Prop. $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ free p.m.p. action. If E_α is amenable, then so is Γ.

Pf. Let λ^n be as in the definition. Define $f_n : \Gamma \to \mathbb{R}$ by
$$f_n(g) = \int_X \lambda^n(x, g\cdot x)\, dx$$

As the action is free, we have that $\|f_n\|_1 = 1$.

$$\cancel{\|f_n(g) - f_n(h^{-1}g)\|_1}\ \|f_n g - h\cdot f_n\|_1 = \sum_g |f_n(g) - f_n(h^{-1}g)|$$

$$\cancel{\le \sum_g \int_X |\lambda^n(x, g\cdot x) - \lambda^n(x, h^{-1}g\cdot x)|\, dx}$$

$$= \sum_g \left| \int \lambda^n(x, g\cdot x)\, dx - \int \lambda^n(x, h^{-1}g\cdot x)\, dx \right| \longrightarrow \text{measure-preserving}$$

$$\le \int \sum_g |\lambda^n(x, g\cdot x) - \lambda^n(h\cdot x, g\cdot x)|\, dx$$

$$= \int \|\lambda^n_x - \lambda^n_{h\cdot x}\|_1\, dx \xrightarrow[n\to\infty]{} 0 \quad \text{(dominated convergence).} \quad \blacksquare$$

Thm. (Dye). All ergodic, hyperfinite, measure-preserving equivalence relations on a standard non-atomic space are isomorphic.

~~Prop.~~ **Def.** E on (X,μ) is <u>ergodic</u> if every measurable E-invariant subset of X has measure 0 or 1.

Prop. E - ergodic, p.m.p. on (X,μ), $A, B \subseteq X$ $\mu(A) = \mu(B)$. Then there exists $\varphi : A \to B$ measurable bijection with graph $\varphi \subseteq E$.

Pf. Let $\Gamma \curvearrowright (X,\mu)$ generate E. Order $\Phi = \{\varphi : \varphi\text{-bijection, graph } \varphi \subseteq E, \text{dom } \varphi \subseteq A, \text{im } \varphi \subseteq B\}$ by a.e. inclusion

$\varphi \subseteq \psi$: ~~dom φ~~

Can apply Zorn. If $C \subseteq \Phi$ is a chain, let $\varphi_1 \subseteq \varphi_2 \subseteq \dots$ be elements of C with $\sup_n \mu(\mathrm{dom}\,\varphi_n) = \sup\{\mu(\mathrm{dom}\,\varphi) : \varphi \in C\}$.

Claim: $\bigcup_n \varphi_n$ is an upper bound for C.

Indeed, for any $\varphi \in C$, $\mathrm{dom}\,\varphi \setminus \mathrm{dom}\,\varphi_n$ is a decreasing sequence of sets with measures converging to 0, so $\mathrm{dom}\,\varphi \setminus \mathrm{dom} \bigcup \varphi_n$ is negligeable.

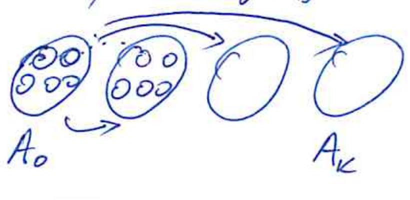
Let φ_0 be a max element of Φ. If $\mu(A \setminus \mathrm{dom}\,\varphi_0) > 0$, then $\mu(B \setminus \mathrm{im}\,\varphi_0) > 0$ and by ergodicity, $\exists \gamma \in \Gamma$ $\mu(\gamma(A \setminus \mathrm{dom}\,\varphi_0) \cap (B \setminus \mathrm{im}\,\varphi_0)) > 0$.

Now $\varphi_0 \cup \gamma|_{A \setminus \mathrm{dom}\,\varphi_0} \supsetneq \varphi_0$ contradicting the maximality of φ_0. ∎

E - equiv rel. on (X, μ). An array for E on X, $\mathcal{A} = \langle A_0, \dots, A_k; \varphi_0, \dots \varphi_k \rangle$ where $A_0, \dots A_k$ is a partition of X in sets of the same measure and $\varphi_i : A_0 \to A_i$, $\varphi_i \in [[E]]$, $\varphi_0 = id$.

$x \in X$: $\mathcal{O}_{\mathcal{A}}(x) = \{\varphi_j \varphi_i^{-1}(x) : j \le k\}$, where $x \in A_i$.

If $\mathcal{A}$ is an array on X and $\mathcal{B} = \langle B_0, \dots, B_n; \psi_0, \dots, \psi_n \rangle$ is an array on A_0, then $\mathcal{A}\mathcal{B}$ is the array on X defined by



$\langle B_{ij}; \psi_{ij} : i \le k, j \le n \rangle$

$B_{ij} = \varphi_i(B_j)$, $\psi_{ij} = \varphi_i \circ \psi_j$.

Lemma 1 E - ergodic. $\mathcal{A}$ - array on X. $C \subseteq X$, $\varepsilon > 0$. There exists an array $\mathcal{B}$ on A_0 s.t. C can be ε-approximated by a union ~~unions~~ of elements of $\mathcal{A}\mathcal{B}$.

Pf. (Let P be the partition ~~gg~~ of A_0 generated by $\{\varphi_i^{-1}(C) : i \le k\}$.)
Let $P^{\varepsilon/k}$ $\| \{B_0, \dots, B_n\}$ be a partition of A_0 in sets of the same measure s.t. every element of the algebra gen. by ~~P~~ $\{\varphi_i^{-1}(C) : i \le k\}$ can be ε/k approximated by elements of P. Let $\psi_j : B_0 \to B_j$, $\psi_j \in [[E]]$ be arbitrary (use the Prop. above). This defines the array $\mathcal{A}\mathcal{B}$.

Now each $\varphi_i^{-1}(C)$ can be ε/k-approx. by unions of elements U_i of $\mathcal{B}$ and C is ε-approximated by $\bigcup_{i\leq k}\varphi_i(U_i)$. $\boxtimes$

<u>Lemma 2</u> E-ergodic, A-arrg on X. ~~$F\in[E] := [[E]]\cap \mathrm{Aut}(X,\mu)$.~~ $F\subseteq E$ - finite equiv. relation $\varepsilon>0$. Then there exists an arrg $\mathcal{B}$ on A_0 s.t.

$$\mu(\{x\in X : [x]_F \not\subseteq \mathcal{O}_{A\mathcal{B}}(x)\}) < \varepsilon.$$

~~Pf. <u>Lemma 3</u> E-aperiodic, hyperfinite. Then there exists finite equiv. relations F_n with $\bigcup_n F_n = E$ and $\forall n$ all classes of F_n have the same size.~~

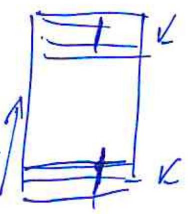
~~Pf. Let E be given by the action of $T\in \mathrm{Aut}(X,\mu)$. By Rokhlin, there exist T_n - periodic with all orbits of size 2^n s.t. $d_u(T_n,T)<2^{-n+1}$.~~
~~Let R_n be the orbit equiv. relation~~

<u>Lemma 3</u> E - ~~aperiodic~~ ergodic, hyperfinite. $R\subseteq E$ - relation s.t. $\forall x\in X$ R_x is finite. $\varepsilon>0$. Then there exists a ~~finite eq. rel. $F\subseteq E$ with all classes of the same size s.t.~~ periodic $T\in[E]$ with all orbits of the same size s.t. $\mu(\{x : R_x \not\subseteq \{T^n x : n\in\mathbb{Z}\}\})<\varepsilon$.

Pf. Let T_0 generate E. There exists a k s.t.

$$\mu(\{x : R_x \not\subseteq \{T_0^i\cdot x : i\in(-k,k)\}\}) < \varepsilon$$

Let N be s.t. $\frac{2k}{N}<\varepsilon$. Apply Rokhlin to T_0 with N,ε.



Define T by going up the tower cyclically and arbitrarily on the remainder. [Cut the remainder in N pieces of the same size and use the Prop.]. Then $\mu(\{x : \{T_0^i\cdot x : i\in(-k,k)\}\not\subseteq \mathcal{O}_T(x)\})$
$\leq \frac{2k}{N}+\varepsilon \quad = 2\varepsilon.$ $\boxtimes$

<u>Pf. of Lemma 2</u>: Define $\pi: X \to A_0$ by $\pi(x) = \varphi_i^{-1}(x)$ if $x \in A_i$.

Note that $E|A_0 = E \cap (A_0 \times A_0)$ is ergodic, hyperfinite.

ergodic: $\forall B_1, B_2 \subseteq A_0$ $\underset{\mu(B_1)=\mu(B_2)}{}$ there exists $\varphi \in [[E]]$ ~~$\mu(\varphi(B_1) \cap B_2)) > 0$~~. $\varphi: B_1 \to B_2$. positive

hyperfinite: $E = \bigcup^{\nearrow} F_n \rightsquigarrow E|A_0 = \bigcup^{\nearrow} F_n|A_0$.

Define $R \subseteq E|A_0$ by: $R_z = \pi([\pi^{-1}(z)]_F)$.

Apply <u>Lemma 3</u> to R and ε/k to find a periodic $T: A_0 \to A_0$ of order n with $\mu(\underbrace{\{z \in A_0 : R_z \not\subseteq \mathcal{O}_T(z)\}}_{C}) < \varepsilon/k$. Let B_0 be a fund. domain for T and let $B_i = T^i B_0$, $i < n$. $\psi_i : B_0 \to B_i$, $\psi_i = T^i|_{B_0}$.

We claim that $\mu(\{x \in X : [x]_F \not\subseteq \mathcal{O}_{AB}(x)\}) < \underline{\varepsilon}$

More precisely, $\{x \in X : [x]_F \not\subseteq \mathcal{O}_{AB}(x)\} \subseteq \pi^{-1}(C)$.

Indeed, let x be such that $[x]_F \not\subseteq \mathcal{O}_{AB}(x)$. ~~Let $z = \pi(x)$. Then~~

Let $y \in [x]_F \setminus \mathcal{O}_{AB}(x)$. Then $\pi(y) \in R_{\pi(x)}$ but $\pi(y) \notin \mathcal{O}_T(\pi(x))$, showing that $\pi(x) \in C$.

$\boxtimes$

<u>Pf. of Theorem</u> Back-and-forth. Let E_1 and E_2 be two ergodic, h.f. equiv. relations on X_1, X_2 respectively. We will define inductively an isomorphism $MALG(X_1) \to MALG(X_2)$ which sends E_1 to E_2.

~~At stage n we have an array~~ Let $(C_n^1), (C_n^2)$ enumerate dense subsets of $MALG(X_1), MALG(X_2)$, respectively. At stage n we will have two arrays A_n^1 on X_1, A_n^2 on X_2 of the same

size and a partial bijection $\theta: MALG(X_1) \to MALG(X_2)$ ... A_n^1 to A_n^2

We also require that C_n^{2} is 2^{-n}-approx. by a union of elements of A_{2n}^1 and C_n^2 by A^2_{2n+1}. This will ensure that $\Theta = \bigcup_n \Theta_n$ extends to an isomorphism. ~~Write~~

At even stages $2n$ go forth. Suppose $\Theta_{2n-1}^{A^1_{2n-1}, A^2_{2n-1}}$ is constructed.

First apply Lemma 1 to find an array B on $A^1_{2n-1,0}$ s.t. $A^1_{2n-1} B$ 2^{-n}-approximates C_n^1.

Write $E_1 = \bigcup^{\uparrow} F_n^1$, $E_2 = \bigcup^{\uparrow} F_n^2$, where F_n^1, F_n^2 are finite eq. rel.

We will also ensure that ~~th~~ $\mu(\{x \in X_1 : [x]_{F_n^1} \not\subseteq \mathcal{O}_{A^1_{2n}}(x)\}) < 2^{-n}$ and similarly for F_n^2 at odd steps.

Then apply Lemma 2 to refine $A^2_{2n-1} B$ to A^1_{2n} so that

$$\mu(\{x \in X_1 : [x]_{F_n^1} \not\subseteq \mathcal{O}_{A^1_{2n}}(x)\}) < 2^{-n}.$$

~~At odd stages go back~~ Define the corresponding array A^2_{2n} by refining A^2_{2n-1} arbitrarily (with the appropriate number of pieces).

At odd stages go back.

<u>Claim 1</u> $\forall^* x \in X_1$ $[x]_{E_1} = \bigcup_n \mathcal{O}_{A_n^1}(x)$ and similarly for E_2.

<u>Claim 2</u> Θ commutes with all φ_i:

$$\Theta(\varphi^1_{n,i}(x)) = \varphi^2_{n,i}(\Theta(x)) \quad \text{for ~~all~~ a.e. } x \in A^1_{n,0}.$$

In particular, $\Theta(\mathcal{O}_{A_n^1}(x)) = \mathcal{O}_{A_n^2}(\Theta(x))$.