Amenability and dynamics – Exercise Sheet 2

Exercises 4, 9 and 10 to turn in on 01/03/22

Variants of Følner's condition

Given a group G and subsets S, T, we define the S-boundary of T as

$$\partial_S T = \{g \in G \colon Sg \cap T \neq \varnothing \text{ and } Sg \cap T^c \neq \varnothing\}.$$

We also define the inner and outer S-boundaries as

$$\partial_S^{\mathsf{in}} T = \{g \in T \colon Sg \cap T^c \neq \varnothing\}, \qquad \partial_S^{\mathsf{out}} T = \{g \in T^c \colon Sg \cap T \neq \varnothing\}.$$

Exercice 1. Prove that a group G satisfies Følner's condition if and only if for every $S \subset G$ finite and $\varepsilon > 0$, there exists $T \subset G$ finite such that $|\partial_S T| \leq \varepsilon |T|$. Prove the same equivalence if $\partial_S T$ is replaced by ∂_S^{in} or $\partial_S^{\mathsf{out}} T$

Amenability as a fixed point property

Exercice 2. Using the characterisation of amenability as a fixed point property for affine actions on compact convex set, give another proof that an extension of amenable groups is amenable.

Exercice 3. Prove that an action on a set $G \curvearrowright X$ is amenable if and only if the following holds: for every affine action of $G \curvearrowright C$ on a compact convex set such that there exists a G-equivariant map $\varphi \colon X \to C$ (that is, a map satisfying $\varphi(gx) = g\varphi(x)$ for every $x \in X$), G fixes a point in C.

Exercice 4 (Turn in on 01/03/22)**.** Let G be a group and H be a subgroup of G. Suppose that for every action $G \curvearrowright X$ by homeomorphism on a compact space there exists a probability measure on X which is fixed by H. Prove that H is amenable.

Exercice 5. Let G be a group.

1. Let N_1, N_2 be normal amenable subgroups of a group G. Prove that $N_1 N_2$ is a normal amenable subgroup of G.

2. Let $A(G)$ be the union of all normal amenable subgroups of G. Prove that $A(G)$ is a normal amenable subgroup of G, and that $G/A(G)$ does not have any normal amenable subgroups.

Definition 1. Given a group G, the normal subgroup $A(G)$ defined in the previous exercise is called the *amenable radical* of G.

Definition 2. Let $G \curvearrowright C$ be an affine action on a non-empty compact convex set. We say that such an action is *irreducible* if the only G-invariant compact convex subsets of C are C itself and $\varnothing$.

By the kernel of an action $G \curvearrowright X$ we mean the set of elements of G that act trivially on X.

Exercice 6.

1. Prove that for every affine action $G \curvearrowright C$ on a non-empty compact convex set, there is a non-empty compact convex invariant subset $C' \subset C$ such that $G \curvearrowright C'$ is irreducible.
2. Prove that if $G \curvearrowright C$ is irreducible, then the amenable radical $A(G)$ is contained in the kernel of the action.
3. Assume that C is a non-empty, G-invariant compact convex subset of the convex set $\mathcal{M}(G)$ of means on G. Prove that the kernel of $G \curvearrowright C$ is equal to $A(G)$.

Around Kesten's criterion

In this part G is a countable group and μ be a symmetric probability measure on G whose support generates G. Recall that the random walk on G associated to μ is the sequence of random variables $Z_n = X_1 \cdots X_n$, where $(X_n)_{n\in\mathbb{N}}$ is a sequence of independent, identically distributed random variables taking values in G with distribution μ. Recall also that the distribution of Z_n coincides with the nth-iterated convolution μ^{*n}. That is, for every subset $A \subset G$, we have $\mathbb{P}(Z_n \in A) = \sum_{g\in A} \mu^{*n}(g)$.

Exercice 7. Let G be a group and μ be a symmetric probability measure on G.

1. Prove that $\mu^{*2n}(1_G) \geq \mu^{*2n}(g)$ for every $g \in G$ (in other words, the probability $\mathbb{P}(Z_{2n} = g)$ is maximized by $g = 1_G$).
2. We say that a sequence of finite subset $A_n \subset G$ *grows subexponentially* if $\lim_{n\to\infty} |A_n|^{\frac{1}{n}} = 1$. Assume that there exists a sequence of finite subsets $A_n \subset G$ which grows subexponentially and such that $\limsup_{n\to\infty} \mathbb{P}(Z_n \in A_n) > 0$. Prove that G is amenable.

Exercice 8 (Speed of random walk and amenability)**.** Assume that G is finitely generated, and fix a finite, symmetric generating set S of G. Recall that the word norm of $g \in G$ with respect to S is defined as

$$|g|_S = \min\{n \colon g = s_1 \cdots s_n, s_i \in S\}.$$

We assume that the measure μ is so that the the expectation $\mathbb{E}|Z_1|_S = \sum_{g\in G} \mu(g)|g|_S$ is finite (we say that μ has *finite first moment*). Let

$$L_\mu(n) = \mathbb{E}|Z_n|_S.$$

1. Prove that $L_\mu(n + m) \leq L_\mu(n) + L_\mu(m)$ for every $n, m \in \mathbb{N}$. Deduce that the limit $\ell_\mu := \lim_{n\to\infty} L_\mu(n)/n$ exists.

2. Assume that $\ell_\mu = 0$. Prove that G is amenable (*Hint*: apply the previous exercise to the sets $A_n = \{g \in G \colon |g|_S \leq 2L_\mu(n)\}$.)

Note: the quantity ℓ_μ is called the *rate of escape* (or *asymptotic speed*) of the random walk associated to μ. The converse to point 2 above is not true: there exists finitely generated amenable groups such that $\ell_\mu > 0$ for any measure μ with finite first moment.

Tilings

Let G be a group. We use the following terminology.

- We say that a finite subset $T \subset G$ *tiles* G if there exists a set $C \subset G$ such that $G = TC$ and $Tc_1 \cap Tc_2 = \varnothing$ for $c_1 c_2 \in C$ distinct.
- Given a p.m.p action $G \curvearrowright (X, \mu)$ on a probability space, a *tower* for the action is a pair (T, A) where $T \subset G$ is a finite set and $A \subset X$ is a measurable subset such that $t_1 A \cap t_2 A = \varnothing$ if $t_1, t_2 \in T$ are distinct. The set A is called the *base* of the tower.
- We say that *Rohlin lemma holds* for a finite subset $T \subset G$ if for every free p.m.p action $G \curvearrowright (X, \mu)$ on a standard probability space and every $\varepsilon > 0$, there exists a tower (T, A) such that $\mu(\bigcup_{t \in T} t(A)) \geq 1 - \varepsilon$.

Exercice 9 (Turn in on 01/03/22)**.** Prove that T tiles G if and only if for every finite subset $F \subset G$, there exists $c_1, \cdots, c_n \in G$ such that $F \subset \bigcup_{i=1,\ldots,n} Tc_i$ and $Tc_i \cap Tc_j = \varnothing$ for $i \neq j$.

Exercice 10 (To turn in on 01/03/22)**.** The goal of this exercise is to show that if Rohlin lemma holds for a finite subset T, then T tiles G.

1. Assume that $G \curvearrowright (X, \mu)$ is a p.m.p action, and (T, A) is a tower, and set $B = \cup_{t \in T} t(A)$. For $x \in X$ let $C_x = \{g \in G \colon gx \in A\}$, and $D_x = \{g \in G \colon gx \in B\}$. Prove that $Tc_1 \cap Tc_2 = \varnothing$ for distinct $c_1, c_2 \in C_x$.
2. Retain the same setting as in the previous question. Let $F \subset G$ be a finite subset. Prove that $\mu(\{x \in X \colon F \not\subset D_x\}) \leq |F|(1 - \mu(B))$.
3. Conclude.

Complement: standard Borel spaces

The following facts were left as exercise in the proof that any uncountable standard Borel spaces are isomorphic.

We denote by $\mathcal{C} = \{0, 1\}^{\mathbb{N}}$ the Cantor space.

Exercice 11. Prove that $\mathcal{C}$ is homeomorphic to $\mathcal{C}^n$.

Exercice 12. Let X be a Polish space without isolated points. Prove that X contains a closed subset homeomorphic to the Cantor space. (*Hint* for every possible choice of $x_1 \cdot x_n$ with $x_i \in \{0, 1\}$ find a ball $B_{x_1 \cdots x_n}$ in X of radius at most $1/n$ in such a way that $B_{x_1 \cdots x_n x_{n+1}} \subset B_{x_1 \cdot x_n}$ for every $x_1, \cdots, x_{n+1}$.)

Exercice 13. Prove that $\mathcal{C}$ is Borel isomorphic to the interval $[0, 1]$.