Amenability and dynamics – Exercise Sheet 1

Exercises 3, 5 and 8 to turn in on 01/03/22

Invariant means

Exercice 1. Let G be an amenable group, and let m be an invariant mean on G. Given a subgroup H of G, we denote by $[G : H]$ the index of H in G. Prove that for every subgroup H, we have $m(H) > 0$ if and only if $[G : H] < +\infty$, and in this case $m(H) = \frac{1}{[G:H]}$.

Exercice 2. Let G be a group. A subgroup $H \leq G$ is called *coamenable* in G if the action of G on G/H is amenable. Consider a pair of subgroups $H \leq K \leq G$.

1. Show that if H is coamenable in G, then so is K. (*Note*: in contrast, this does not imply that H is co-amenable in K; see Exercise 11.)

2. Assume that H is coamenable in K and K is coamenable in G. Show that H is coamenable in G. Deduce that if G admits a subgroup H which is both coamenable and amenable, then G is amenable.

Exercice 3 (*To turn in on 01/03/22*)**.** Let G be an amenable group.

1. Prove that there exists a mean $m \in \mathcal{M}(G)$ which is *right-invariant*, that is $m(Ag) = m(A)$ for every $A \subset G$ and $g \in G$.

2. Prove that there exists a mean $m \in \mathcal{M}(G)$ which is *bi-invariant*, that is $m(gAh) = m(A)$ for every $g, h \in G$ and $A \subset G$.

Equidecomposability and the Banach–Tarski paradox

The goal of this exercise is to construct a free subgroup of $\mathrm{SO}(3, \mathbb{R})$. This subgroup was used in the lectures to prove the Banach-Tarski paradox.

Exercice 4. Let $R, S \in \mathrm{SO}(3, \mathbb{R})$ be the rotations given by:

$$R = \begin{pmatrix} 1/3 & -2\sqrt{2}/3 & 0 \\ 2\sqrt{2}/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & -2\sqrt{2}/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{pmatrix}$$

and let $\phi \colon \mathbb{F}_2 \to \mathrm{SO}(3, \mathbb{R})$ be the homomorphism defined by $\phi(a) = R, \phi(b) = S$, where a and b are the generators of the free group $\mathbb{F}_2$. Let e_1, e_2, e_3 denote the standard basis vectors of $\mathbb{R}^3$.

1. Show that if $w \in \mathbb{F}_2$ is a reduced word whose last letter (on the right) is a or a^{-1}, then

$$\phi(w) \cdot e_1 = \frac{1}{3^k}(xe_1 + y\sqrt{2}e_2 + ze_3)$$

with $x, y, z \in \mathbb{Z}$, $k \in \mathbb{N}_{>0}$, and y not divisible by 3. In particular, $\phi(w) \neq 1$ for all such w. (*Hint:* Argue by induction on the length of w.)

2. Conclude that the homomorphism ϕ is injective.

Exercice 5 (*To turn in on 01/03/22*)**.** Show that the two intervals $[0, 1]$ and $[0, 1)$ are equidecomposable subsets of $\mathbb{R}$ with respect to the group of translations.

Wreath products

The next exercises require the following notions.

Definition (Wreath product)**.** Let L, B be two groups, with identity elements $1_L, 1_B$. Denote by $\oplus_B L$ the group of all functions $f \colon B \to L$ of finite support, that is such that $f(b) = 1_L$ for all but finitely many $b \in B$. The group operation on $\oplus_B L$ is given by pointwise product (within the group L), namely if $f_1, f_2 \in \oplus_B L$ then $f_1 f_2(b) = f_1(b) f_2(b)$. The group B acts on $\oplus_B L$ by automorphisms by shifting functions, via the formula $b \cdot f(c) = f(b^{-1}c)$. The *wreath product* of L and B, denoted $L \wr B$, is defined as the semi-direct product

$$L \wr B = (\oplus_B L) \rtimes B$$

taken with respect to this action. Explicitly, an element of $L \wr B$ is a pair (f, b) with $f \in \oplus_B L$ and $b \in B$, and the group operation on $L \wr B$ is given by

$$(f_1, b_1)(f_2, b_2) = (f_1(b_1 \cdot f_2), b_1 b_2).$$

Definition (Lamplighter group)**.** Denote $C_2 = \{0, 1\}$ the cyclic group of order 2. The group $G = C_2 \wr \mathbb{Z}$ is called the *lamplighter group*.

Comment. The terminology *lamplighter* comes from a suggestive interpretation which helps thinking about that group. Imagine that $\mathbb{Z}$ is an infinite street, and at each point of it there is a lamp, which can be on or off. A city worker (the *lamplighter*) moves around the street and can switch lamps on and off. Initially all lamps are off and the lamplighter is at position 0. At each step, assuming the lamplighter is at position n, she can chose between the following two moves:

(M) *Move* from position n to a neighbouring position $n \pm 1$

(S) *Switch* the status (on/off) of the lamp at position n

An element $g = (f, n) \in C_2 \wr \mathbb{Z}$ can be interpreted as the result of finitely many moves of such type: the number n is the current position of the lamplighter, and the configuration $f \colon \mathbb{Z} \to C_2$ indicates which lamps are on.

Exercice 6 (Finite generation of wreath products)**.**

1. Let first $G = C_2 \wr \mathbb{Z}$ be the lamplighter group. Let $h \in G$ be the element $h = (f, 0)$, where $f : \mathbb{Z} \to C_2$ is given by $f(0) = 1$ and $f(n) = 0$ for $n \neq 0$. Let $t = (0, 1)$ (the first cordinate of t denotes the constant function $\mathbb{Z} \to C_2$ whose value is equal to 0). Show that h and t generate G. (*Hint*: given $g = (f, n) \in G$, observe that replacing g by $gt^{\pm 1}$ corresponds to a move (M), while replacing g by gh corresponds to (S)).

2. Generalise to show that if L and B are finitely generated groups, then $L \wr B$ is finitely generated.

Exercice 7. Prove that $L \wr B$ is amenable (respectively elementary amenable, respectively solvable) if and only if L and B are both amenable (respectively elementary amenable, respectively solvable).

In the following exercise, we say that a sequence $(F_n)_n$ of finite subsets of a group G is a *right* Følner sequence if for every $g \in G$ we have $\frac{|F_n g \triangle F_n|}{|F_n|} \to 0$ as $n \to \infty$. Note that $(F_n)_n$ is a right Følner sequence if and only if $(F_n^{-1})_n$ is a left Følner sequence.

Exercice 8 (*To turn in on 01/03/22*)**.** Let L and B be countable amenable groups, and set $G = L \wr B$. Let $(F_n)_n$ and $(T_n)_n$ be right Følner sequences for L and B respectively. For $n, m \in \mathbb{N}$, set

$$K_{m,n} = \{(f, b) \in L \wr B : b \in T_n, f(T_n) \subset F_m, f(x) = 1_L \text{ if } x \notin T_n\}.$$

Prove that there exists an increasing sequence $(m_n)_n$ such that $(K_{m_n,n})_n$ is a right Følner sequence for G.

Exercice 9. Let $G = C_2 \wr \mathbb{Z}$, the lamplighter group. Consider the set $E \subset G$ given by

$$E = \{(f, n) : n \geq 0 \text{ and } f(x) = 0 \text{ if } x < 0 \text{ or } x > n\}.$$

Let $h, t \in G$ be the elements as in Exercise 6(1). Show that $ht(E), t(E)$ are disjoint subsets of E such that $E = ht(E) \sqcup t(E)$. Does this contradict the existence of an invariant mean on G?

Complementary exercises

Exercice 10 (Amenability and growth)**.** Let G be a finitely generated group, and S a finite *symmetric* (i.e., such that $S = S^{-1}$) generating set of G. The *word length* $\ell_S(g)$ of $g \in G$ (with respect to S) is minimal n such that g can be written as $g = s_1 \cdots s_n$, with $s_i \in S$. For $n \in \mathbb{N}$. Denote by $B_S(n) = \{g \in G : \ell_S(g) \leq n\}$. The function

$$b_G(n) = |B_S(n)|$$

is called the *growth* of G (with respect to S). We will say that G has *exponential growth* if there exists a constant $\lambda > 1$ such that $b_{G,S}(n) \geq \lambda^n$ for every n large enough. Otherwise, we say that G has *subexponential growth.*

1. Let T be another finite symmetric generating set of G. Prove that there exists a constants $C > 0$ such that $C^{-1}\ell_S(g) \leq \ell_T(G) \leq C\ell_S(g)$ for every $g \in G$ and $b_T(C^{-1}n) \leq b_S(n) \leq b_T(Cn)$ for every n. Deduce that the property whether G has exponential growth or not does not depend on the choice of S.

2. Check that the free group $\mathbb{F}_2$ on 2 generators has exponential growth.

3. Prove that the sequence $b_S(n)^{\frac{1}{n}}$ admits a limit $1 \leq v < +\infty$, and that $v = 1$ if and only if G has subexponential growth.

4. Assume that G has subexponential growth. Show that there exists an increasing sequence (k_n) of integers such that $(B_S(k_n))_n$ is a Følner sequence for G. Deduce that groups of subexponential growth are amenable.

5. Show that the lamplighter group $G = C_2 \wr \mathbb{Z}$ has exponential growth. Deduce that there exist amenable groups of exponential growth.

Exercice 11 (Coamenability does not pass to subgroups)**.** The goal of this exercise is to give a construction, due to N. Monod and S. Popa, of a group G together with two subgroups $H < K < G$ such that H is coamenable in G but not in K (compare with Exercise 2).

Let L be an arbitrary group, and set $G = L \wr \mathbb{Z}$. Let $H = \{(f, 0) \colon f(m) = 1_L \text{ for } m > 0\}$ and $K = \{(f, 0) \colon f(m) = 1_L \text{ for } m > 1\}$.

1. Observe that $H \leq K \leq G$.

2. For $n \in \mathbb{N}_+$, let $F_n \subset G/H$ be the collection of cosets $F_n = \{(1_L, j)H \colon n \leq j \leq 2n\}$. Prove that (F_n) is a Følner sequence for the action of G on G/H. Deduce that H is coamenable in H (see Exercise 2).

3. Prove that H is coamenable in K if and only if the group L is amenable. Deduce that there exists a group G with subgroups $H < K < G$ such that H is coamenable in G but not in K

Exercice 12 (An elementary amenable, non virtually solvable group)**.** Denote $\mathrm{Sym}(\mathbb{Z})$ the group of all permutations of $\mathbb{Z}$, and $\mathrm{Sym}_f(\mathbb{Z})$ its subgroup of *finitely supported* permutations, that is those permutations σ such that the set $\{n \colon \sigma(n) \neq n\}$ is finite. Let $G \subset \mathrm{Sym}(\mathbb{Z})$ be the set of all permutations $g \in \mathrm{Sym}(\mathbb{Z})$ with the property that there exists $N > 0$ and $\alpha \in \mathbb{Z}$ such that $g(n) = n + \alpha$ for every $n \notin [-N, N]$.

1. Check that G is a subgroup of $\mathrm{Sym}(\mathbb{Z})$.

2. Prove that there exists a surjective homomorphism $\tau \colon G \to \mathbb{Z}$ such that $\ker \tau = \mathrm{Sym}_f(\mathbb{Z})$.

3. Prove that G is elementary amenable.

4. Prove that G is not virtually solvable, that is, it does not admit a solvable subgroup of finite index. (*Hint*: prove first that the subgroup of G consisting of finitely supported alternating permutations is not virtually solvable).

5. Prove that G is finitely generated.