(\mathbf{q}\,\delta\boldsymbol{\lambda}\,\mathbf{q}\,\delta)\right]. \qquad (\mathrm{A}1.3)$$

The quadratic form in the exponent can be diagonalised by performing a rotation in the $\delta\boldsymbol{\lambda}$ space. This leads to

$$\int_{-\infty}^{+\infty} \prod_{\alpha<\beta} \frac{N}{2\pi}\, d\lambda_{\alpha\beta} \int_{-\infty}^{+\infty} \prod_{\alpha} \frac{\sqrt{N}}{2\pi}\, d\lambda_{\alpha\alpha}$$

$$\times \exp\left[-\frac{N}{4}\sum_{\alpha} q_\alpha q_\beta (\delta\lambda_{\alpha\beta})^2\right]$$

$$=\left(\frac{N}{2\pi}\right)^{n(n-1)/4} \pi^{-n/2}\left(\prod_{\alpha<\beta} q_\alpha q_\beta\right)^{-1/2}\left(\prod_{\alpha} q_\alpha^2\right)^{-1/2} \tag{A1.4}$$

where q_α are the eigenvalues of the matrix $\mathbf{q}$. Using the identities

$$\prod_{\alpha} q_\alpha = \det(\mathbf{q}), \qquad \prod_{\alpha<\beta} q_\alpha q_\beta = (\det(\mathbf{q}))^{n-1} \tag{A1.5}$$

(3.15) follows.

Appendix 2. Parisi RSB solution

In this appendix we show that within the Parisi RSB scheme, the most general solution for the spherical pSG is the one-step RSB. The Parisi ansatz for $q_{\alpha\beta}$ can be described by means of the following recursive algorithm:

(i) First breaking: the n replicas are grouped in n/m_1 clusters of m_1 replicas. Any two replicas $\alpha \neq \beta$ within the same cluster have overlap $q_{\alpha\beta}=q_1$, whereas replicas in different clusters have overlap $q_{\alpha\beta}=q_0 \leqq q_1$. The $n\times n$ matrix $\mathbf{q}$ is hence divided into $(n/m_1)\times(n/m_1)$ blocks of size $m_1\times m_1$. Inside the blocks on the diagonal $q_{\alpha\beta}=q_1$, whereas in the others $q_{\alpha\beta}=q_0$.
(ii) Second breaking: each cluster of size m_1 is broken into m_1/m_2 sub-clusters of m_2 replicas. Any two replicas $\alpha\neq\beta$ in the same sub-cluster have overlap $q_{\alpha\beta}=q_2 \geqq q_1$, the others overlaps remain unchanged. As a consequence each of the n/m_1 diagonal blocks is divided in $(m_1/m_2)\times(m_1/m_2)$ sub-blocks of size $m_2\times m_2$. In the diagonal sub-blocks q_1 is replaced by q_2.

Iterating this procedure one obtains a general k-breaking situation defined by,

$$\begin{aligned} &n\geqq m_1 \geqq m_2 \geqq \cdots \geqq m_k \geqq 1 \\ &q_0 \leqq q_1 \leqq q_2 \leqq \cdots \leqq q_{k-1} \leqq q_k . \end{aligned} \tag{A2.1}$$

Note that in the limit $n\to 0$ the m_i become continuous variables between 0 and 1, and the inequalities in (A2.1) are reversed, i.e. for $n=0$: $0\leqq m_1 \leqq m_2 \leqq \cdots \leqq m_k \leqq 1$.

The matrix $\mathbf{q}$ obtained after k steps of this procedure is conveniently parametrised by the function $x(q)$, which equals the fraction of pairs of replicas with $q_{\alpha\beta}\leqq q$,

$$\begin{aligned} &x(q)=n+\sum_{i=0}^{k}(m_{i+1}-m_i)\,\theta(q-q_i), \\ &m_0\equiv n;\, m_{k+1}\equiv 1 . \end{aligned} \tag{A2.2}$$

In the limit of infinite k, q becomes continuous and we can define $q(x)$ as the inverse of $x(q)$.

For a generic k-step RSB, the eigenvalues of $\mathbf{q}$ are

$$\begin{aligned} &\int_0^1 \mathrm{d}q\, x(q), \quad \text{deg.: } 1 \\ &\int_{q_i}^1 \mathrm{d}q\, x(q), \quad \text{deg.: } n\left(\frac{1}{m_{i+1}}-\frac{1}{m_i}\right). \end{aligned} \tag{A2.3}$$

Inserting these into (3.17), neglecting the last term of order $O(n^2)$, and replacing the sums by integrals, one gets after a little of algebra,

$$\frac{2}{n}G_0 = -\mu\int_0^1 \mathrm{d}q\, x(q)\, q^{p-1} - b^2\int_0^1 \mathrm{d}q\, x(q) - \int_0^{q_k}\frac{\mathrm{d}q}{\int_q^1 \mathrm{d}q'\, x(q')} - \ln(1-q_k). \tag{A2.4}$$

The saddle point equation is obtained by varying this expression with respect to $x(q)$,

$$\frac{2}{n}\delta G_0 = \int_0^1 \mathrm{d}q\, \delta x(q) F(q) \tag{A2.5a}$$

where

$$F(q) = -\mu q^{p-1} - b^2 + \int_0^q \frac{\mathrm{d}q'}{\left(\int_{q'}^1 \mathrm{d}q''\, x(q'')\right)^2} \tag{A2.5b}$$

and

$$\delta x(q) = \sum_{i=0}^{k}(\delta m_{i+1}-\delta m_i)\,\theta(q-q_i) - \sum_{i=0}^{k}(m_{i+1}-m_i)\,\delta(q-q_i)\,\delta q_i . \tag{A2.6}$$

By requiring the stationarity of G_0 with respect to the q_i and the m_i one gets, respectively

$$F(q_i)=0, \quad 0\leqq i \leqq k \tag{A2.7}$$

$$\int_{q_{i-1}}^{q_i} \mathrm{d}q\, F(q) = 0, \quad 1\leqq i\leqq k . \tag{A2.8}$$

The function $F(q)$ is continuous in q, hence (A2.8) implies that betwen any two successive q_i there must by at least two extrema of $F(q)$. If we denote these by Q_j, then the extrema condition $F'(Q_j)=0$ reads

$$\Lambda(Q_j) \equiv -\mu(p-1)Q_j^{p-2} + \left(\int_{Q_j}^1 \mathrm{d}q\, x(q)\right)^{-2} = 0 \tag{A2.9}$$

or

$$\int_Q^1 \mathrm{d}q\, x(q) = [\mu(p-1)]^{-1/2} Q^{1-p/2} \quad \text{at } Q_j . \tag{A2.10}$$

Alternatively, the solutions of (A2.10) define the Q_j. This equation, however, admits not more than two solutions, since the left hand side is a concave function, whereas the right hand side is convex. As a consequence only one-step RSB is possible.

A continuous replica symmetry breaking is also not allowed since in this case (A2.9) would be identically valid. For $p>2$ this would imply that $x(q)$ is a decreasing function, which we exclude since $\mathrm{d}x/\mathrm{d}q$ is the probability density of the overlaps [3]. The $\Lambda(Q)$ turn out to be the relevant eigenvalues of the fluctuation matrix calculated in Appendix 3.

Note that for the RS solution the m_i do not appear, and (A2.7) reduces to the RS saddle point equation.

We finally note that from the above equations it follows also that $q_0=0$ for $h=0$. In fact, otherwise there exist three extrema of $F(q)$ since (A2.7) implies

$$F(0)=F(q_0)=F(q_1)=0$$

which is not possible.

Appendix 3. Eigenvalues of the Gaussian q-fluctuations for 1 RSB

In this appendix we briefly report the derivation of the eigenvalues of the Gaussian **q**-fluctuations for the 1 RSB solution. The quadratic form to be diagonalised is obtained by inserting (5.2) and (5.4) into (3.21),

$$\begin{aligned}2\delta^2 G_0 = & -\mu(p-1) \\ & \times \sum_{\alpha\beta} [q_0^{p-2}+\varepsilon_{\alpha\beta}(q_1^{p-2}-q_0^{p-2})](\delta q_{\alpha\beta})^2 \\ & + A^2 \sum_{\alpha\beta} (\delta q_{\alpha\beta})^2 + B^2 \operatorname{Tr}(\varepsilon\delta\mathbf{q})^2 \\ & + (C^2+b^4)\left(\sum_{\alpha\beta} \mathrm{d}q_{\alpha\beta}\right)^2 + 2AB\operatorname{Tr}(\delta\mathbf{q}\varepsilon\delta\mathbf{q}) \\ & + 2AC\sum_{\alpha\beta}(\delta\mathbf{q}\delta\mathbf{q})_{\alpha\beta} + 2BC\sum_{\alpha\beta}(\delta\mathbf{q}\varepsilon\delta\mathbf{q})_{\alpha\beta}.\end{aligned} \tag{A3.1}$$

The eigenvalues of this quadratic form are solutions of the eigenvalue equation

$$\begin{aligned}&[A^2-\mu(p-1)q_0^{p-2}]\delta q_{\alpha\beta} \\ &-\mu(p-1)(q_1^{p-2}-q_0^{p-2})\varepsilon_{\alpha\beta}\delta q_{\alpha\beta} + B^2(\varepsilon\delta\mathbf{q}\varepsilon)_{\alpha\beta} \\ &+(C^2+b^4)\sum_{\gamma\delta}\delta q_{\gamma\delta} + AB[(\varepsilon\delta\mathbf{q})_{\alpha\beta}+(\delta\mathbf{q}\varepsilon)_{\alpha\beta}] \\ &+AC\sum_{\gamma}(\delta q_{\alpha\gamma}+\delta q_{\beta\gamma}) \\ &+BC\sum_{\gamma}[(\varepsilon\delta\mathbf{q})_{\alpha\gamma}+(\varepsilon\delta\mathbf{q})_{\beta\gamma}] = \Lambda\delta q_{\alpha\beta}\end{aligned} \tag{A3.2}$$

which is valid for $\alpha \neq \beta$. The diagonal term of $\delta\mathbf{q}$ are identically zero from the spherical constraint.

The eigenvectors can be devided in three classes, corresponding to different types of fluctuations. This can be seen by representing the replicas on a tree-diagram. Here, since we have only one replica symmetry breaking, there will be only n/m branches (clusters), and each terminates with m smaller branches (replicas). Then, roughly speaking, there are three different types of fluctuations: *1)* one which involves the overlaps of one replica with other m replicas; *2)* one which involves the overlaps of m replicas with other m replicas; *3)* one which involves the overlaps of one cluster as a whole with the other clusters as a whole. Consequently we will call the first type "1-m" fluctuations, the second "m-m" fluctuations, and the last "cluster-cluster" fluctuations.

1. The 1-m fluctuations. These fluctuations satisfy the constraint

$$(\varepsilon\delta\mathbf{q})_{\alpha\beta}=0, \quad \forall\alpha,\beta \tag{A3.3}$$

and covers a sub-space of dimension

$$\frac{n(n-1)}{2}-\frac{n^2}{m}+\frac{n}{2m}\left(\frac{n}{m}-1\right). \tag{A3.4}$$

This sub-space contains two eigenvalues.

The first,

$$\begin{aligned}&\Lambda_0^{(1)} = -\mu(p-1)q_0^{p-2}+A^2, \\ &n_0^{(1)} = \frac{n}{2m^2}(n-m)(m-1)^2\end{aligned} \tag{A3.5}$$

corresponds to fluctuations between two different clusters for which have all diagonal blocks of $\delta\mathbf{q}$ equal to zero, i.e.

$$\varepsilon_{\alpha\beta}\delta q_{\alpha\beta}=0, \quad \forall\alpha,\beta. \tag{A3.6}$$

The second,

$$\begin{aligned}&\Lambda_1^{(1)} = -\mu(p-1)q_1^{p-2}+A^2, \\ &n_1^{(1)} = \frac{n(m-3)}{2}\end{aligned} \tag{A3.7}$$

is associate to the orthogonal fluctuations, i.e., fluctuations in the same cluster. In this case the off-diagonal blocks of $\delta\mathbf{q}$ are zero,

$$(1-\varepsilon_{\alpha\beta})\delta q_{\alpha\beta}=0, \quad \forall\alpha,\beta. \tag{A3.8}$$

2. The m-m fluctuations. The constraint for these fluctuations is

$$(\varepsilon\delta\mathbf{q}\varepsilon)_{\alpha\beta}=0, \quad \forall\alpha,\beta. \tag{A3.9}$$

If we also require that (A3.3) is not satisfyed, then these fluctuations are orthogonal to the first class. This leads to a sub-space of dimension

$$\frac{n^2}{m}-\frac{n^2}{m^2}. \tag{A3.10}$$

The eigenvalue equation can be reduced to this sub-space by multiplying it by $\varepsilon_{\alpha\beta}$ and summing over β. This sub-space contains again two eigenvalues corresponding to diagonal and off-diagonal fluctuations.

The fluctuations between different clusters, for which

$$\varepsilon_{\alpha\beta}(\varepsilon\delta\mathbf{q})_{\alpha\beta}=0, \quad \forall\alpha,\beta, \tag{A3.11}$$

give the eigenvalue

$$\begin{aligned}&\Lambda_0^{(2)} = -\mu(p-1)q_0^{p-2}+A(A+mB), \\ &n_0^{(2)} = \frac{n}{m}\left(\frac{n}{m}-1\right)(m-1).\end{aligned} \tag{A3.12}$$

The second eigenvalue corresponds to fluctuations in the same cluster,

$$(1-\varepsilon_{\alpha\beta})(\varepsilon\delta\mathbf{q})_{\alpha\beta}=0, \quad \forall\alpha,\beta, \tag{A3.13}$$

and reads

$$\Lambda_1^{(2)}=-\mu(p-1)q_1^{p-2} + A[A+(m-2)(B+C)], \tag{A3.14}$$

$$n_1^{(2)}=\frac{n}{m}(m-1).$$

3. *The cluster-cluster fluctuations.* For this fluctuations the clusters are considered as single entities. The space dimensionality of this sub-space, orthogonal to the first two classes, is

$$\frac{n}{2m}\left(\frac{n}{m}+1\right). \tag{A3.15}$$

The eigenvalue equation can be reduced to this sub-space by multiplying it on both sides by ε and summing on α and β. This sub-space for finite n contains five different eigenvalues, of which only one corresponds to off-diagonal fluctuations. The other four are combinations of diagonal and off-diagonal fluctuations. In the limit $n\to 0$ these four eigenvalues reduce to two. Since the clusters are considered as single entities, the fluctuations can be described by defining the cluster matrix $\mathbf{Q}_{\nu\eta}$ as

$$\mathbf{Q}_{\nu\eta}=(\varepsilon\delta\mathbf{q}\varepsilon)_{\alpha\beta} \quad \text{with} \quad \alpha\in\nu, \beta\in\eta \tag{A3.16}$$

where ν and η are cluster indices.

The first eigenvalue is

$$\Lambda_0^{(3)}=-\mu(p-1)q_0^{p-2}+(A+mB)^2, \qquad n_0^{(3)}=\frac{n}{2m}\left(\frac{n}{m}-3\right). \tag{A3.16}$$

This corresponds to the only purely off-diagonal fluctuations in this sub-space for which

$$\begin{cases} \mathbf{Q}_{\nu\nu}=0 \\ \sum_{\eta}\mathbf{Q}_{\nu\eta}=0. \end{cases} \quad \forall\nu \tag{A3.17}$$

For finite n there are two different sub-spaces which correspond to mixed fluctuations. The first is given by the vector for which

$$\begin{cases} \sum_{\nu}\mathbf{Q}_{\nu\nu}=0 \\ \sum_{\eta\neq\nu}\mathbf{Q}_{\nu\eta}=0. \end{cases} \tag{A3.18}$$

These eigenvectors give the two eigenvalues

$$\Lambda_{1,2}^{(3)}=\tfrac{1}{2}\left[T\pm\sqrt{T^2-4\Delta}\right] \tag{A3.19}$$

with degeneracies

$$n_{1,2}^{(3)}=\frac{n}{m}-1 \tag{A3.20}$$

where

$$T=\Lambda_0^{(3)}+\Lambda'+D+R \tag{A3.21}$$

$$\Delta=(\Lambda_0^{(3)}+D)(\Lambda'+R)-DR \tag{A3.22}$$

and

$$\Lambda'=-\mu(p-1)q_1^{p-2} + (A+mB)^2-B(2A+mB) \tag{A3.23}$$

$$D=(n-2m)C(A+mB) \tag{A3.24}$$

$$R=2(m-1)C(A+mB). \tag{A3.25}$$

Finally the last two eigenvalues correspond to eigenvectors for which (A3.18) are not satisfied. These eigenvalues $\Lambda_{3,4}^{(3)}$ have the same form (A3.19) of $\Lambda_{1,2}^{(3)}$ but with degeneracies

$$n_{3,4}^{(3)}=1 \tag{A3.26}$$

and D replaced by

$$D=2(n-m)C(A+mB) + n(n-m)(C^2+b^4). \tag{A3.27}$$

Note that these two eigenvalues become equal to $\Lambda_{1,2}^{(3)}$ in the limit $n\to 0$. However, their dependence on n is relevant for the $O(1/N)$ correction to the free energy since their degeneracy is finite for $n\to 0$.

References

1. Sherrington, D., Kirkpatrick, S.: Phys. Rev. Lett. **32**, 792 (1974); Kirkpatrick, S., Sherrington, D.: Phys. Rev. B**17**, 4384 (1978)
2. Parisi, G.: Phys. Rev. Lett. **50**, 1946 (1983)
3. Parisi, G.: Phys. Rev. Lett. **43**, 1754 (1979); Parisi, G.: J. Phys. A**13**, L115, 1101, 1887 (1980)
4. de Dominicis, C., Kondor, I.: Phys. Rev. B**27**, 606 (1983); de Dominicis, C., Kondor, I.: J. Phys. A**16**, L73 (1983)
5. Mézard, M., Parisi, G., Sourlas, N., Toulouse, G., Virasoro, M.A.: Phys. Rev. Lett. **52**, 1156 (1984); Mézard, M., Parisi, G., Sourlas, N., Toulouse, G., Virasoro, M.A.: J. Phys. (Paris) **45**, 843 (1984)
6. Gross, D.J., Mézard, M.: Nucl. Phys. B**240**, 431 (1984)
7. Gardner, E.: Nucl. Phys. B**257**, 747 (1985)
8. Kirkpatrick, T.R., Thirumalai, D.: Phys. Rev. B**36**, 5388 (1987)
9. Crisanti, A., Sommers, H.-J.: (in preparation)
10. Kosterlitz,. J.M., Thouless, D.J., Jones, R.C.: Phys. Rev. Lett. **36**, 1217 (1976)
11. See e.g., Crisanti, A., Paladin, G., Vulpiani, A.: Product of random matrices in statistical physics. Berlin, Heidelberg, New York: 1992
12. Crisanti, A., Paladin, G., Sommers, H.-J., Vulpiani, A.: J. Phys. (in press 1992)
13. de Almeida, J.R.L., Thouless, D.J.: J. Phys. A**11**, 983 (1978)