Amenability and dynamics – Exercise Sheet 3

Exercice 1. Let (X, μ) and (Y, ν) be two standard probability spaces and denote by $\mathrm{MALG}(X)$ and $\mathrm{MALG}(Y)$ the corresponding measure algebras. Let $\phi \colon \mathrm{MALG}(X) \to \mathrm{MALG}(Y)$ be an isometric isomorphism (i.e., ϕ preserves the Boolean operations and the measure). Show that there exists a measure-preserving isomorphism $f \colon X \to Y$ such that for every $A \subseteq X$, $\phi([A]) = [f(A)]$. Here $[A] \in \mathrm{MALG}(X)$ is the equivalence class of the set A under the eqiuvalence relation $\sim$ given by

$$A \sim B \iff \mu(A \triangle B) = 0.$$

Exercice 2. A measure-preserving action $\Gamma \curvearrowright (X, \mu)$ is said to have *almost invariant sets* if there exists a sequence $A_1, A_2, \ldots \subseteq X$ with $\mu(A_n) = 1/2$ for all n such that for all $\gamma \in \Gamma$, $\mu(\gamma A_n \triangle A_n) \to 0$. Prove that if Γ is amenable, then all of its free actions admit almost invariant sets.

Exercice 3. Let E be a countable equivalence relation on the standard probability space (X, μ). Define the σ-finite measures M_l and M_r on E by

$$M_l(A) = \int_X |A_x| \, \mathrm{d}\mu(x)$$
$$M_r(A) = \int_X |A^x| \, \mathrm{d}\mu(x),$$

where $A \subseteq E$. Here $A_x = \{y : (x, y) \in A\}$ and $A^x = \{(y : (y, x) \in A\}$. Show that E preserves μ iff $M_l = M_r$.

A measure-preserving equivalence relation E on the standard probability space (X, μ) is called *amenable* if there exists a sequence of Borel functions $\lambda^n \colon E \to \mathbb{R}$, $n \in \mathbb{N}$ such that

- $\lambda^n \geq 0$;
- For a.e. $x \in X$
$$\|\lambda^n_x\|_1 \coloneqq \sum_{y \, E \, x} |\lambda^n(x, y)| = 1;$$
- For M_l-almost all $(x, y) \in E$, we have that $\lim_{n \to \infty} \|\lambda^n_x - \lambda^n_y\|_1 = 0$.

Compare this with the Reiter condition for groups.

Exercice 4.
1. Show that every hyperfinite equivalence relation is amenable.
2. Show directly that if $\Gamma \curvearrowright^{\alpha} (X, \mu)$ is a measure-preserving action and Γ is an amenable group, then the orbit equivalence relation E_α is amenable.

The Connes–Feldman–Weiss theorem states that every amenable equivalence relation is hyperfinite a.e.

Exercice 5. Let $\Gamma \curvearrowright^{\alpha} (X, \mu)$ be a *free*, measure-preserving action. Show that if the orbit equivalence relation is amenable, then Γ is amenable. In particular, the orbit equivalence relation of the Bernoulli shift $\mathbb{F}_2 \curvearrowright 2^{\mathbb{F}_2}$ is not amenable.