Genericity in Topological Dynamics
Aussois, June 2025

Conventions:
- Ω is the Cantor space $\{0,1\}^{\mathbb{N}}$
- G is an infinite countable group.

$\mathrm{Homeo}(\Omega)$ is endowed with the topology of uniform convergence; one obtains subbases for this topology by considering $\{g \in \mathrm{Homeo}(\Omega) : ga = b\}$ $(a, b \in \mathrm{Clopen}(\Omega))$

or $\{g \in \mathrm{Homeo}(\Omega) : ga \cap b \neq \emptyset\}$ $(a, b \in \mathrm{Clopen}(\Omega))$.

Def: A *G-flow* is an action $G \curvearrowright X$ by homeomorphisms, with X a compact Hausdorff space.

We denote $\mathcal{A}(G)$ the space of G-flows with $X = \Omega$.
It can be thought of as a subset of $\mathrm{Homeo}(\Omega)^G$; it is closed for the product topology, hence Polish.

As above, we have two natural subbases for the topology of $\mathcal{A}(G)$:

$\{\xi : \xi(g)a = b\}$ $\left(\begin{matrix} g \in G \\ a, b \in \mathrm{Clopen}(\Omega)\end{matrix}\right)$ or $\{\xi : \xi(g)a \cap b \neq \emptyset\}$

We are concerned with the action of $\mathrm{Homeo}(\Omega)$ on $\mathcal{A}(G)$ by conjugation.

Def: $\xi_1 : G \curvearrowright X_1$ is a *factor* of $\xi_2 : G \curvearrowright X_2$ if there exists a G-equivariant surjection $\pi : X_2 \to X_1$.

Lemma: If $\xi_1, \xi_2 \in \mathcal{A}(G)$ are s.t. ξ_1 is a factor of ξ_2, then $\xi_1 \in \overline{\mathrm{Conj}(\xi_2)}$.

Proof: Let U be a neighborhood of ξ_1.

wlog $U = \{ \xi : \forall g \in F \ \forall a \in A \ \xi(g)a = \xi_1(g)a \}$

with $F \subseteq G$ finite, $A \subseteq \mathrm{Clope}(\Omega)$ finite.

Denote $\pi : (\Omega, \xi_2) \longrightarrow (\Omega, \xi_1)$ a factor map.

There exists $f \in \mathrm{Homeo}(\Omega)$ s.t

$\forall g \in F \ \forall a \in A \ f(\pi^{-1}(\xi_1(g)a)) = \xi_1(g)a$.

Then $\forall g \in F \ \forall a \in A$
$$f \xi_2(g) f^{-1}(a) = f \xi_2(g) \pi^{-1}(a) = f \pi^{-1} \xi_1(g) a = \xi_1(g) a$$ $\square$

Lemma: There is a dense conjugacy class in $A(G)$.

Proof: Take $(\xi_n)_n$ dense in $A(G)$.

Then $\prod_n \xi_n : G \curvearrowright \Omega^{\mathbb{N}} \cong \Omega$ has a dense conjugacy class since it factors on each ξ_n. $\square$

We are also interested in some (conjugacy invariant) subsets of $A(G)$, namely

$A_{tr}(G) = \{ \xi \in A(G) : \xi \text{ is (topologically) transitive} \}$

$A_{min}(G) = \{ \xi \in A(G) : \xi \text{ is minimal} \}$

Both subsets are G_δ, hence Polish.

There is a dense conjugacy class in $A(G)$: take $(\xi_n)_n$ dense in $A_{min}(G)$, then consider a minimal subflow of $\prod_n \xi_n$.

Thus, in $A(G)$ as well as $A_{min}(G)$, any "definable" (e.g., Borel) conjugacy invariant subset is either meager or comeager.

This applies in particular to conjugacy classes, which are always Borel (for any Polish group action).

* The case of $\mathbb{Z}$: The universal odometer has a comeager conjugacy class in $\mathcal{A}_{min}(G)$ (Hochman); $\overline{\mathcal{A}_{min}(G)} = \overline{\mathcal{A}_{tr}(G)}$; there is a comeager conjugacy class in $\mathcal{A}(G)$

* The case of $\mathbb{F}_p$, $2 \le p < +\infty$: there is a comeager conjugacy class in $\mathcal{A}(\mathbb{F}_p)$ for all p (Kwiatkowska).

To study these problems for general groups, it is convenient to work with subshifts (following Hochman, then Doucha)

Def: Let A be a finite set (alphabet)
The full shift on A is $G \curvearrowright A^G$, $g \cdot x(h) = x(g^{-1}h)$
A subshift is a shift-invariant closed subset of A^G, $\neq \emptyset$.
We denote $\mathcal{S}(A^G)$ the set of subshifts of A^G.

Def: A pattern is an element of A^F for some finite $F \subseteq G$.
We say that a pattern $p \in A^F$ occurs in $x \in A^G$ if

$$\exists g \in G \quad \forall f \in F \quad x(g^{-1}f) = p(f).$$

Every subshift X is determined by a set P of patterns which are forbidden in X, i.e $X = \{x \in A^G : \text{no } p \in P \text{ occurs in } x\}$.
(because $A^G \setminus X$ is open, and G-invariant).

Def: $X \in \mathcal{S}(A^G)$ is a subshift of finite type (SFT) if it is determined by a finite set of forbidden patterns.

Think of forbidden patterns as analogues of relations that a group must satisfy. Following this analogy, SFTs are analogues of finitely presented groups; subshifts correspond to quotients; and minimal subshifts to simple groups.

The topology on the space of marked groups also has an analogue for subshifts.

Def: Let X be a compact space. The Vietoris topology on the set of nonempty closed subsets of X is induced by $\{F : F \subseteq U\}$ and $\{F : F \cap U \neq \emptyset\}$, where U runs over open subsets of X.
(If X is 0-dimensional, one can take U above clopen)

Given a set of patterns P, denote:

$\mathcal{U}_P^+ = \{X \in \mathcal{S}(A^G) : \text{every element of } P \text{ occurs in } X\}$

$\mathcal{U}_P^- = \{X \in \mathcal{S}(A^G) : \text{no element of } P \text{ occurs in } X\}$

Then: * Being contained in an SFT amounts to being an element of some $\mathcal{U}_P^-$ for a finite P.

* Intersecting a given clopen subset of A^G can be encoded by some $\mathcal{U}_P^+$ for finite P (corresponding to clopen cylinders)

Thus: A basis for the topology of $\mathcal{S}(A^G)$ is given by sets of the form $\{X \in \mathcal{S}(Z) : X \in \mathcal{U}_P^+\}$ for Z an SFT and P a finite set of patterns.

Note: <u>SFTs are dense in $\mathcal{S}(A^G)$</u>, by the previous considerations.

For a minimal subshift X, and open sets $U, U_1 \dots U_n$ of A^G, $(X \subseteq U \wedge \forall i \in \{1, \dots, n\}\ X \cap U_i \neq \emptyset) \iff (X \subseteq U \cap G.U_1 \cap \dots \cap G.U_n)$

Thus a neighborhood basis of a minimal subshift is given by sets of the form $\{Y \in \mathcal{S}(G) : Y \subseteq U\}$ for U clopen which is encoded by $\{Y \in \mathcal{S}(G) : Y \in \mathcal{U}_P^+\}$ for some finite set of patterns P.

Back to $\mathcal{A}(G)$:

Given $\xi \in \mathcal{A}(G)$ and a clopen partition of Ω, define $\pi^A_\xi : \Omega \to A^G$ by

$$\pi^A_\xi(\omega)(g) = a \iff \xi(g^{-1}\omega) \in a .$$

Then π^A_ξ is G-equivariant, so $\pi^A_\xi[\Omega]$ is a subshift. Denote $\pi^A(\xi) = \pi^A_\xi[\Omega]$. The map $\pi^A : \mathcal{A}(G) \to \mathcal{S}(A^G)$ is continuous, and the topology of $\mathcal{A}(G)$ is induced by the maps π^A as A runs over all clopen partitions of Ω.

As hinted ~~at~~ by the analogy with marked groups, this setup fits into a broader framework, leading to a criterion for the existence of a comeager conjugacy class in either $\mathcal{A}(G)$ or $\mathcal{A}_{min}(G)$.

Def:
- $X \in \mathcal{S}(A^G)$ is **isolated** if it is an isolated point in $\mathcal{S}(A^G)$ for the Vietoris topology.
- $X \in \mathcal{S}(A^G)$ is **projectively isolated** if there exists a finite B, a map $\varphi : B \to A$ and an open subset $U \neq \emptyset$ in $\mathcal{S}(B^G)$ such that $\forall Y \in U$ $\Phi(Y) = X$, where $\Phi : B^G \to A^G$ is the factor map induced by φ.

Remark: Every isolated subshift is an SFT, and every projectively isolated subshift is sofic (i.e., it is a factor of an SFT).

Similarly, one can define minimal subshifts which are projectively isolated among minimals (p.i.o.m)

Examples: · Any minimal SFT is isolated; if X is an SFT with an isolated transitive point then X is isolated.

· If X is sofic and minimal, then it is projectively isolated; if X is sofic with an isolated transitive point then it is projectively isolated.

This applies for instance to $\left\{ x \in \{0,1\}^{\mathbb{N}} : \begin{array}{l} \text{there exists at most} \\ \text{one } i \text{ s.t. } x(i)=1 \end{array} \right\}$

Theorem: (i) [Doucha, which we recover here with our general setup]

There exists a comeager conjugacy class in $\mathcal{A}(G)$ iff projectively isolated subshifts are dense in $\mathcal{S}(A^G)$ for all finite A.

(ii) There exists a comeager conjugacy class in $\mathcal{A}_{min}(G)$ iff p.i.m subshifts are dense in $\mathcal{S}_{min}(A^G)$ for all finite A.

Question: Does this ever happen for a non-finitely generated G?

Prop: Assume that G is not finitely generated.

Then: (i) $X \in \mathcal{S}(A^G)$ is isolated $\iff$ X is a minimal SFT.

(ii) $X \in \mathcal{S}(A^G)$ is projectively isolated $\iff$ X is minimal sofic.

To prove this, one uses co-induction.

Def: Assume $H \leq G$ is a subgroup, and $X \in \mathcal{S}(A^H)$.

Then the co-induced subshift $\tilde{X} \in \mathcal{S}(A^G)$ is

$$\tilde{X} = \{ x \in A^G : \forall g \in G \quad h \mapsto x(gh) \in X \}$$

Remark: If $[G:H]$ is infinite then $\tilde{X}$ is a transitive subshift (use Neumann's lemma) and $\tilde{X}$ is homeomorphic to $X^{G/H} \cong \Omega$ as soon as X is nontrivial.

By definition of the topology, subshifts which are coinduced from a f.g subgroup are dense in $\mathcal{S}(A^G)$ for any A; so transitivity is generic among subshifts in that case.

Similarly, one obtains:

Theorem: Assume G is not fg. Then a generic element of $\mathcal{S}(G)$ is transitive.

Proof: Let $U \subseteq \mathcal{S}(G)$ be nonempty open. Take a nontrivial subshift X in U (subshifts are dense) then co-induced $X_{|H}$ for a sufficiently large f.g H. The co-induced subshift is then an element of U, and is transitive since $[G:H]=\infty$. □

In the amenable, non fg case, one obtains the following results (the proofs are not very difficult but I skip them because of time constraints)

Thm: (i) If G is amenable and is not locally finite, then a generic element of $\mathcal{S}(G)$ is not minimal.

(ii) If G is (infinite and) locally finite, then a generic element of is minimal and uniquely ergodic.
There are no nontrivial minimal sofic subshifts.
In particular there is no comeager conjugacy class in $\mathcal{S}(G)$ whenever G is amenable and not finitely generated.

(In both cases, projectively isolated subshifts cannot be dense, since they are minimal because G is not fg)

We conclude by a brief discussion of the case of $\mathcal{S}_{tr}(G)$, and why it is completely different from the cases of $\mathcal{S}(G)$ and $\mathcal{S}_{min}(G)$.

If one wishes to prove that $\mathcal{A}_{tr}(G)$ has a dense conjugacy class, the one naturally tries the following:

Pick U, V nonempty open in $\mathcal{A}_{tr}(G)$, $\xi_1 \in U$, $\xi_2 \in V$.

We want to find $\xi \in \mathcal{A}_{tr}(G)$ which factors onto both ξ_1 and ξ_2.

For this, consider the diagonal action $\xi : G \curvearrowright \Omega \times \Omega$, then choose x_1 a transitive point for ξ_1, x_2 a transitive point for ξ_2, and consider $Y = \overline{X_1 \times X_2}$.

The $G \curvearrowright Y$ is a transitive flow which factors onto both $G \overset{\xi_1}{\curvearrowright} \Omega$ and $G \overset{\xi_2}{\curvearrowright} \Omega$.

However: Y may have isolated points!

If Y is a factor of a transitive action $G \curvearrowright \Omega$, then we can still conclude. When does that happen?

Def: Let X be a G-flow.

For $x \in X$ and U open, denote $\mathrm{Ret}_U(x) = \{g : g \cdot x \in U\}$

Say that x is recurrent if $\mathrm{Ret}_U(x)$ is infinite for any open $U \ni x$.

Prop: Let $G \curvearrowright Z$ be a 0-dimensional, metrizable G-flow, and $z_0 \in Z$ be a transitive point. Then

(i) Z is a factor of a transitive flow on Ω

$\Updownarrow$

(ii) z_0 is recurrent

$\Updownarrow$

(iii) z_0 is not isolated, or z_0 has an infinite stabilizer.

So the problematic situation is when there is no recurrent point in $X_1 \times X_2$ which projects onto transitive points of X_1, X_2.

For instance, for $G = \mathbb{Z}^2$, consider the following subshifts on $\{0,1\}^G$: $= \langle a, b \rangle$

X_a is obtained by saying that a acts trivially, and that on each vertical line there is at most one 1.

It is not hard to see that X_a is sofic with an isolated transitive point, hence projectively isolated.

X_b is defined similarly, switching the roles of a and b.

(There is no recurrent point in $X_a \times X_b$ which projects to elements of X_a, X_b other than 0^∞.)

So we obtain open subsets of $\mathcal{A}_{tr}(\mathbb{Z}^2)$ (give by the SFTs which isolate X_a, X_b) whose conjugates do not intersect, since otherwise we would obtain a transitive action which factors onto $\mathbb{Z}^2 \cdot (x_a, x_b)$, where x_a, x_b are transitive points in X_a, X_b.

Hence these two "incompatible" transitive subshifts give us the following.

Prop: There is no dense conjugacy class in $\mathcal{A}_{tr}(\mathbb{Z}^2)$.

This contradicts a result of Hochman (which is based on the proof strategy outlined in the previous page).

This phenomenon appears to be fairly widespread.

Thm:
- There is no dense conjugacy class in $\mathcal{A}_{tr}(\mathbb{F}_p)$, $p \geq 2$
- If G is nilpotent, then there is a dense conjugacy class in $\mathcal{A}_{tr}(G)$ iff G is virtually cyclic.

The end

(Before Michal Doucha's talk...)