A SMOOTHING INEQUALITY FOR HIERARCHICAL PINNING MODELS

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Abstract. We consider a hierarchical pinning model introduced by B. Derrida, V. Hakim and J. Vannimenus in [3], which undergoes a localization/delocalization phase transition. This depends on a parameter $B > 2$, related to the geometry of the hierarchical lattice. We prove that the phase transition is of second order in presence of disorder. This implies that disorder smooths the transition in the so-called *relevant disorder case*, i.e., $B > B_c = 2 + \sqrt{2}$.

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1. Model

The model we are going to introduce can be interpreted either as an infinite-dimensional dynamical system or as a pinning model on a hierarchical lattice. We refer to [3], where the model was introduced, or to [7], for the latter interpretation, and we mention only that the parameter B in (1.2) is related to the geometry of the lattice. The mathematical understanding of disordered pinning models, hierarchical or not, has witnessed a remarkable progress lately, cf. in particular [1], [2] and [4]–[8].

Let $\{\omega_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables (with related probability distribution denoted by $\mathbb{P}$), with zero mean, unit variance and satisfying

$$M(\beta) := \mathbb{E}[\exp(\beta\omega_1)] < \infty \quad \forall \beta > 0.$$

In Section 3 we will state our result in the particular case of Gaussian random variables, $\omega_1 \sim \mathcal{N}(0,1)$.

We are interested in the dynamical system defined by the following initial condition and recurrence equation

$$R_0^{(i)} = \exp\left(\beta\omega_i - \log M(\beta) + h\right), \tag{1.1}$$

$$R_{n+1}^{(i)} = \frac{R_n^{(2i)} R_n^{(2i-1)} + (B-1)}{B}, \tag{1.2}$$

where $B > 2$, $\beta > 0$ and $h \in \mathbb{R}$ are fixed parameters.

One can easily note that for every n, the random variables $\{R_n^{(i)}\}_{i\geq 1}$, are i.i.d.. We are mainly interested in the evolution of the probability law of $R_n^{(1)}$ (let us call it $\mathcal{L}_n$). $R_n^{(1)}$ is essentially the partition function of the pinning model at generation n [3]. One can

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re-interpret this system as a dynamical system on probability laws, $\mathcal{L}_{n+1}$ being the law of R_{n+1} given by

$$R_{n+1} = \frac{R_n^{(1)} R_n^{(2)} + (B-1)}{B}, \tag{1.3}$$

where $R_n^{(i)}$ $i = 1, 2$ are i.i.d. random variables with probability law $\mathcal{L}_n$.

One can also study this dynamical system in a non-random set up, choosing $\beta = 0$ or, equivalently, considering $r_n = \mathbb{E} R_n$ (respectively pure or annealed system). The recursion becomes then

$$r_{n+1} = \frac{r_n^2 + (B-1)}{B}, \tag{1.4}$$

(with initial condition $r_0 = \exp(h)$), which is nothing but a particular case of the well-known *logistic map*.

In any case, there exists a (non-random) quantity $\textsc{f}(\beta, h)$ such that

$$\lim_{n\to\infty} \frac{1}{2^n} \log R_n = \textsc{f}(\beta, h) \quad \text{almost surely w.r.t. } \mathbb{P}.$$

The convergence also holds in $\mathbb{L}_1$, and $\textsc{f}(\beta, h) \geq 0$. For a proof of these statements see [4, Theorem 1.1]. Observe that the non-negativity of $\textsc{f}$ just follows from $R_n^{(i)} \geq (B-1)/B$ (cf. (1.2)). We call $\textsc{f}(\beta, h)$ the free energy of the system, and we remark that $\textsc{f}(\beta, .)$ is increasing. We define the critical point

$$h_c(\beta) = \inf\{h \text{ such that } \textsc{f}(\beta, h) > 0\}.$$

For an interpretation of $h_c(\beta)$ as the transition point between a delocalized and a localized phase ($h < h_c(\beta)$ and $h > h_c(\beta)$, respectively) we refer to [4] and [3].

Our aim in this paper is to describe the influence of the disorder on the shape of the curve $\textsc{f}(\beta, .)$ around $h_c(\beta)$, and in particular to obtain a bound on the critical exponent which governs the vanishing of $\textsc{f}$ when $h_c(\beta)$ is approached from the localized phase.

2. Known results

In [4] various results have been obtained about the influence of disorder on the free energy curve. We summarize the situation in the two following theorems. The first one describes the behavior of the free energy around h_c and the value of h_c for the pure system ($\beta = 0$). The second one concerns the behavior of $h_c(\beta)$ for small values of β, making a distinction between two different situations: irrelevant disorder for $B < 2 + \sqrt{2}$ (critical properties of the system are unchanged, for β small, with respect to $\beta = 0$) and relevant disorder for $B > 2 + \sqrt{2}$ (the disordered system behaves very differently from the pure one, for every $\beta > 0$). For this reason we will refer to $B_c = 2 + \sqrt{2}$ as the critical value of B.

Theorem 2.1. [4] *(Annealed system estimates). The function $h \mapsto \textsc{f}(0, h)$ is real analytic except at $h = h_c := h_c(0)$. Moreover $h_c = \log(B-1)$ and there exists $c = c(B) > 0$ such that for all $h \in (h_c, h_c + 1)$*

$$c(B)^{-1}(h - h_c)^{1/\alpha} \leq \textsc{f}(0, h) \leq c(B)(h - h_c)^{1/\alpha}, \tag{2.1}$$

where

$$\alpha := \frac{\log(2(B-1)/B)}{\log 2}. \tag{2.2}$$

Theorem 2.2. [4] *When $B < 2+\sqrt{2} = B_c$, there exists β_0 such that for all $0 \le \beta \le \beta_0$, $h_c(\beta) = h_c(0)$. Moreover, for any $\varepsilon > 0$, one can find $h_\varepsilon > h_c(0)$ such that for any $h \in (h_c(0), h_\varepsilon)$ we have*

$$(1-\varepsilon)\mathrm{F}(0,h) \le \mathrm{F}(\beta,h) \le \mathrm{F}(0,h).$$

When $B > 2+\sqrt{2}$, we have for any $\beta > 0$, $h_c(\beta) > h_c(0)$. Moreover we can estimate the difference between the two: there exists $C(B) > 0$ such that for all $\beta \le 1$

$$C(B)^{-1}\beta^{\frac{2\alpha}{2\alpha-1}} \le h_c(\beta) - h_c(0) \le C(B)\beta^{\frac{2\alpha}{2\alpha-1}}.$$

Finally, if $B = B_c$ then

$$0 \le h_c(\beta) - h_c(0) \le e^{-C/\beta^2} \tag{2.3}$$

for some positive constant C.

It is a very interesting open problem to close the gap between the upper and lower bound in (2.3). Let us mention that the arguments in [3] suggest that the upper bound is the correct one, with an explicit prediction for the constant C.

When $B \ge B_c$ the previous theorem says nothing on the shape of the free energy around $h_c(\beta)$; in particular, one may wonder if it is different from the one of the pure system. In this spirit, we prove a general theorem on the effect of disorder on the phase transition of the system. This theorem is the analog of what has been proved in [5]-[6] for the non-hierarchical pinning model based on a renewal process.

3. Smoothness of the phase transition in presence of disorder

Assume that $\omega_1 \sim \mathcal{N}(0,1)$. We have:

Theorem 3.1. *For every $B > 2$ there exists $c(B) < \infty$ such that for every $\beta > 0$ and $\delta > 0$ one has*

$$\mathrm{F}(\beta, h_c(\beta)+\delta) \le \frac{\delta^2}{\beta^2} c(B). \tag{3.1}$$

Remark 3.2. In view of Theorem 2.1 and the definition of α, this shows that if $B > B_c$ (which corresponds to $\alpha > 1/2$) and $\beta > 0$ the critical exponent of the transition is different from that of the pure model, i.e. we have

$$\liminf_{h \to h_c(\beta)^+} \frac{\log \mathrm{F}(\beta,h)}{\log(h-h_c(\beta))} \ge 2 > \lim_{h \to h_c(0)^+} \frac{\log \mathrm{F}(0,h)}{\log(h-h_c(0))} = \frac{1}{\alpha}.$$

Proof of Theorem 3.1. Fix $\beta > 0$ and let $h = h_c(\beta)$. Let $N \in \mathbb{N}$, $\ell \in \mathbb{N}$ with $\ell < N$ and

$$\mathcal{I}_N(\omega) := \{1 \le j \le 2^{N-\ell} \ : \ R_\ell^{(j)} \ge \exp(2^{\ell-1}\mathrm{F}(\beta, h_c(\beta)+\delta))\}. \tag{3.2}$$

Defining

$$p_\ell := \mathbb{P}(1 \in \mathcal{I}_N(\omega)), \tag{3.3}$$

one has $\mathbb{P}(\,\mathrm{d}\omega)$-a.s.

$$\lim_{N\to\infty} \frac{|\mathcal{I}_N(\omega)|}{2^{N-\ell}} = p_\ell \tag{3.4}$$

from the strong law of large numbers and, for ℓ sufficiently large,

$$p_\ell \ge e^{-2^\ell \delta^2/\beta^2}. \tag{3.5}$$

To prove the latter estimate we proceed as in [5] and we use the classical entropic inequality

$$\mathbb{P}(A) \geq \widetilde{\mathbb{P}}(A) \exp\left(-\frac{1}{\widetilde{\mathbb{P}}(A)}\left(\mathrm{H}(\widetilde{\mathbb{P}}|\mathbb{P}) + e^{-1}\right)\right) \tag{3.6}$$

which holds for every event A if the laws $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are mutually absolutely continuous, and

$$\mathrm{H}(\widetilde{\mathbb{P}}|\mathbb{P}) := \mathbb{E}\left(\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} \log \frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right)$$

denotes the relative entropy. Equation (3.5) then easily follows if we apply (3.6) with $A = \{1 \in \mathcal{I}_N(\omega)\}$ and $\widetilde{\mathbb{P}} = \widetilde{\mathbb{P}}_\ell$ being the law under which $\{\omega_n\}_{n\in\mathbb{N}}$ are independent Gaussian variables of unit variance and mean $\widetilde{\mathbb{E}}\,\omega_n = (\delta/\beta)$ for $n \leq 2^\ell$ and $\widetilde{\mathbb{E}}\omega_n = 0$ otherwise. Indeed, in that case $\widetilde{\mathbb{P}}(A) \geq (3/4)$ for ℓ sufficiently large by the fact that $2^{-\ell}\log R_\ell$ converges to $\textsc{f}(\beta, h_c(\beta)+\delta)$ in $\widetilde{\mathbb{P}}$-probability for $\ell \to \infty$ and the relative entropy is immediately computed: $\mathrm{H}(\widetilde{\mathbb{P}}|\mathbb{P}) = 2^\ell(\delta/\beta)^2/2$.

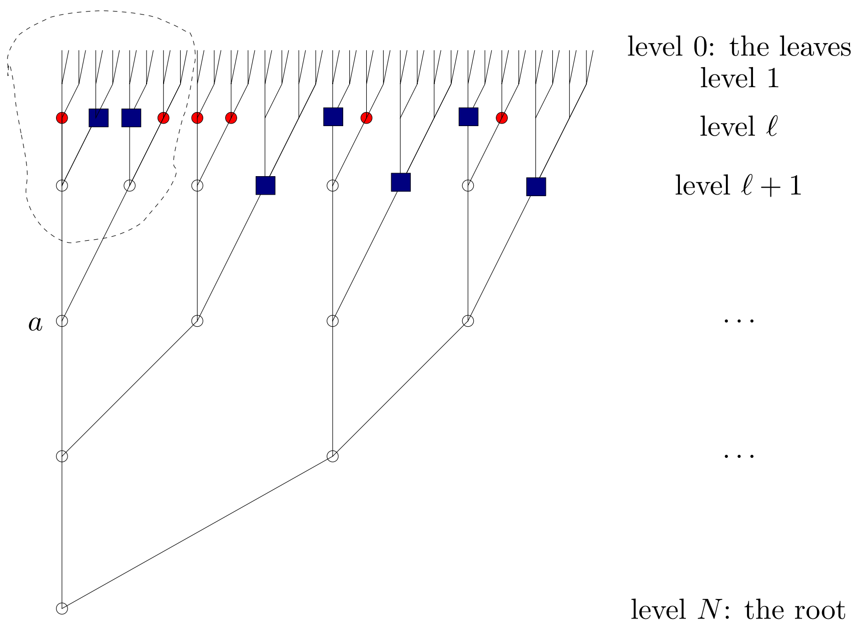

FIGURE 1. The binary tree $\mathcal{T}_N$ for $N = 6$. At level 0 are the leaves. Nodes (ℓ, j) with $j \in \mathcal{I}_N(\omega)$ are marked by full dots (here we have taken $\ell = 2$), good nodes by empty dots and bad nodes by squares. The dashed line includes all the descendents of the node a.

Consider the binary tree $\mathcal{T}_N$ with $N+1$ levels (see Fig. 1): at level 0 are the leaves, at level N is the root. Pairs (n, j) with $0 \leq n \leq N$ and $1 \leq j \leq 2^{N-n}$ are called nodes (leaves are also considered to be nodes). The number of nodes in $\mathcal{T}_N$ is $2^{N+1}-1$. Given a node $(n,j) \in \mathcal{T}_N$ with $n < N$ we call $(n+1, \lceil j/2 \rceil)$ its father. The descendents of a node $(n, j) \in \mathcal{T}_N$ are defined in the natural way.

Given a node $(n, j) \in \mathcal{T}_N$ with $n > \ell$ we say that it is a *good node* if there exists $i \in \mathcal{I}_N(\omega)$ such that (ℓ, i) is a descendent of (n, j). A node (n, j) with $n \geq \ell$ will be called bad if its father is good but he himself is not good (cf. Fig. 1). Note that descendents of bad nodes are neither good nor bad. If $\mathcal{I}_N(\omega) = \emptyset$, by convention we say that the root is a bad node, so that the root is always either bad or good. Let $g_N(\omega)$ and $b_N(\omega)$ be the number of good and bad nodes in $\mathcal{T}_N$.

Given ω, one has the inequality

$$R_N \geq \exp(\mathrm{F}(\beta, h_c(\beta) + \delta) 2^{\ell-1} |\mathcal{I}_N(\omega)|) B^{-g_N(\omega)} \left(\frac{B-1}{B}\right)^{b_N(\omega)}, \tag{3.7}$$

which will be proved in a moment. As a consequence, $\mathbb{P}(\,\mathrm{d}\omega)$-almost surely from (3.4)

$$0 = \mathrm{F}(\beta, h_c(\beta)) \geq \frac{p_\ell}{2} \mathrm{F}(\beta, h_c(\beta) + \delta) - \liminf_{N\to\infty} 2^{-N} \left(g_N(\omega) \log B + b_N(\omega) \log\left(\frac{B}{B-1}\right)\right) \tag{3.8}$$

Now it is obvious that, if $\mathcal{I}_N(\omega)$ is not empty, $g_N(\omega)$ is bounded above by the total number of points of $\mathcal{T}_{\lceil \log_2 |\mathcal{I}_N(\omega)| \rceil}$, plus $|\mathcal{I}_N(\omega)|(N - \ell - \lfloor \log_2 |\mathcal{I}_N(\omega)| \rfloor)$. In formulas,

$$g_N(\omega) \leq |\mathcal{I}_N(\omega)|(2 + N - \lfloor \log_2 |\mathcal{I}_N(\omega)| \rfloor - \ell), \tag{3.9}$$

with the convention that $0 \log 0 = 0$. Letting $N \to \infty$, almost surely one has via (3.4) and (3.5)

$$\liminf_{N\to\infty} 2^{-N} g_N(\omega) \leq 2^{-\ell} p_\ell \left(2 - \log_2 p_\ell\right) \leq p_\ell \frac{\delta^2}{\beta^2 \log 2}(1 + o_\ell(1)), \tag{3.10}$$

where $o_\ell(1)$ denotes a quantity which vanishes for $\ell \to \infty$.

As for the last term in (3.8), write $\mathcal{I}_N(\omega) = \{x_1(\omega), x_2(\omega), \ldots,\}$ with $x_r(\omega) < x_{r+1}(\omega)$. Then we have

$$b_N(\omega) \leq 2 \sum_{r=1}^{|\mathcal{I}_N(\omega)|+1} \log_2(x_r(\omega) - x_{r-1}(\omega) + 1) \tag{3.11}$$

with the convention that $x_0(\omega) := 1$ and $x_{|\mathcal{I}_N(\omega)|+1} := 2^{N-\ell}$. This can be proven as follows (see Fig. 2). Let $1 < r \leq |\mathcal{I}_N(\omega)|$ and a_r be the first common ancestor of $(\ell, x_{r-1}(\omega))$ and $(\ell, x_r(\omega))$. Let π_1 (respectively π_2) denote the unique path without self-intersections which leads from a_r to $(\ell, x_{r-1}(\omega))$ (resp. from a_r to $(\ell, x_r(\omega))$). Let L be the number of times the walk π_1 makes a move to the left, and R the number of times π_2 makes a move to the right. Then the number of bad nodes which are enclosed by the paths π_1 and π_2, call it b_r, equals $b_r = L + R - 2$ (cf. Figure 2). On the other hand,

$$x_r(\omega) - x_{r-1}(\omega) - 1 \geq 2^{L-1} + 2^{R-1} - 2 \geq 2^{(L+R)/2} - 2 \tag{3.12}$$

from which one deduces that $b_r \leq 2 \log_2(x_r(\omega) - x_{r-1}(\omega) + 1)$. Similarly one proves that the number of bad nodes which are descendents of the first common ancestor of $(\ell, 1)$ and $(\ell, x_1(\omega))$ (respectively, of $(\ell, x_{|\mathcal{I}_N(\omega)|})$ and $(\ell, 2^{N-\ell})$) is at most $2\log_2(x_1(\omega))$ (resp. at most $2 \log_2(2^{N-\ell} - x_{|\mathcal{I}_N(\omega)|} + 1)$) and Eq. (3.11) is proven.

Using Jensen's inequality for the logarithm, from Eqs. (3.11) and (3.5) one has almost surely

$$\liminf_{N\to\infty} 2^{-N} b_N(\omega) \leq 2^{-\ell+1} p_\ell \log_2\left(\frac{1}{p_\ell}\right) \leq 2 p_\ell \frac{\delta^2}{\beta^2 \log 2}. \tag{3.13}$$

Putting together Eqs. (3.8), (3.10) and (3.13) and taking ℓ large one obtains then (3.1) for a suitable $c(B)$.

It remains to prove (3.7), and we will do that by induction on $N - \ell$. We need only consider the case where $\mathcal{I}_N(\omega) \neq \emptyset$ (i.e., the root is a good node) otherwise (3.7) reduces to $R_N \geq (B-1)/B$ which is evident from (1.3). For $N - \ell = 1$, Eq. (3.7) is easily checked

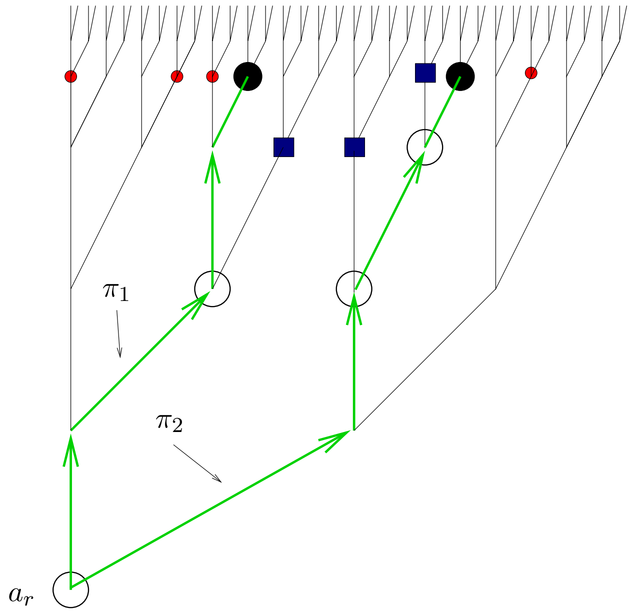

FIGURE 2. Again, the binary tree $\mathcal{T}_N$ for $N = 6$, $\ell = 2$ and nodes (ℓ, j) with $j \in \mathcal{I}_N(\omega)$ marked by full dots. The two big full dots denote nodes $x_{r-1}(\omega)$ and $x_r(\omega)$ ($r = 5$ in the figure) and their first common ancestor a_r is the root. The paths π_1, π_2 are marked by thicker lines and follow the arrows. Empty circles mark left turns in π_1, and right turns in π_2 (note that a_r contributes both to L and to R). Squares mark bad nodes which are between paths π_1 and π_2.

using the definition of good and bad sites and the basic recursion (1.3). Assume therefore that the statement holds for $N - \ell < k$, and let $N = \ell + k$. Call $\mathcal{T}^{(1)}_{N-1}$ and $\mathcal{T}^{(2)}_{N-1}$ the two trees of depth $N - 1$ which originate from the root. Since the root is assumed to be good, the following statements are true:

- $g_N(\omega)$ equals the number $g^{(1)}_{N-1}(\omega)$ of good sites in $\mathcal{T}^{(1)}_{N-1}$ plus the number $g^{(2)}_{N-1}(\omega)$ of good sites in $\mathcal{T}^{(2)}_{N-1}$ plus one;
- $b_N(\omega)$ equals $b^{(1)}_{N-1}(\omega) + b^{(2)}_{N-1}(\omega)$ (with obvious notations).

From (1.3) we have $R_N \geq R^{(1)}_{N-1} R^{(2)}_{N-1}/B$ which, in view of the induction hypothesis, reads

$$\begin{aligned} R_N \quad &\geq \quad \exp(F(\beta, h_c(\beta) + \delta) 2^{\ell-1} |\mathcal{I}_N(\omega)|) \\ &\times \left(\frac{B-1}{B}\right)^{b^{(1)}_{N-1}(\omega) + b^{(2)}_{N-1}(\omega)} B^{-[1 + g^{(1)}_{N-1}(\omega) + g^{(2)}_{N-1}]}. \end{aligned} \tag{3.14}$$

Thanks to the two observations above, this coincides with inequality (3.7) and the proof is complete. $\square$

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References

[1] K. S. Alexander, *The effect of disorder on polymer depinning transitions*, Commun. Math. Phys. **279** (2008), 117-146.

[2] B. Derrida, G. Giacomin, H. Lacoin, F.L. Toninelli, *Fractional moment bounds and disorder relevance for pinning models*, preprint (2007). arxiv.org/abs/0712.2515 [math.PR].

[3] B. Derrida, V. Hakim and J. Vannimenus, *Effect of disorder on two-dimensional wetting*, J. Statist. Phys. **66** (1992), 1189–1213.

[4] G. Giacomin, H. Lacoin and F. L. Toninelli, *Hierarchical pinning models, quadratic maps and quenched disorder*, preprint (2007), arXiv:0711.4649 [math.PR].

[5] G. Giacomin and F. L. Toninelli, *Smoothing effect of quenched disorder on polymer depinning transitions*, Commun. Math. Phys. **266** (2006), 1–16.

[6] G. Giacomin, F.L. Toninelli, *Smoothing of Depinning Transitions for Directed Polymers with Quenched Disorder*, Phys. Rev. Lett **96**, 060702 (2006).

[7] C. Monthus, T. Garel, *Critical points of quadratic renormalizations of random variables and phase transitions of disordered polymer models on diamond lattices*, arXiv:0710.0735v1.

[8] F. L. Toninelli, *A replica-coupling approach to disordered pinning models*, Commun. Math. Phys. **280**, 389-401 (2008).

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