METRIC FRAÏSSÉ LIMITS VIA JOININGS

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The goal of this note is to provide a new proof of the existence and uniqueness of metric Fraïssé limits. The original theorem is due to Ben Yaacov [BY] and while the underlying ideas of the proof are similar, our approach uses a different formalism and relies on joinings (inspired from ergodic theory) and the Baire category theorem.

We quickly recall the definitions. Let $\mathcal{L}$ be a metric language and let $\mathcal{K}$ be a class of finitely generated $\mathcal{L}$-structures. We will suppose that $\mathcal{K}$ is *hereditary* (i.e., closed under substructures) and *directed* (i.e., every two structures in $\mathcal{K}$ embed into a third). If M is a structure and $\bar{a}$ is a tuple from M, $\langle \bar{a} \rangle$ denotes the (closed) substructure generated by $\bar{a}$. We denote by $\mathrm{S}_n(\mathcal{K})$ the space of quantifier-free n-types in $\mathcal{K}$, that is

$$\mathrm{S}_n(\mathcal{K}) = \{\operatorname{tp} \bar{a} : \bar{a} \in A^n, A \in \mathcal{K}\}.$$

Here and below, tp always means quantifier-free type. $\operatorname{tp} \bar{a}$ is nothing but the isomorphism type of $\langle \bar{a} \rangle$. We will also use the notation $\mathrm{S}_I(\mathcal{K})$ instead of $\mathrm{S}_n(\mathcal{K})$ if I is a set of variables of size n.

Define the function $\partial \colon \mathrm{S}_n(\mathcal{K}) \times \mathrm{S}_n(\mathcal{K}) \to \mathbf{R}^+$ by

$$\partial(p, q) = \inf\{d^C(\bar{a}, \bar{b}) : \bar{a}, \bar{b} \in C^n, C \in \mathcal{K}, \bar{a} \models p, \bar{b} \models q\}.$$

Note that $\partial(p, q) < \infty$ because $\mathcal{K}$ is directed. Note also that $\partial(p, q) = 0$ implies that $p = q$. This follows from the fact that a quantifier-free $\mathcal{L}$-formula has the same modulus of continuity in all elements of $\mathcal{K}$ (this is part of the definition of a metric language). If $I \subseteq J$ are sets of variables, we denote by $p \mapsto p|_I$ the natural projection $\mathrm{S}_J(\mathcal{K}) \to \mathrm{S}_I(\mathcal{K})$. Note that this map is surjective.

In order for ∂ to be a metric, we need an additional condition. $\mathcal{K}$ satisfies the *near amalgamation property (NAP)* if for every I_1, J_1, I_2, J_2 finite, $p \in \mathrm{S}_{I_1 \cup J_1}$, $q \in \mathrm{S}_{I_2 \cup J_2}$, and $\epsilon > 0$, if $p|_{I_1} = q|_{I_2}$, then there exists $r \in \mathrm{S}_{I_1 \cup J_1 \cup I_2 \cup J_2}(\mathcal{K})$ with $r|_{I_1 \cup J_1} = p$, $r|_{I_2 \cup J_2} = q$, and $d^r(I_1, I_2) < \epsilon$. (Here and below d^r denotes the metric as evaluated in the type r. When we evaluate the metric on tuples, the sup metric is assumed.)

If M is an $\mathcal{L}$-structure, we denote by $\operatorname{Age}(M)$ the class of finitely generated substructures of M. The structure M is called *ultrahomogeneous* if for all n and all $\bar{a}, \bar{b} \in M^n$,

$$\operatorname{tp} \bar{a} = \operatorname{tp} \bar{b} \implies \bar{b} \in \overline{\operatorname{Aut}(M) \cdot \bar{a}}.$$

The following is the main theorem, generalizing well-known results of Fraïssé in the classical setting.

Theorem 1 (Ben Yaacov)**.** *Let $\mathcal{L}$ be a metric language and $\mathcal{K}$ be a class of finitely generated $\mathcal{L}$-structures. Then the following are equivalent:*

(i) *$\mathcal{K}$ is hereditary, directed, satisfies NAP, and for every n, $(\mathrm{S}_n(\mathcal{K}), \partial)$ is a complete, separable metric space.*
(ii) *There exists a unique separable ultrahomogeneous structure M with* $\operatorname{Age}(M) =$ *$\mathcal{K}$.*

Suppose that $\mathcal{K}$ is a class that satisfies condition (i) of Theorem 1. We define $S_\omega(\mathcal{K})$ as $\varprojlim S_n(\mathcal{K})$ and equip it with the complete metric

$$\partial(p,q) = \sum_{n=0}^{\infty} 2^{-n} \min\big(\partial(p|_n, q|_n), 1\big).$$

Note that any $p \in S_\omega(\mathcal{K})$ is realized in some structure with age contained in $\mathcal{K}$; this is basically because the class of structures with age contained in $\mathcal{K}$ is closed under direct limits. We denote by M_p the isomorphism type of the (closed) structure generated by any realization of p.

Definition 2. Let $X \cap Y = \emptyset$ be sets of variables, $p \in S_X(\mathcal{K}), q \in S_Y(\mathcal{K})$. A *joining of p and q* is an element $r \in S_{X\cup Y}$ such that $r|_X = p$ and $r|_Y = q$. We will denote the set of all joinings of p and q by $J(p,q)$.

Note that $J(p,q)$ is a closed subset of $S_{X\cup Y}$ and thus a Polish space (if X and Y are countable). The fact that $\mathcal{K}$ is directed implies that for all X, Y finite, $p \in S_X(\mathcal{K}), q \in S_Y(\mathcal{K})$, $J(p,q)$ is non-empty. Proposition 3 will imply this for countable X, Y.

The following proposition is the main fact about extension of types that we will need. It follows easily from an iterated application of NAP.

Proposition 3. *Let $I \subseteq X$, $J \subseteq Y$ with $|I| = |J|$ finite and X, Y countable, $X \cap Y = \emptyset$, $\epsilon > 0$. Let $p \in S_X(\mathcal{K})$, $q \in S_Y(\mathcal{K})$ with $\partial(p|_I, q|_J) < \epsilon$. Then there exists $r \in J(p,q)$ with $d^r(I,J) < \epsilon$.*

Proof. Let $X = \bigcup_n I_n$, $Y = \bigcup J_n$ with $I_0 = I$, $J_0 = J$, $I_0 \subseteq I_1 \subseteq \cdots$, $J_0 \subseteq J_1 \subseteq \cdots$ finite. Fix $\epsilon' < \epsilon$ such that $\partial(p|_I, q|_J) < \epsilon'$. We construct by induction $r_n \in J(p|_{I_n}, q|_{J_n})$ such that for all n:

- $\partial(r_{n+1}|_{I_n \cup J_n}, r_n) < 2^{-n}$; and
- $d^{r_n}(I,J) < \epsilon'$.

r_0 exists by definition because $\partial(p|_I, q|_J) < \epsilon'$. Suppose that r_n is already constructed. We amalgamate r_n and $p|_{I_{n+1}}$ over $r_n|_{I_n} = p|_{I_n}$ to obtain $s \in S_{I_{n+1}\cup I_n' \cup J_n}$ that satisfies $s|_{I_{n+1}} = p|_{I_{n+1}}$, $s|_{I_n' \cup J_n} = r_n$, and $d^s(I_n, I_n') < \min\big(2^{-(n+1)}, \epsilon' - d^{r_n}(I,J)\big)$. Let $r_n' \in J(p|_{I_{n+1}}, q|_{J_n})$ be defined by $r_n' = s|_{I_{n+1}\cup J_n}$. Then r_n' satisfies $d^{r_n'}(I,J) < \epsilon'$ and $\partial(r_n'|_{I_n\cup J_n}, r_n) < 2^{-(n+1)}$.

Similarly, by amalgamating r_n' and $q|_{J_{n+1}}$ over $r_n'|_{J_n} = q|_{J_n}$, find r_{n+1} a joining of $p|_{I_{n+1}}$ and $q|_{J_{n+1}}$ satisfying $d^{r_{n+1}}(I,J) < \epsilon'$ and $\partial(r_{n+1}|_{I_{n+1}\cup J_n}, r_n') < 2^{-(n+1)}$. This together with the fact that projections are ∂-contractions implies the two required properties for r_{n+1}.

Once the construction is finished, extend r_n to $\hat{r}_n \in S_{X\cup Y}(\mathcal{K})$ arbitrarily, so that $\hat{r}_n|_{I_n \cup J_n} = r_n$. Observe that the sequence $(\hat{r}_n)_n$ is Cauchy, so it converges to some $r \in J(p,q)$. By continuity, we have that $d^r(I,J) \leq \epsilon' < \epsilon$. □

Definition 4. Let X be a countably infinite set of variables. An element $p \in S_X(\mathcal{K})$ is called *$\mathcal{K}$-existentially closed* (or *$\mathcal{K}$-e.c.* for short) if for every finite $I \subseteq X$, finite J with $J \cap I = \emptyset$, $\epsilon > 0$ and $q \in S_{I\cup J}(\mathcal{K})$ with $\partial(p|_I, q|_I) < \epsilon$, there exists $J' \subseteq X$ with $|J'| = |J|$ such that $\partial(p|_{I\cup J'}, q) < \epsilon$.

Proposition 5. *Let X be a countably infinite set of variables. Then*

$$\{p \in S_X(\mathcal{K}) : p \text{ is } \mathcal{K}\text{-e.c.}\}$$

is dense G_δ in $S_X(\mathcal{K})$.

Proof. For every finite J, choose a countable, dense subset $\mathrm{S}'_J(\mathcal{K}) \subseteq \mathrm{S}_J(\mathcal{K})$. Then p is $\mathcal{K}$-e.c. iff

(1) $$\forall I \subseteq X \text{ finite } \forall J \text{ finite } \forall \epsilon > 0\ \forall q \in \mathrm{S}'_{I\cup J}(\mathcal{K})$$
$$\partial(p|_I, q|_I) \geq \epsilon \text{ or } \exists J' \subseteq X\ \partial(p|_{I\cup J'}, q) < \epsilon.$$

As for fixed I, J, ϵ, q, the set of p defined on the second line of (1) is G_δ, by the Baire category theorem, it suffices to check that it is dense. A basic open set U in $\mathrm{S}_X(\mathcal{K})$ is given by
$$U = \{p \in \mathrm{S}_X(\mathcal{K}) : \partial(p|_L, p_0|_L) < \delta\},$$
where $L \subseteq X$ is finite, $\delta > 0$ and $p_0 \in \mathrm{S}_X(\mathcal{K})$. We may assume that $I \subseteq L$ and $\delta < \epsilon$. We may also assume that $\partial(p_0|_I, q|_I) < \epsilon$ in order to prove that there exist $J' \subseteq X$ and $p \in U$ with $\partial(p|_{I\cup J'}, q) < \epsilon$. Let $J' \subseteq X$ be arbitrary with $|J'| = |J|$ and $J' \cap L = \varnothing$. As $\partial(p_0|_I, q|_I) < \epsilon$, by Proposition 3, there exists $r \in \mathrm{S}_{L\cup I'\cup J'}$ such that $r|_L = p_0|_L$, $r|_{I'\cup J'} = q$, and $d^r(I, I') < \epsilon$. Finally, take $p \in \mathrm{S}_X(\mathcal{K})$ to be any extension to X of $r|_{L\cup J'}$. We will have that $p|_L = p_0|_L$ (so $p \in U$) and $\partial(p|_{I\cup J'}, q) \leq d^r(IJ', I'J') < \epsilon$. □

Proposition 6. *Suppose that $p \in \mathrm{S}_X(\mathcal{K})$ is $\mathcal{K}$-e.c., $Y \cap X = \varnothing$, $|Y| \leq \aleph_0$, $q \in \mathrm{S}_Y(\mathcal{K})$. Then for comeagerly many $r \in J(p,q)$, $\langle r|_Y\rangle \subseteq \langle r|_X\rangle$.*

Proof. By uniform continuity of terms, we have that $\langle r|_Y\rangle \subseteq \langle r|_X\rangle$ iff
$$\forall i \in Y\ \forall \epsilon > 0\ \exists i' \in X \quad d^r(i, i') < \epsilon.$$
Fix $i \in Y$ and $\epsilon > 0$ in order to show that $V_{i,\epsilon} = \{r \in J(p,q) : \exists i' \in X\ d^r(i,i') < \epsilon\}$ is dense in $J(p,q)$ (it is clearly open). Let
$$U = \{r \in J(p,q) : \partial(r|_{I\cup J}, r_0|_{I\cup J}) < \delta\},$$
where $r_0 \in J(p,q)$, $I \subseteq Y$ and $J \subseteq X$ are finite, and $\delta > 0$ be an open set. We will construct $r \in V_{i,\epsilon} \cap U$. We may assume that $i \in I$ and $\delta < \epsilon$. As p is $\mathcal{K}$-e.c., there exists $I' \subseteq X$ such that $\partial(p|_{J\cup I'}, r_0|_{J\cup I}) < \delta$. By Proposition 3, there exists $r_1 \in \mathrm{S}_{X_1\cup X_2\cup Y_2}$ such that $r_1|_{X_1} = p$, $r_1|_{X_2\cup Y_2} = r_0$ and $d^{r_1}(J_1 I_1', J_2 I_2) < \delta$. (Here we consider X_1 and X_2 as copies of X, Y_2 as a copy of Y and I_1, I_1', J_1, I_2, J_2 are the corresponding copies of I, I', J.) Take $r = r_1|_{X_1\cup Y_2} \in J(p,q)$. Then $d^r(I, I') = d^{r_1}(I_2, I_1') < \delta < \epsilon$ and
$$\partial(r|_{I\cup J}, r_0|_{I\cup J}) \leq d^{r_1}(J_1 I_2, J_2 I_2) = d^{r_1}(J_1, J_2) < \delta.$$
So $r \in V_{i,\epsilon} \cap U$. □

Corollary 7. (i) *If $p \in \mathrm{S}_X(\mathcal{K})$ is $\mathcal{K}$-e.c., and A is any structure with* $\mathrm{Age}(A) \subseteq$ *$\mathcal{K}$, then A embeds in M_p.*
(ii) *If $p_1, p_2 \in \mathrm{S}_X(\mathcal{K})$ are $\mathcal{K}$-e.c., then M_{p_1} and M_{p_2} are isomorphic.*

Proof. (i) Let Y be countable and $q \in \mathrm{S}_Y(\mathcal{K})$ enumerate a dense subset of A. Then there exists a joining $r \in J(p,q)$ with $\langle r|_Y\rangle \subseteq \langle r|_X\rangle$, showing that $A \cong \langle r|_Y\rangle$ embeds in $\langle r|_X\rangle = M_p$.

(ii) We have that for comeagerly many $r \in J(p_1,p_2)$, $\langle r|_X\rangle \subseteq \langle r|_Y\rangle$ and for comeagerly many $r \in J(p_1,p_2)$, $\langle r|_Y\rangle \subseteq \langle r|_X\rangle$. Thus there exists $r \in J(p,q)$ such that $M_{p_1} \cong \langle r|_X\rangle = \langle r|_Y\rangle \cong M_{p_2}$, whence the conclusion. □

Proposition 8. *Let X be countable, infinite and $p \in \mathrm{S}_X(\mathcal{K})$. The following are equivalent:*

(i) *p is $\mathcal{K}$-e.c.;*
(ii) $\mathrm{Age}(M_p) = \mathcal{K}$, *$M_p$ is ultrahomogeneous, and p* enumerates *M_p, i.e., for any realization $\bar{a} \models p$, the set $\{a_0, a_1, \dots\}$ is dense in $\langle \bar{a}\rangle$.*

Proof. (i) $\Rightarrow$ (ii). Let $q \in S_n(\mathcal{K})$ be arbitrary. By Proposition 6, for comeagerly many $r \in J(p,q)$, $\langle r|_Y \rangle \subseteq \langle r|_X \rangle$; in particular, M_p realizes q. This shows that $\mathrm{Age}(M_p) = \mathcal{K}$.

Next we show that p enumerates M_p. Let $\bar{a} \models p$ and let $t(\bar{v})$ be an $\mathcal{L}$-term. Let $I \subseteq X$ and $q \in S_{I \cup \{i_0\}}$ be defined by $q = \mathrm{tp}(a_I, t(a_I))$. As p is $\mathcal{K}$-e.c., this means that for every $n \in \mathbf{N}$, there exists $j_n \in X$ with $\partial(p|_{I \cup \{j_n\}}, q) < 2^{-n}$. By the equicontinuity of t in $\mathcal{K}$, this implies that $\lim_{n \to \infty} a_{j_n} = t(a_I)$.

Finally we check that M_p is ultrahomogeneous. Let $I, J \subseteq X$ be finite with $\partial(p|_I, p|_J) < \epsilon$. By Proposition 3, this implies that $\{r \in J(p,p) \subseteq S_{X_1 \cup X_2} : d^r(I_1, J_2) < \epsilon\} \neq \emptyset$. (Here, as before, I_1 and J_2 denote the copies of I and J in X_1 and X_2, respectively.) By Proposition 6, there exists $r \in J(p,p)$ with $\langle r|_{X_1} \rangle = \langle r|_{X_2} \rangle$ and $d^r(I_1, J_2) < \epsilon$. This r yields an automorphism $g \in \mathrm{Aut}(M_p)$ with $d^{M_p}(I, J) < \epsilon$.

(ii) $\Rightarrow$ (i). Let $I \subseteq X$ and $q \in S_{I \cup J}(\mathcal{K})$ be such that $\partial(p|_I, q|_I) < \epsilon$. Let $\bar{a} \models p$. As $\mathrm{Age}(M_p) = \mathcal{K}$, and by Proposition 3, there exists $b_{I_1 \cup I_2 \cup J_2} \in M_p^{2|I|+|J|}$ with $\mathrm{tp}\, b_{I_1} = p|_I$, $\mathrm{tp}\, b_{I_2 \cup J_2} = q$, and $d(b_{I_1}, b_{I_2}) < \epsilon$. As M_p is ultrahomogeneous, there exist $g \in \mathrm{Aut}(M_p)$ such that $d(g \cdot b_{I_2}, a_I) < \epsilon$. Finally, as p enumerates M_p, there exists $J' \subseteq X$ with $d(a_{J'}, g \cdot b_{J_2}) < \epsilon$. This implies that $\partial(q, p|_{I \cup J'}) \leq d(g \cdot b_{I_2 \cup J_2}, a_{I \cup J'}) < \epsilon$. ∎

Proof of Theorem 1. (i) $\Rightarrow$ (ii). Let $p \in S_X(\mathcal{K})$ be $\mathcal{K}$-e.c. (such a p exists by Proposition 5). Then M_p is ultrahomogeneous by Proposition 8. Suppose now that M_1 and M_2 are separable, ultrahomogeneous with age $\mathcal{K}$. Let p_1 and p_2 be types in $S_X(\mathcal{K})$ that enumerate dense subsets of M_1, M_2, respectively. By Proposition 8, p_1 and p_2 are $\mathcal{K}$-e.c. and by Corollary 7, $M_1 \cong M_{p_1} \cong M_{p_2} \cong M_2$.

(ii) $\Rightarrow$ (i). Let M be separable ultrahomogeneous with age $\mathcal{K}$ and let $G = \mathrm{Aut}(M)$. Every age is hereditary and directed. Next we check NAP. Let $p \in S_{I_1 \cup J_1}(\mathcal{K})$, $q \in S_{I_2 \cup J_2}(\mathcal{K})$ with $p|_{I_1} = q|_{I_2}$ and let $\epsilon > 0$. Let $a_{I_1 \cup J_1}$ and $b_{I_2 \cup J_2}$ be realizations of p and q in M. By ultrahomogeneity, there exists $g \in G$ with $d(g \cdot a_{I_1}, b_{I_2}) < \epsilon$. Then $\langle g \cdot a_{I_1 \cup J_1}, b_{I_2, J_2} \rangle$ is the required amalgam of p and q. Finally one easily checks that $(S_n(\mathcal{K}), \partial) \cong (M^n /\!\!/ G, \bar{d})$, where $M^n /\!\!/ G = \{\overline{G \cdot \bar{a}} : \bar{a} \in M^n\}$ and

$$\bar{d}(\overline{G \cdot \bar{a}}, \overline{G \cdot \bar{b}}) = \inf\{d(g \cdot \bar{a}, \bar{b}) : g \in G\}.$$

This implies that $S_n(\mathcal{K})$ is separable and complete. ∎

References

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