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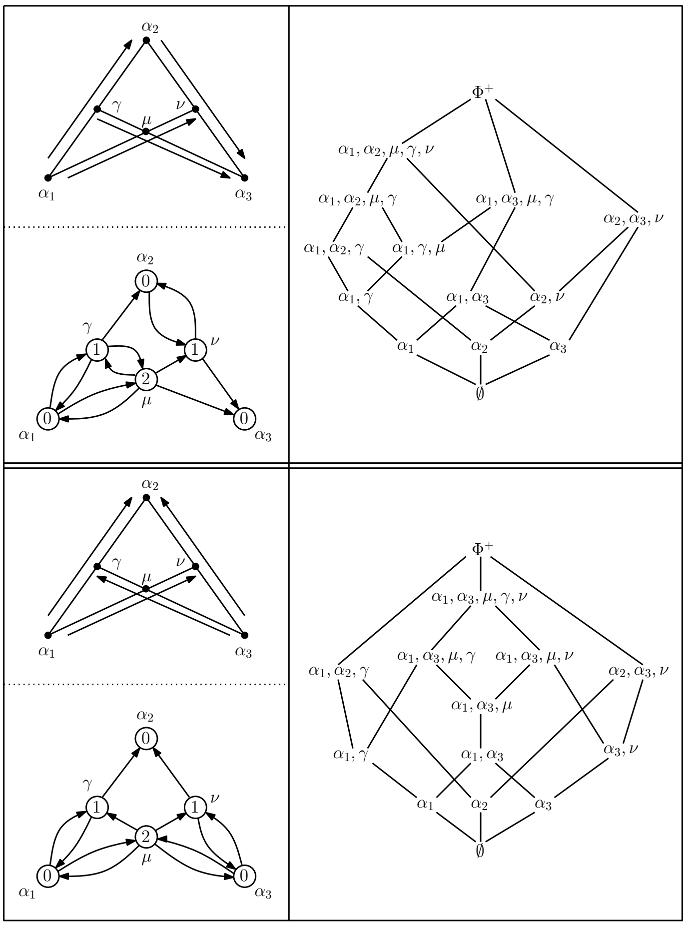
Des graphes orientés aux treillis complets : une nouvelle approche de l'ordre faible sur les groupes de Coxeter



François Viard

Thèse de doctorat

Université Claude Bernard Lyon 1

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CHAPITRE 1

Introduction en français

Prenons un espace euclidien, quelques hyperplans de cet espace, et considérons le groupe engendré par les réflexions orthogonales par rapport à ces hyperplans. Le groupe ainsi obtenu est appelé un *groupe de réflexions*. De nombreux groupes classiques en combinatoire, comme en algèbre, peuvent être réalisés comme des groupes de réflexions. L'exemple peut-être le plus parlant est celui du groupe symétrique, absolument central en combinatoire. En effet, le groupe des permutations d'un ensemble à n éléments peut être réalisé comme le groupe engendré (dans l'espace de dimension n) par les réflexions par rapport aux hyperplans d'équations $x_{i+1}-x_i = 0$, pour tout $1 \leq i \leq n-1$.

Ce point de vue a de nombreux avantages. Par exemple, il fournit une représentation du groupe symétrique, nous permettant d'étudier sa structure d'un point de vue géométrique. Mais surtout, il nous mène à la généralisation des propriétés combinatoires du groupe symétrique à d'autres groupes de réflexions qui lui ressemblent. Ainsi, on peut voir l'introduction au XX$^{\text{e}}$ siècle du concept de *groupe de Coxeter* comme une conséquence de cette démarche généralisatrice.

Un groupe de Coxeter est définit comme étant un groupe engendré par un ensemble de générateurs vérifiant certaines relations mimant les propriétés des réflexions : le carré de tout générateur est l'identité, et le produit de deux générateurs est d'ordre fixé, potentiellement infini (ceci imite le fait que le produit de deux reflexions est une rotation). Les groupes de Coxeter interviennent de manière fondamentale dans de nombreux domaines des mathématiques et forment un terrain de jeu apprécié de nombreux combinatoristes.

Cette thèse est dédiée à l'étude d'une structure combinatoire associée à tout groupe de Coxeter : *l'ordre faible* (voir l'article fondateur [**B**]) . L'ordre faible est, comme son nom l'indique, un ordre partiel sur le groupe, et qui a de nombreuses propriétés intéressantes. Il intervient, par exemple, dans des domaines aussi variés que la théorie des fonctions symétriques ([**S2**] et [**L2**], la géométrie de la grassmanienne et le calcul de Schubert (voir [**FGRS**] et [**BJS**]), ou encore les algèbres amassées (voir [**FZ1**], [**FZ2**], [**RS2**] et [**RS1**]).

Dans cette thèse nous développons un nouveau cadre théorique pour l'étude de l'ordre faible sur tout groupe de Coxeter, et nous l'appliquons à l'étude de diverses problématiques associées. Plus précisément, nous suivons un cheminement qui a fait ses preuves pour l'étude des groupes de Coxeter : partir d'un problème énumératif fondamentalement relié au groupe symétrique (ici, l'énumération de certains tableaux), puis extraire de la construction combinatoire faite à l'étape précédente un modèle applicable à une plus grand classe de groupes de Coxeter, et enfin se servir de ce nouveau point de vue pour aborder des problèmes ouverts. Dans le cadre de cette thèse, le premier problème étudié est un ensemble de conjectures de Dyer à propos d'une extension de l'ordre faible, qui permettraient en principe d'étendre à tous les groupes de Coxeter des propriétés combinatoires et géométriques usuellement restreintes au cas des groupes de Coxeter finis. Bien que l'origine de ces conjectures remontent à 1993 (voir [**D1**], ou encore [**D4**]), elles restent encore largement ouvertes. Elles ont cependant récemment fait l'objet d'études intensives, conduisant à l'étude du comportement asymptotique des systèmes de racine des groupes de Coxeter (voir [**DHR**], [**HLR**]), ou encore à l'étude de constructions abstraites (voir [**D2**] et [**D3**]) aussi bien que géométriques (voir [**HL**] et [**L1**]).

L'une des motivations possibles pour l'étude des conjectures de Dyer concerne les treillis cambriens (voir [**R1**] et [**R3**]). Les treillis Cambriens sont des quotients de l'ordre faible sur tout groupe de Coxeter fini, et ils sont reliés à la combinatoire des algèbres amassées. En particulier le diagramme de Hasse d'un treillis cambrien est isomorphe au graphe des échanges de l'algèbre amassé associé au groupe de Coxeter considéré (voir [**R2**] et [**RS2**]). Leur construction peut-être étendue à tout groupe de Coxeter (voir [**RS3**]), mais la correspondance avec les algèbres amassés n'est alors plus que partielle, et par exemple le diagramme de Hasse des posets ainsi obtenu (appelé semi-treillis cambriens) n'est isomorphe qu'à un sous-graphe du graphe des échange de l'algèbre amassé correspondante (voir [**RS2**]). Bien que dans le cas des groupes de Coxeter affine, Reading et Speyer aient montré dans [**RS1**] comment compléter les semi-treillis cambriens pour retrouver les propriétés attendues à propos des algèbres amassé, ce problème reste ouvert dans le cas général. Or, les conjectures de Dyer fournissant un candidat d'extension de l'ordre faible sur tout groupe de Coxeter, et il est donc naturel de voir si la construction des semi-treillis cambriens peut être étendue à cette extension, dans l'espoir d'obtenir toutes les informations attendues sur l'algèbre amassée. Ceci constitue le deuxième problème dont l'étude est commencée dans cette thèse.

Dans la suite de cette introduction nous détaillons le contenu de la thèse.

Le chapitre 2 est dédié à quelques rappels fondamentaux sur les tableaux, les graphes orientés, les ensembles partiellement ordonnés (aussi appelés *posets*) et plus particulièrement les treillis et semi-treillis, et enfin les groupes de Coxeter.

Commençons par introduire le problème énumératif à l'origine du contenu de cette thèse, *i.e.* l'étude des tableaux équilibrés. Cette famille de tableaux a été introduite par Edelman et Greene dans [**EG**], et ils sont définis comme étant des diagrammes de Ferrers remplis par tous les entiers de 1 à n, où n est la taille du diagramme, de telle façon qu'une certaine condition sur les équerres de chaque case du diagramme soit satisfaite. Edelman et Greene ont démontré le résultat suivant : pour une partition $\lambda \vdash n$ donnée, il y a autant de tableaux équilibrés de forme λ que de tableaux standards de même forme. De façon assez surprenante, la preuve proposée est à la fois bijective et, a priori, peu satisfaisante. En effet, elle utilise de manière fondamentale l'ordre faible sur le groupe symétrique comme intermédiaire, alors que la similarité entre tableaux standards et équilibrés est frappante et que l'on pourrait d'attendre à une bijection directe entre ces deux familles d'objets.

C'est en tentant de construire une bijection directe entre tableaux équilibrés et tableaux standards que nous avons introduit les objets fondamentaux de ce chapitre. Plus précisément : pour travailler sur cette question, il était nécessaire de trouver un moyen efficace de construire des exemples de tableaux équilibrés. Ceci nous amène à l'introduction d'un algorithme de remplissage, qui démarre avec un diagramme de Ferrers vide, le remplit case par case avec les entiers de 1 à n, et donne à la fin un tableau équilibré.

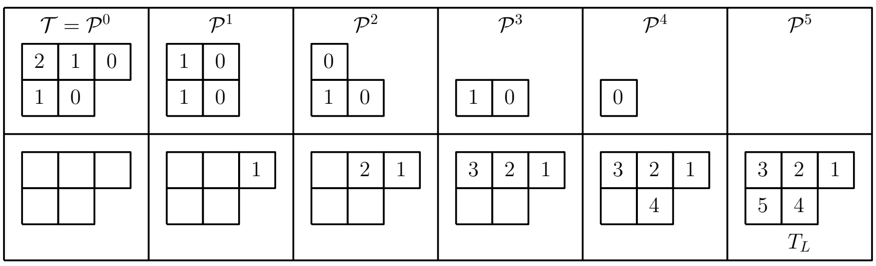


FIGURE 1.1. L'algorithme en action.

Même si tous les tableaux équilibrés peuvent être obtenus par ce biais, et malgré la richesse combinatoire apparente de l'algorithme, cette approche n'a pas permis d'aboutir à la bijection désirée. Néanmoins, la forme de l'algorithme amène naturellement à une généralisation du concept de tableaux équilibrés. Ainsi, on associe à chaque tableau (qu'il soit équilibré, standard, ou aucun des deux) un objet combinatoire appelé le *type* de ce tableau. Cette notion peut-être formellement définie de la manière suivante.

DÉFINITION 1. Soit $T = (T_{\mathfrak{c}})_{\mathfrak{c}}$ un tableau de Young de forme S, on appelle le *type* de T le tableau de forme S dont chaque case $\mathfrak{c}$ est remplit par l'entier $\theta(\mathfrak{c})$ définie par

$$\theta(\mathfrak{c}) := |\{\mathfrak{d} \in S \mid \mathfrak{d} \in H_S(\mathfrak{c}) \text{ et } t_{\mathfrak{c}} < t_{\mathfrak{d}}\}| \text{ pour tout } \mathfrak{c} \in S,$$

où $H_S(\mathfrak{c})$ désigne l'équerre basée en $\mathfrak{c}$, c'est-à-dire l'ensemble des cases situées à droite et en dessous de $\mathfrak{c}$ dans le diagramme S.

Tous les tableaux possibles se trouvent alors classifiés en fonction de leur type, et les tableaux standard et équilibrés forment deux classes particulières de cette classification. L'algorithme

7	5	2	0
5	3	0	
3	1		
2	0		
0			

3	2	1	0
2	1	0	
1	0		
1	0		
0			

FIGURE 1.2. Types associés aux tableaux standards (à gauche) et équilibrés (à droite) de forme $(4, 3, 2, 2, 1)$.

s'adapte naturellement à ce nouveau contexte, et cela nous permet de généraliser les résultats de Edelman et Greene à de nombreuses autres classes de tableaux.

Plus précisément, on montre qu'à toute permutation *vexillaire* σ (*i.e.* évitant le motif 2134) on peut associer un type $\mathcal{T}(\sigma)$, tel que les tableaux associés soient énumérés par les tableaux standards de même forme (voir théorème 6.3.11). En particulier, on retrouve le fait que l'ensemble des tableaux standards et l'ensemble des tableaux équilibrés ont même cardinal. Il est important de noter que cette généralisation ne repose pas sur une bijection directe, et utilise de manière fondamentale l'ordre faible sur les permutations comme intermédiaire. En utilisant cette connexion, on montre également que le nombre tableaux de type $\mathcal{T}(\sigma)$ tels que les entiers $1, 2, \ldots, k$ apparaissent à des positions fixés, est donné par le nombre de décompositions réduites d'une certaine permutation ω explicite. Ceci généralise un résultat implicite dans **[FGRS]** à propos des tableaux équilibrés.

On peut également noter que ce modèle fournit un modèle combinatoire nouveau pour les expressions réduites d'une permutation. On peut ainsi faire correspondre à toute permutation une famille de tableaux, chacun obtenu à l'aide de l'algorithme, et surtout chacun étant canoniquement associé à une expression réduite de la permutation. Sur la figure suivante on représente les tableaux associés aux expressions réduites de $[4, 1, 3, 5, 2]$.

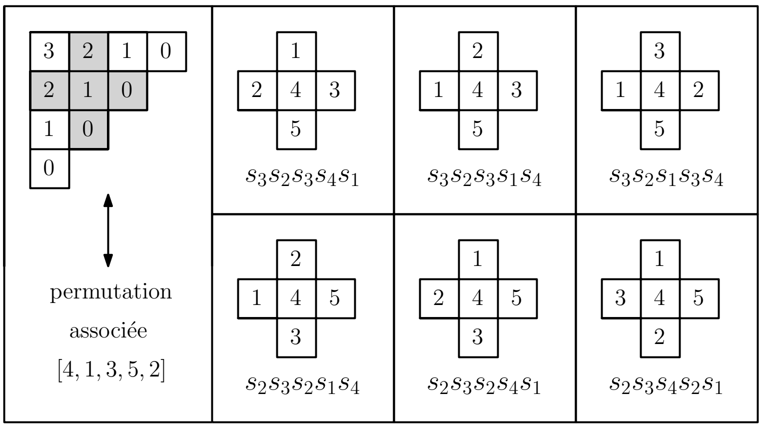


FIGURE 1.3. Les décomposition réduites de $[4, 1, 3, 5, 2]$ vues comme des tableaux.

Dans la mesure où ces résultats peuvent être reformulés en utilisant les généralisations introduites ultérieurement, ces notions sont développés dans le chapitre 6.

Dans le chapitre 3 on change de regard sur les objets mentionné précédemment. En effet, l'argument clé dans notre étude des tableaux est que l'algorithme aboutit à un modèle combinatoire des décompositions réduites des permutations. Il fournit donc également un modèle combinatoire de l'ordre faible sur le groupe symétrique. On en vient donc naturellement à la question suivante : est-il possible de généraliser ce modèle à l'ordre faible sur d'autres groupes de Coxeter ? Il apparait que la description à partir de tableaux est insuffisante pour être appliquée à d'autres cas, et ceci même sur de petits exemples comme celui du groupe B_3. Cela nous amène à nous placer dans un contexte bien plus général : celui des graphes orientés.

L'objet fondamental de ce chapitre est ce que nous appelons un *graphe valué* $\mathcal{G} = (G, \theta)$, où G est un graphe orienté simple et acyclique (pas nécessairement fini), et θ est une application θ définie sur les sommets du graphe $\mathcal{V}(G)$ et à valeurs entière. On impose également que θ vérifie l'inégalité :

$$\text{pour tout } z \in \mathcal{V}(G),\ 0 \leq \theta(z) \leq d^+(z),$$

où $d^+(z)$ désigne le degré sortant du sommet z. Dans ce nouveau cadre, l'algorithme ne va plus être employé pour construire des tableaux, mais pour générer des suites injectives d'éléments de $\mathcal{V}(G)$, appelées des *épluchages* de $\mathcal{G}$, ceci en épluchant le graphe sommet par sommet en respectant la contrainte donnée par la valuation θ. Plus précisément, on va d'abord choisir un sommet z dit *épluchable*, c'est à dire tel que :

- $\theta(z) = 0$;
- pour tout $y \in \mathcal{V}(G)$, si il y a un arc allant de y à z, alors $\theta(y) > 0$,

On épluche ensuite ce sommet, c'est à dire :

(1) on diminue d'un la valuation de tous les sommets y tel qu'il y ai un arc allant de y à z ;
(2) puis on efface le sommet z ainsi que tous les arcs ayant z comme point de départ ou d'arrivée.

Le résultat après cette opération est encore un graphe valué, et on peut répéter ce processus. Sur la figure 1.4, on montre le déroulement de l'algorithme sur un exemple simple.

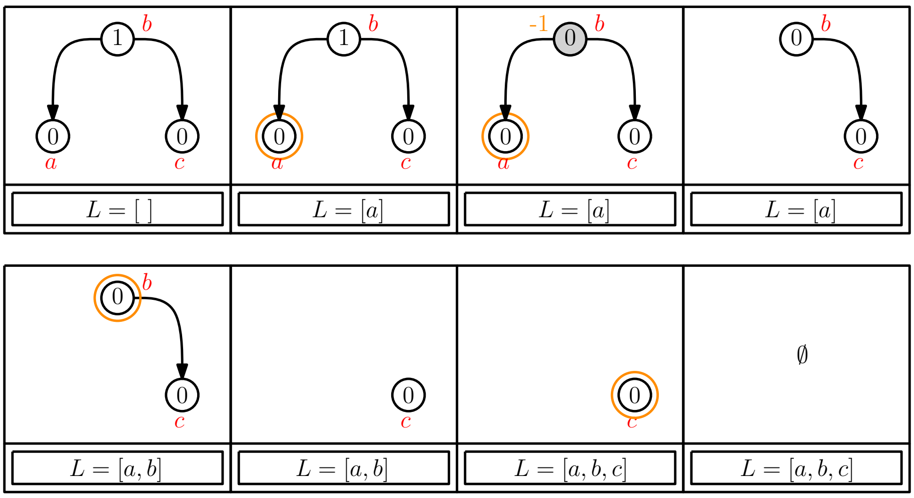


FIGURE 1.4. L'algorithme d'épluchage en action. La suite L obtenue à la fin est ce qu'on appelle un épluchage de $\mathcal{G}$.

On note $PS(\mathcal{G})$ l'ensemble des épluchages ainsi obtenus, et on considère l'ensemble $IS(\mathcal{G})$ des sections initales des éléments de $PS(\mathcal{G})$, c'est à dire les ensembles de la forme $\{z_1, z_2, \ldots, z_k\}$, où la suite $[z_1, \ldots, z_k]$ est le début d'un épluchage de $\mathcal{G}$. On montre alors que le poset $(IS(\mathcal{G}), \subseteq)$ a de nombreuses propriétés similaires à celles de l'ordre faible.

Théorème 1. Soit $\mathcal{G} = (G, \theta)$ un graphe valué, le poset $(IS(\mathcal{G}), \subseteq)$ est un semi-treillis inférieur gradué, dont la graduation est donnée par $\rho : A \to |A|$. De plus, si G est fini, alors $(IS(\mathcal{G}), \subseteq)$ est un treillis complet.

On rappelle que la fonction de Möbius μ sur un poset $(P, \leq)$ est la fonction de P^2 dans $\mathbb{Z}$ définie recursivement par :

- pour tout $x \in P$, $\mu(x, x) = 1$,
- pour tout $x, y \in P$, $\mu(x, y) = -\sum_{x \leq c < y} \mu(x, c)$.

Théorème 2. Soit μ la fonction de Möbius de $(IS(\mathcal{G}), \subseteq)$ et $A \in IS(\mathcal{G})$, on définit les deux ensembles suivants :

$$\mathcal{N}(A) := \{x \in A \mid \theta(x) = 0\},$$
$$\mathcal{F}(A) := \{x \in A \mid A \setminus \{x\} \in IS(\mathcal{G})\}.$$

On a :

(1) si $\mathcal{F}(A) = \mathcal{N}(A)$, alors $\mu(\emptyset, A) = (-1)^{|\mathcal{N}(A)|}$,
(2) sinon, on a $\mu(\emptyset, A) = 0$.

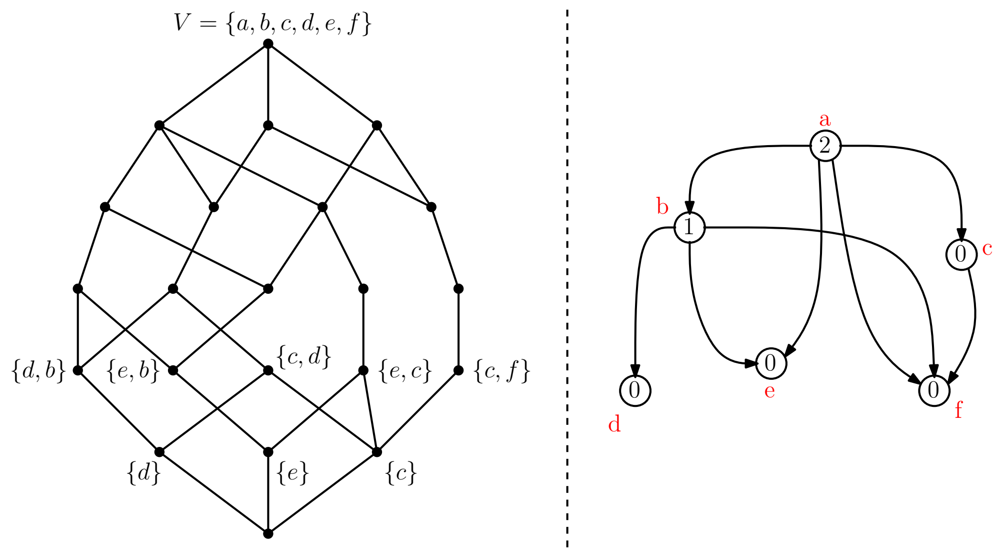


Figure 1.5. Un exemple de poset $(IS(\mathcal{G}), \subseteq)$ (à gauche) et le graphe valué associé (à droite).

On s'attache ensuite à montrer, en construisant explicitement le graphe valué associé, que l'ordre faible sur les groupes de Coxeter de type A, B et $\widetilde{A}$ peut être décrit par ce modèle. Ceci fournit une preuve alternative du fait qu'il s'agit de treillis gradués, un nouveau modèle combinatoire pour leur étude, ainsi qu'une nouvelle formule pour calculer les valeurs de la fonction de Möbius associée à chacun de ces cas. Précisons un peu la méthode employée pour montrer que l'ordre faible en type A proviens d'un graphe valué (la méthode est similaire pour les types B et $\widetilde{A}$). Il est classique que le groupe A_{n-1} s'identifie au groupe des permutations S_n, et on peut associer à toute permutation σ un ensemble, appelé son ensemble d'inversion, défini par

$$\mathrm{Inv}(\sigma) := \{(a, b) \in [n]^2 \mid a < b \text{ et } \sigma^{-1}(a) > \sigma^{-1}(b)\}.$$

Notons qu'il ne s'agit pas ici de la définition standard (voir section 3.3.1), mais il s'agit de celle qui s'adapte le mieux à notre étude. En particulier, on a la propriété suivante (voir par exemple **[BB]** pour une preuve précise de cette proposition)

$$\text{pour tout } \sigma, \omega \in S_n,\ \sigma \leq_R \omega \text{ si et seulement si } \mathrm{Inv}(\sigma) \subseteq \mathrm{Inv}(\omega),$$

où $\leq_R$ désigne l'ordre faible sur les permutations. On peut alors construire un graphe valué $\mathcal{A}$ dont les sommets sont indéxés par les couples d'entiers (a, b) tels que $1 \leq a < b \leq n$, et tel que

$$IS(\mathcal{A}) = \{\mathrm{Inv}(\sigma) \mid \sigma \in S_n\},$$

ce qui entraine immédiatement que $(IS(\mathcal{A}), \subseteq)$ et $(A_{n-1}, \leq_R)$ sont isomorphes. On montre également qu'une généralisation de l'ordre faible au cas des groupes de permutations colorées, appelé le *flag weak order* et introduit par Adin, Brenti et Roichman dans [**ABR**], peut être décrit dans ce contexte. Il est intéressant de noter que ces graphes peuvent tous être vu comme des tableaux munis d'une structure de graphe implicite, comme représenté dans la figure 1.6.

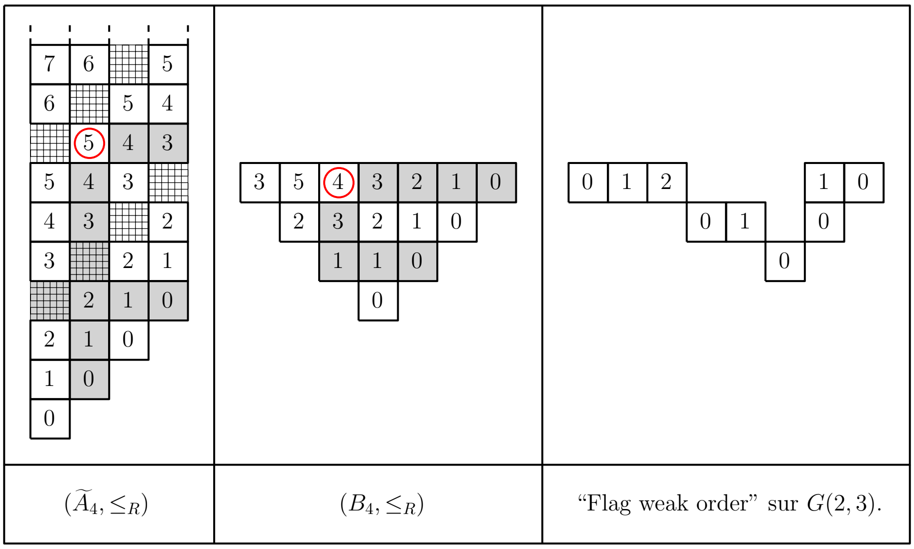


FIGURE 1.6. Les graphes associées à différents ordres. La structure de graphe est donnée implicitement par une notion d'équerre adaptée à chaque diagramme (voir chapitre 3).

Toutefois, l'exemple le plus riche de conséquence est aussi le plus simple : celui du treillis des idéaux inférieurs (et supérieurs) de tout poset fini. En effet, on montre que pour tout poset fini P, il existe un graphe valué $\mathcal{G}(P)$ dont les sommets sont les éléments de P, tel que $IS(\mathcal{G}(P))$ soit constitué exactement des idéaux inférieurs de P, et tel que $PS(\mathcal{G}(P))$ soit exactement l'ensemble des extensions linéaires de P. Ainsi, notre construction peut être vue comme une généralisation du concept d'extension linéaire au cas des graphes valués. Or les extensions linéaires sont intimement reliées à la théorie des fonctions quasi-symétriques, via la notion de P-partition (voir [**G**]). Il est donc naturel de chercher à utiliser notre construction pour définir des fonctions quasi-symétriques. Ceci est l'objet de la fin de ce chapitre, où on montre comment associer une famille de fonctions quasi-symétriques à tout élément d'un poset construit à partir d'un graphe valué. Si, dans la majeure partie des cas, ces fonctions ne semblent pas pertinentes, a priori, pour l'étude de la structure du poset sous-jacent, il arrive que la forme même du graphe valué désigne un choix canonique de fonction quasi-symétrique dans cette famille. C'est par exemple le cas pour les graphes valués associés à l'ordre faible sur les groupes de Coxeter de type A et $\widetilde{A}$. Dans ces cas on montre (en suivant une méthode similaires à celle présentée dans [**FGRS**] et [**YY**]) que cette fonction canonique est exactement la fonction symétrique de Stanley en type A (voir [**S2**]), et la généralisation de Lam en type $\widetilde{A}$ (voir [**L2**]). Notons également que cette description aboutit naturellement à une description des séries de Stanley comme une somme de monômes associés à une certaine famille de tableaux, généralisant au passage la notion de tableaux standard et semi-standard.

On en vient alors à la question de la généralisation de ces résultats à tout groupe de Coxeter. Néanmoins, dans cette arène bien plus large, de nombreuses difficultés apparaissent, et pour aborder ce problème nous allons devoir encore une fois élargir notre point de vue sur les objets en

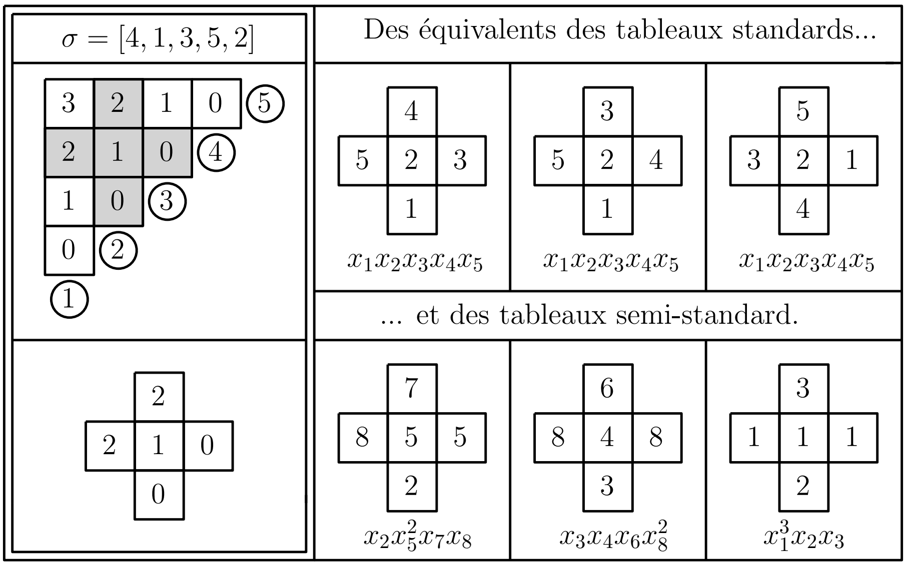


FIGURE 1.7. Certains tableaux associés à la permutation $[4, 1, 3, 5, 2]$ et leurs monômes associés

jeu. Arrêtons nous d'abord quelques instant sur la méthode employée pour construire les graphes valués associés aux groupes de Coxeter de types B et $\widetilde{A}$. Dans chacun de ces cas on procède en deux étapes : tout d'abord trouver une bonne notion *d'ensemble d'inversion* associée aux types B et $\widetilde{A}$ (*i.e.* généralisant les propriétés des ensembles d'inversions des permutations) afin de construire un candidat de graphe valué, puis utiliser l'interprétation classique des éléments de B_n (resp. $\widetilde{A}_n$) comme des permutations signées (resp. affines) afin de prouver à l'aide de méthodes combinatoires que les graphes valués obtenus décrivent bien l'ordre faible.

Si nous n'avons pas en général d'interprétation des groupes de Coxeter en tant qu'ensemble de permutations (ou d'objets ressemblants), on a par contre bien une notion d'ensemble d'inversion via les *systèmes de racines*. Il s'agit d'un objet essentiellement géométrique, qui permet de voir un groupe de Coxeter comme étant effectivement un groupe de réflexion, mais dans un cadre qui n'est généralement plus euclidien. Plus concrètement, si W est un groupe de Coxeter de rang n, un système de racines de W est une sous partie Φ de $\mathbb{R}^n$ sur lequel W agit, qui caractérise W complètement, et qui nous permet d'intépréter géométriquement de nombreuses propriétés de W. En particulier, il existe une partition de Φ en deux sous-ensembles Φ^+ et Φ^-, appelés respectivement les *racines positives* et les *racines négatives* de Φ, avec laquelle on peut définir une notion d'ensemble d'inversion. Pour tout élément $\omega \in W$, on définit son ensemble d'inversion par

$$\mathrm{Inv}(\omega) := \Phi^+ \cap \omega(\Phi^-).$$

On peut alors montrer (voir [**BB**]) que pour tout τ dans W, ω est plus petit que τ pour l'ordre faible si et seulement si $\mathrm{Inv}(\omega) \subseteq \mathrm{Inv}(\tau)$.

L'idée est alors de construire un graphe valué $\mathcal{G}$ dont les sommets sont les éléments de Φ^+, et tel que $IS(\mathcal{G})$ soit constitué exactement des ensembles d'inversion des éléments de W. On se retrouve néanmoins confronté à deux difficultés : même dans le cas des groupes dihédraux, il y a de nombreux graphes qui conviennent (donc on n'a pas de choix canonique) ; et quand on veut généraliser la construction au cas des groupes de Coxeter en général, la méthode naturelle ne donne généralement pas un graphe acyclique (donc la théorie développée au chapitre 3 ne s'applique pas). Ces deux remarques nous amènent au contenu du chapitre 4.

Dans le chapitre 4, on commence par généraliser la construction du chapitre précédent, en la rendant applicable au cas de graphes non acycliques. Cette nouvelle construction est en fait une conséquence naturelle de la théorie du chapitre 3. En effet, afin de simplifier certaines preuves on y a introduit une caractérisation intrinsèque des éléments de $IS(\mathcal{G})$ qui n'utilise pas les épluchages, et qui ne dépend ni du caractère acyclique du graphe, ni du fait que les éléments de $IS(\mathcal{G})$ soient finis. On peut donc prendre cette proposition comme définition des éléments de $IS(\mathcal{G})$, en retirant les conditions sur le graphe ainsi que la condition de finitude. Plus précisément, on propose pour $IS(\mathcal{G})$ la définition suivante.

Définition 2. Soit $\mathcal{G} = (G, \theta)$ un couple formé d'un graphe quelconque et d'une application $\theta : \mathcal{V}(G) \longrightarrow \mathbb{N}$ tel que

$$0 \leq \theta(z) \leq d^+(z) \text{ pour tout } z \in \mathcal{V}(\mathcal{G}).$$

On note alors $IS(\mathcal{G})$ l'ensemble définit par

$$IS(\mathcal{G}) := \left\{ A \subseteq \mathcal{V}(\mathcal{G}) \;\middle|\; \begin{array}{l} \theta(z) \leq d_A^+(z) \text{ pour tout } z \in A \\ \theta(z) \geq d_A^+(z) \text{ pour tout } z \in \mathcal{V}(\mathcal{G}) \setminus A \end{array} \right\}.$$

Notons que cette nouvelle définition généralise la définition proposée au chapitre 3. Avec une petite subtilité néanmoins : quand le graphe est infini, simple et acyclique, on prend en compte avec cette nouvelle définition des ensembles infinis (par exemple $\mathcal{V}(\mathcal{G})$ est toujours dans $IS(\mathcal{G})$), qui sont inaccessibles en utilisant la définition à base de sections initiales d'épluchages. Dans ce nouveau cadre élargit, on montre le théorème suivant.

Théorème 3. Pour tout graphe valué $\mathcal{G}$, $(IS(\mathcal{G}), \subseteq)$ est un treillis complet.

En particulier, cela nous donne un plongement naturel des posets introduits au chapitre 3 dans un treillis complet (quand le graphe sous-jacent est infini).

Ce dernier point est particulièrement important pour la suite de ce chapitre. En effet, Dyer a conjecturé, dans **[D1]** et **[D4]**, qu'il était possible de compléter l'ordre faible sur n'importe quel groupe de Coxeter. Cette conjecture fait intervenir la notion d'ensemble bi-clos, définie comme suit. Un ensemble $A \subseteq \Phi^+$ est dit *clos* si et seulement si

$$\forall \alpha, \beta, \gamma \in \Phi^+ \text{ tel que } \gamma = a\alpha + b\beta,\ a > 0,\ b > 0,\ \text{si } \alpha, \beta \in A,\ \text{alors } \gamma \in A.$$

On dit que A est *bi-clos* si et seulement si A et $\Phi^+ \setminus A$ sont tous les deux clos. On note $\mathcal{B}(\Phi^+)$ l'ensemble des bi-clos de Φ^+. On peut montrer (voir, par exemple, **[P]**) que les ensembles bi-clos finis sont exactement les ensembles d'inversions des éléments de W. Ainsi, le poset $(\mathcal{B}(\Phi^+), \subseteq)$ est une extension de l'ordre faible, et la conjecture de Dyer est que ce poset est en fait un treillis complet. On représente sur la prochaine figure le poset $(\mathcal{B}(\Phi^+), \subseteq)$ obtenu à l'aide de l'exemple le plus simple de groupe de Coxeter infini : le groupe dihédral infini.

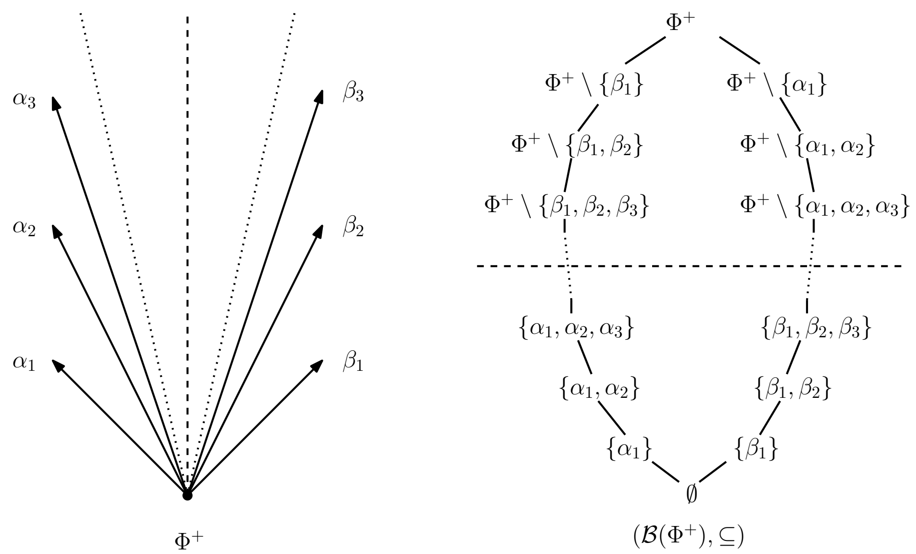


Figure 1.8. On représente sur cette figure le poset $(\mathcal{B}(\Phi^+), \subseteq)$ obtenu en considérant un ensemble de racine du groupe dihédral infini.

L'idée est donc de chercher un graphe valué $\mathcal{G}$ qui a pour ensemble de sommets Φ^+, et tel que $IS(\mathcal{G}) = \mathcal{B}(\Phi^+)$. Dans le cas du groupe dihédral, cette nouvelle contrainte restreint considérablement le choix de graphes possibles, et on introduit alors une famille de graphes valués appelés les *échafaudages*. Sur la figure suivante, on représente le treillis obtenu à l'aide d'un de ces échaffaudages.

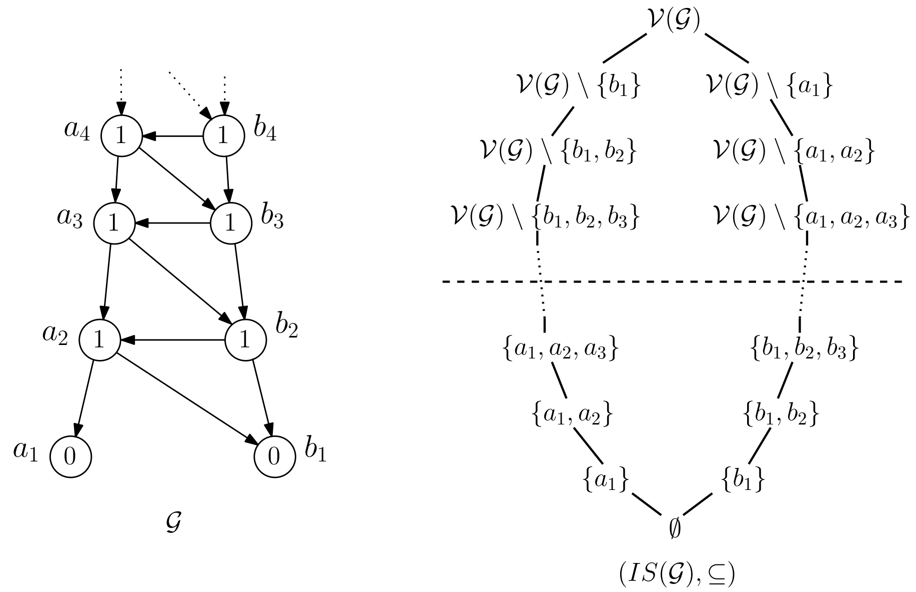


FIGURE 1.9. Un échaffaudage infini $\mathcal{G}$, et le treillis $(IS(\mathcal{G}), \subseteq)$ associé.

On remarque alors que le treillis obtenu avec cet échaffaudage et le treillis des bi-clos du groupe dihédral infini sont isomorphe, en identifiant α_i avec a_i et β_i avec b_i. On va alors se servir de ces échaffaudage (plus précisément, on va les assembler ensemble en respectant certaines contraintes données par la géométrie du système de racine) pour construire des graphes valués, qui sont dit *bien assemblés* sur Φ^+. On montre alors que pour tout graphe valué $\mathcal{G}$ bien assemblé, on a $\mathcal{B}(\Phi^+) \subseteq IS(\mathcal{G})$, et pour tout $A \subseteq \Phi^+$, si $A \in IS(\mathcal{G})$ alors $\Phi^+ \setminus A$ est aussi dans $IS(\mathcal{G})$. Ainsi, on a un plongement de $(\mathcal{B}(\Phi^+), \subseteq)$ dans un ortho-treillis complet.

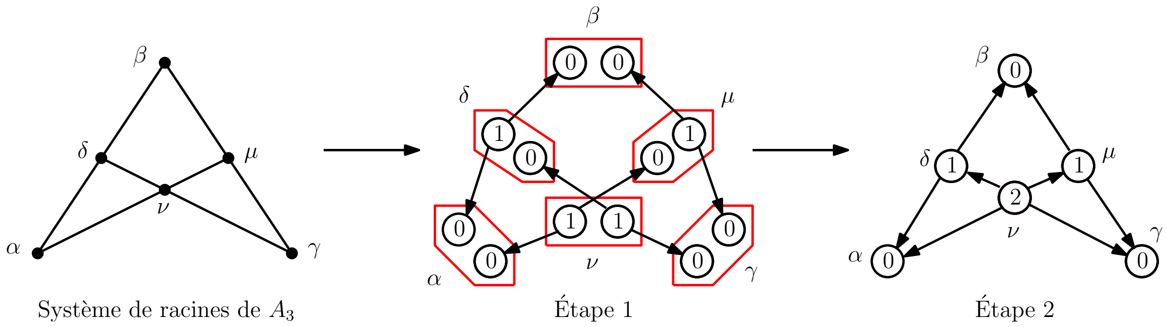


FIGURE 1.10. Construction d'un graphe valué bien assemblé sur un système de racine de A_3. On peut vérifier que le treillis obtenu est bien isomorphe à $(A_3, \leq_R)$. Voir le chapitre 4 pour plus de détails sur cette construction.

La difficulté principale qui apparait à cette étape est le caractère infini des graphes ainsi construits. Il est de fait difficile de faire des tests pour identifier les éléments de $IS(\mathcal{G})$, et ceci nous amène à considérer une nouvelle famille de graphes valués, les graphes valués *projectifs*. Les graphes valués projectifs (notés $\mathcal{G}_\infty$) sont la limite (en un certain sens) d'une suite de graphes valués finis et acycliques $(\mathcal{G}_i)_{i \geq 0}$. Dans ce cas, on a que les ensembles $IS(\mathcal{G}_i)$ forment un système projectif, et $IS(\mathcal{G}_\infty)$ peut être pensé comme en étant la limite projective. Ainsi, on peut complètement étudier le treillis $(IS(\mathcal{G}_\infty), \subseteq)$ au travers des treillis finis $(IS(\mathcal{G}_i), \subseteq)$, ce qui nous permet de les étudier et de faire des tests.

Un autre avantage considérable des graphes valués projectifs est leur similarité avec le cas fini acyclique. En effet, à tout graphe projectif on peut associer un ensemble d'ordres totaux sur ses sommets, noté $PS(\mathcal{G}_\infty)$ et généralisant les épluchages. C'est-à-dire que tout élément de $IS(\mathcal{G}_\infty)$ est une section initiale d'un élément de $PS(\mathcal{G}_\infty)$. De plus, $PS(\mathcal{G}_\infty)$ encode totalement les chaines maximales de $(IS(\mathcal{G}_\infty), \subseteq)$, dans le sens où une chaine $\mathcal{C}$ de ce treillis est maximale si et seulement si il existe $L \in PS(\mathcal{G}_\infty)$ tel que $\mathcal{C}$ soit exactement l'ensemble des sections initiales de L. Ces propriétés évoquent fortement une autre conjecture de Dyer (voir [**D4**]), qui porte sur la structure des chaines maximales de $(\mathcal{B}(\Phi^+), \subseteq)$. En effet, il est conjecturé que toute chaine $\mathcal{C}$ de $(\mathcal{B}(\Phi^+), \subseteq)$ est maximale si et seulement si il existe un ordre de réflexion L (*i.e.* un ordre total sur Φ^+ vérifiant certaines conditions) tel que $\mathcal{C}$ soit constituée exactement des sections initiales de L. Ainsi, la suite du chapitre 4 est consacrée à la construction d'un graphe valué projectif et bien assemblé sur Φ^+. On montre alors qu'un tel graphe $\mathcal{G}_\infty$ vérifie bien que tout ordre de réflexion est dans $PS(\mathcal{G}_\infty)$. On montre également que, dans ce nouveau contexte, les deux conjectures de Dyer sont équivalentes : on a $\mathcal{B}(\Phi^+) = IS(\mathcal{G}_\infty)$ si et seulement si $PS(\mathcal{G}_\infty)$ est l'ensemble des ordres de réflexion de Φ^+. De plus, la structure projective nous permet de faire de nombreux tests, ce qui nous amène à conjecturer qu'il existe au moins un graphe valué projectif $\mathcal{G}$ bien assemblé sur Φ^+ tel que $IS(\mathcal{G})$ soit effectivement égal à $\mathcal{B}(\Phi^+)$, et on propose un candidat d'un tel graphe.

La fin du chapitre est consacrée à l'étude de quelques applications possibles de notre construction. On commence d'abord par un rapide survol des connexions qui existent entre cette théorie et la géométrie convexe abstraite. On introduit notamment plusieurs *opérateurs de clôture*, et on montre quelques-unes de leurs propriétés.

En se servant d'un de ces opérateurs de clôture, on présente quelques résultats en direction de l'extension des semi-treillis cambriens mentionnée plus tôt dans l'introduction. On montre comment associer à tout treillis et semi-treillis cambrien un graphe valué, puis on montre que le (semi-)treillis cambrien est un sous-poset du treillis obtenu. On constate sur des exemples en type A que le treillis obtenu est exactement le treillis cambrien associé, mais nous n'avons pas encore d'explication générale de ce fait. Au passage, on introduit un nouveau concept intéressant, celui de *treillis induit*. Plus précisément, si on considère un graphe valué $\mathcal{G}$ et un autre graphe valué $\mathcal{G}'$ obtenu à partir de $\mathcal{G}$ en lui ajoutant des arcs, on définit une relation d'équivalence $\sim_{\mathcal{G}'}$ sur $IS(\mathcal{G})$. On montre ensuite que toute classe d'équivalence de cette relation admet un élément maximal, et que le poset obtenu en ordonnant ces éléments maximaux est un treillis complet (mais pas nécessairement un sous-treillis de $(IS(\mathcal{G}), \subseteq)$).

Dans le chapitre 5 on s'intéresse à l'étude du cas particulier du treillis de Tamari (voir [**MHPS**]), en introduisant un graphe valué $\mathcal{A}_n^\uparrow$, puis en montrant que le treillis $(IS(\mathcal{A}_n^\uparrow), \subseteq)$ est isomorphe au $(n+1)$-ième treillis de Tamari. Notre démonstration repose sur la construction d'une bijection directe entre $IS(\mathcal{A}_n^\uparrow)$ et les *chemins de Dyck* de taille $n+1$, et ceci nous amène naturellement à une description des treillis de m-Tamari à l'aide d'un graphe valué. Notons que le graphe valué $\mathcal{A}_n^\uparrow$ provient directement de la famille de graphes valués introduit à la fin du chapitre 4, et il s'agit pour l'instant du seul cas particulier pour lequel nous avons démontré que le graphe valué donnait bien le treillis cambrien associé.

CHAPTER 1

Introduction

Let us consider an Euclidean space, some hyperplanes of this vector space, and consider the group generated by orthogonal reflections across these hyperplanes. The resulting group is called a *reflection group*. Many classical groups appearing in algebraic combinatorics can be realized as reflection groups. For instance, the symmetric group S_n is a reflection group. Indeed, it can be seen as the reflection group (in a vector space of dimension n) generated by the reflections across hyperplanes of equation $(x_{i+1} - x_i = 0)$ for all $i \in [n-1]$.

This point of view has many advantages. For instance, it naturally provides a representation of S_n that allows to study this group using geometric methods. Furthermore, it leads us to a generalization of combinatorial properties of the symmetric group to other reflections groups. The notion of Coxeter group, introduced during the XX$^{\text{th}}$ century, provides a general context in which these generalizations can be studied.

A Coxeter group is defined as a group generated by a set of generators satisfying some relations imitating properties of reflections: the square of any generator equals the identity, and the product of any given pair of generators has a given order which can be infinite. Coxeter groups are fundamental in many areas of mathematics and are studied by many combinatorists.

This thesis is dedicated to the study of a combinatorial structure associated with each Coxeter group: the *weak order* (see the seminal article [**B**]). The weak order on a Coxeter group is a partial order on the elements of this group, having many interesting properties. It appears, for instance, in the study of Symmetric functions ([**S2**] and [**L2**]), Grassmanian geometry and Schubert calculus (see [**FGRS**] and [**BJS**]), or in the study of Cluster algebras (see [**FZ1**], [**FZ2**], [**RS2**] and [**RS1**]).

In this thesis we develop a new theoretic framework for the study of weak order on any Coxeter group, and we apply this to the study of various associated questions. More precisely, we follow a classical method in the study of Coxeter groups: we begin with the study of an enumerative problem fundamentally connected with symmetric group (here, the enumeration of some tableaux); we then extract from the combinatorial construction made at the previous step a general model that we apply to a wider class of Coxeter groups; finally, we use this new framework to study various problems related to the weak order. In this thesis, the first problem we focus on is a set of several conjectures of Dyer about an extension of the weak order, which would allow us to extend combinatorial and geometrical properties usually confined to finite Coxeter groups to all Coxeter groups. Even if these conjectures first appear more than 20 years ago (see [**D1**] and [**D4**]), little is known about them. However, they recently become the object of intensive studies, motivating the study of asymptotical behavious of roots systems (see, for instance, [**DHR**] and [**HLR**]), of abstract constructions (see [**D2**] and [**D3**]) and of geometrical properties of root systems (see [**HL**] and [**L1**]).

One of the possible motivations for the study of Dyer's conjectures is the study of Cambrian lattices (see [**R1**] and [**R3**]). Cambrian lattices are quotients of the weak order of any finite Coxeter groups, which are related to combinatorics of cluster algebras. In particular, the Hasse diagram of a Cambrian lattice is isomorphic to the exchange graph of the cluster algebra associated with the considered Coxeter group (see [**R2**] and [**RS2**]). Their definition can be extended to any Coxeter group (see [**RS3**]), but the correspondence with cluster algebra is only partial. For example, the Hasse diagram of the resulting poset (called Cambrian semi-lattice) is isomorphic to a sub-graph of the corresponding cluster algebra (see [**RS2**]). Even if in the case of affine Coxeter groups Reading and Speyer showed in [**RS1**] how to extend these Cambrian semi-lattices in order to obtain all the expected informations about Cluster algebras, this problem remains open in general. However, Dyer's conjectures provide a candidate for an extension of the weak order on any Coxeter group. Thus, it is natural to look for a way to extend the construction of Cambrian semi-lattice to these extensions, hoping that one could recover more informations about cluster algebras from it. This constitutes the second problem whose study is started in this thesis.

In the sequel of this introduction, we detail the content of the thesis.

In Chapter2, we recall some fundamental results and notations about tableaux, directed graphs (digraphs), partially ordered sets (posets), lattices and Coxeter groups.

Let us begin with introducing the problem that motivated the development of the notion introduced here: the study of balanced tableaux. This family of tableaux has been introduced by Edelman and Greene in [**EG**]. They are Ferrers diagrams filled with all the integers from 1 to n, where n is the size of the diagram, satisfying a specific condition on each hook of the diagram. Edelman and Greene showed that, for any partition $\lambda \vdash n$, balanced tableaux and standard tableaux of shape λ are equinumerous. Surprisingly, the proof is simultaneously bijective and a bit unsatisfying. Indeed, it uses the weak order on the symmetric group as intermediary.

Our first goal was to find a bijective proof of the previous result. This originated many of the tools introduced in this thesis. We introduced an algorithm to construct balanced tableaux, starting with an empty Ferrers diagram, filling it box by box with the integers from 1 to n and ending with a balanced tableau.

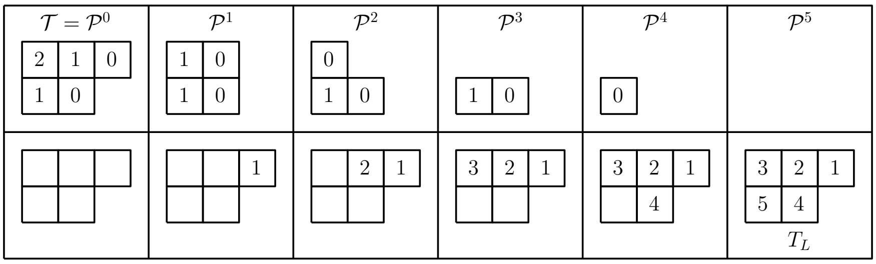


FIGURE 1.1. Example of the action of the algorithm.

Even if all balanced tableaux can be obtained by this way, and despite of the combinatorial character of the algorithm, this approach did not lead to the expected bijection. However, the form of the algorithm naturally leads to a generalization of the concept of balanced tableaux. Hence, we associate to each tableaux (which are not required to be standard or balanced) a combinatorial object called its *type*.

DEFINITION 1. Let $T = (T_{\mathfrak{c}})_{\mathfrak{c}}$ be a Young tableaux of shape $\lambda \vdash n$, the *type* of T is the tableau of shape λ whose each box $\mathfrak{c}$ if filled by the integer $\theta(\mathfrak{c})$ defined by

$$\theta(\mathfrak{c}) := |\{\mathfrak{d} \in S \mid \mathfrak{d} \in H_S(\mathfrak{c}) \text{ et } t_{\mathfrak{c}} < t_{\mathfrak{d}}\}| \text{ for all } \mathfrak{c} \in S,$$

where $H_S(\mathfrak{c})$ is the hook based on $\mathfrak{c}$, that is the set of all the boxes being below and on the right of $\mathfrak{c}$ in the Ferrers diagram of λ.

By this way, tableaux are classified according to their type, and balanced and standard tableaux form two particular classes of our classification. We can naturally adapt the algorithm to this new context, and this allows us to generalize Edelman and Greene results to a wider class of tableaux.

More precisely, we show that one can associate a type $\mathcal{T}_\sigma$ with each *vexillary* permutation $\sigma \in S_n$ (*i.e.* avoiding the pattern 2134), whose shape is given by a partition $\lambda(\sigma)$ canonically associated with σ, such that the tableaux of type $\mathcal{T}_\sigma$ and the standard tableaux of shape $\lambda(\sigma)$

7	5	2	0
5	3	0	
3	1		
2	0		
0			

3	2	1	0
2	1	0	
1	0		
1	0		
0			

FIGURE 1.2. Type associated with standard (on the left) and balanced (on the right) tableaux of shape $(4, 3, 2, 2, 1)$.

are equinumerous. It is important to notice that this result does not rely on a direct bijection, and uses the weak order on S_n as an intermediary. Using this connection, we are able to show that the number of tableaux of type $\mathcal{T}_\sigma$ such that the integers $1, 2, \ldots, k$ appear in some fixed position is given by the number of reduced decomposition of an explicit permutation $\omega \in S_n$, generalizing an implicit result of [**FGRS**] about balanced tableaux.

Note that this approach also leads to a new combinatorial model for the study of the reduced decompositions of any permutation. Indeed, one can associate each permutation with a family of tableaux obtained using the algorithm and such that each one of these tableaux is canonically associated with a reduced decomposition of the permutation. We depict in Figure 1.3 the tableaux associated with the reduced decompositions of $[4, 1, 3, 5, 2]$.

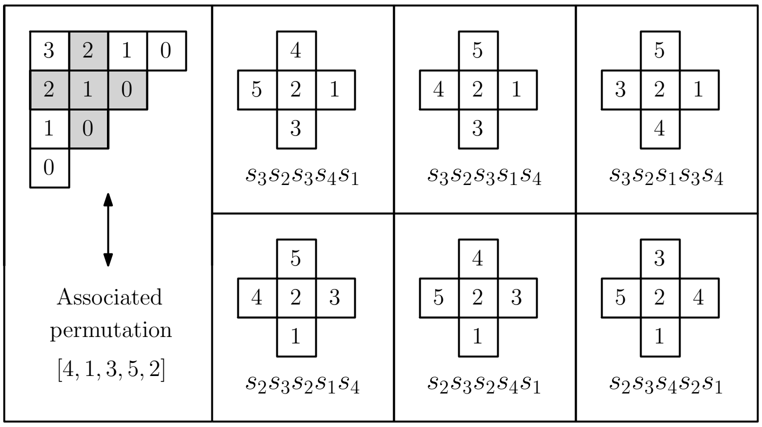


FIGURE 1.3. Reduced decompositions of $[4, 1, 3, 5, 2]$ seen as tableaux.

Since all the results mentioned earlier can be reformulated using the generalizations introduced after, these notions will be developed in Chapter 6.

In Chapter 3 we change our point of view on the objects mentioned earlier. Indeed, the key argument in our study of combinatorics of tableaux is that the algorithm leads to a combinatorial model for the reduced decompositions of any permutation. Thus, it also provides a combinatorial model for the weak order on the symmetric group. This naturally leads us to the following question: is it possible to generalize this model to the weak order on other Coxeter groups ? It appears that our framework using tableaux is not sufficient to describe most of other cases, even if we consider small examples such as the weak order on B_3. We thus have to consider a wider family of objects (containing the notion of tableaux, as it is explained in Chapter 6), namely digraphs.

The fundamental object of this chapter is what we call a *valued digraph* $\mathcal{G} = (G, \theta)$, where G is a simple acyclic digraph (not necessarily finite) and θ is a function from $\mathcal{V}(G)$, the vertices of the digraph, to $\mathbb{N}$, satisfying the following inequality

$$\text{for all } z \in \mathcal{V}(G),\ 0 \leq \theta(z) \leq d^+(z),$$

where $d^+(z)$ denotes the out-degree of the vertex z. In this new context, the algorithm will not be used to construct tableaux any more, but it will be used to generate sequences of elements of $\mathcal{V}(G)$, called *peeling sequences* of $\mathcal{G}$, by peeling the digraph vertex by vertex with respect to the constraint given by the valuation θ. More precisely, we first chose an erasable vertex z of $\mathcal{V}(G)$, that is a vertex such that:

- $\theta(z) = 0$;
- for all $y \in \mathcal{V}(G)$, if there is an arc from y to z in G, then $\theta(y) > 0$,

We then peel this vertex, that is:

(1) we decrease by one the valuation on each vertex y such that there is an arc from y to z;
(2) we erase the vertex z and the arcs having z as starting or ending point.

What we obtain after this operation is again a valued digraph, so that we can iterate this process. An iteration of this peeling process is depicted in Figure 1.4.

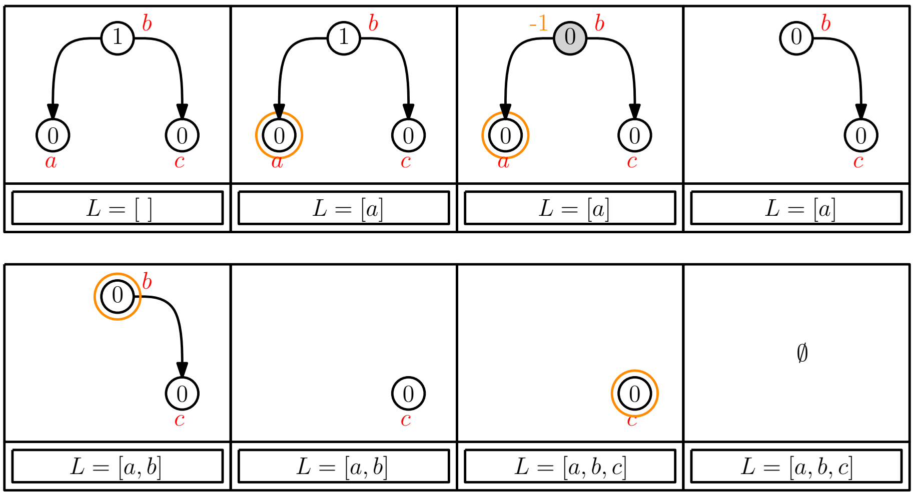


FIGURE 1.4. Peeling process in action. The resulting sequence L is what we call a peeling sequence of $\mathcal{G}$.

We denote by $PS(\mathcal{G})$ the set of all the peeling sequences, and we consider the set $IS(\mathcal{G})$ of all the initial sections of the elements of $PS(\mathcal{G})$, that are the sets of the form $\{z_1, \dots, z_k\}$ where the sequence $[z_1, \dots, z_k]$ is the beginning of a peeling sequence of $\mathcal{G}$. We then show that the poset $(IS(\mathcal{G}), \subseteq)$ has many properties similar to those of the weak order.

THEOREM 1. Let $\mathcal{G} = (G, \theta)$ be a valued digraph, the poset $(IS(\mathcal{G}), \subseteq)$ is a graded meet semi-lattice, with rank function given by $A \mapsto |A|$. Moreover, if G is finite, then $(IS(\mathcal{G}), \subseteq)$ is a complete lattice.

We recall that the Möbius function μ on a locally finite poset $(P, \leq)$ is the function from P^2 to $\mathbb{Z}$ recursively defined by:

- for all $x \in P$, $\mu(x, x) = 1$,
- for all $x, y \in P$, $\mu(x, y) = -\sum_{x \leq c < y} \mu(x, c)$.

THEOREM 2. Let $A \in IS(\mathcal{G})$, $\mathcal{N}(A) := \{x \in A \mid \theta(x) = 0\}$, $\mathcal{F}(A) := \{x \in A \mid A \setminus \{x\} \in IS(\mathcal{G})\}$ and μ be the Möbius function of $(IS(\mathcal{G}), \subseteq)$. Then, we have:

(1) if $\mathcal{F}(A) = \mathcal{N}(A)$, then $\mu(\emptyset, A) = (-1)^{|\mathcal{N}(A)|}$,
(2) else, we have $\mu(\emptyset, A) = 0$.

We then prove that the weak order on Coxeter groups of type A, B and $\widetilde{A}$ can be described within this theory, constructing explicitly an associated valued digraph. This provides an alternative proof of the fact that they are graded meet semi-lattices, a new combinatorial model for their study and a new formula for the values of their associated Möbius functions. Let us explain a little bit the method we follow to prove that the weak order on A_{n-1} can be

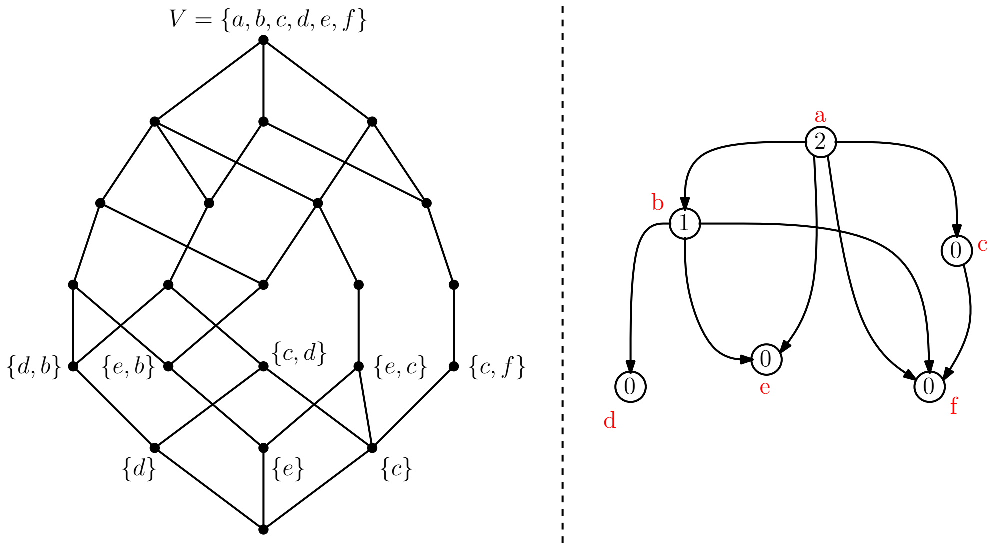


FIGURE 1.5. An example of poset $(IS(\mathcal{G}), \subseteq)$ (on the left) and the associated valued digraph (on the right).

described using a valued digraph (the method in types B and $\widetilde{A}$ is similar). It is well-known that A_{n-1} can be identified with the symmetric group S_n, and there is a canonical set associated with each permutation $\sigma \in S_n$: its inversion set, defined by

$$\mathrm{Inv}(\sigma) := \{(a, b) \in [n]^2 \mid a < b \text{ et } \sigma^{-1}(a) > \sigma^{-1}(b)\}.$$

We then have the following classical property.

$$\text{for all } \sigma, \omega \in S_n, \;\; \sigma \leq_R \omega \text{ if and only if } \mathrm{Inv}(\sigma) \subseteq \mathrm{Inv}(\omega),$$

where $\leq_R$ denotes the weak order on A_{n-1}. We then construct a valued digraph $\mathcal{A}$ whose vertices are indexed by all pairs of integers (a, b) such that $1 \leq a < b \leq n$, and such that $\mathcal{A}$ satisfies the following property:

$$IS(\mathcal{A}) = \{\mathrm{Inv}(\sigma) \mid \sigma \in S_n\},$$

implying immediately that $(IS(\mathcal{A}), \subseteq)$ and $(A_{n-1}, \leq_R)$ are isomorphic. We also show that a generalisation of the weak order to the group of r-colored permutations $G(r, n)$, called the *flag weak order*, introduced by Adin, Brenti and Roichman in [**ABR**] can be described with a valued digraph. It is interesting to note that all previous digraphs can be seen as diagrams endowed with an implicit digraph structure, as depicted on Figure 1.6

However, the example of the down-set lattice of any finite poset is perhaps the one giving the more insights on our construction. Indeed, we show that for all finite poset P, there exists a valued digraph $\mathcal{G}(P)$ whose vertices are the elements of P such that $IS(\mathcal{G}(P))$ is the set of the lower sets of P and $PS(\mathcal{G}(P))$ is the set the linear extensions of P. Thus, our construction can be seen as a generalization of the concept of linear extensions to valued digraphs. But linear extensions are connected to the theory of quasi-symmetric functions thanks to the notion of P-partition (see [**G**] or [**S1**]). Then, it is natural to look for a way to use valued digraphs to define quasi-symmetric functions. This is the point of the end of Chapter 3, in which we show how one can associate a family of quasi-symmetric functions with each element of a poset coming from a valued digraph. In most cases, these quasi-symmetric functions do not seem to be interesting, but sometimes the form of the digraph designates a canonical choice of a quasi-symmetric function within this family. This is the case with the valued digraphs associated with the weak order on types A and $\widetilde{A}$, and we show that the arising series are exactly the Stanley symmetric functions in type A (see [**S2**]) and Lam's generalization in type $\widetilde{A}$ (see [**L2**]), by following a similar method as in [**FGRS**] and [**YY**]. Notice that this description naturally leads to a characterization of Stanley's series as a sum over a set of tableaux generalizing

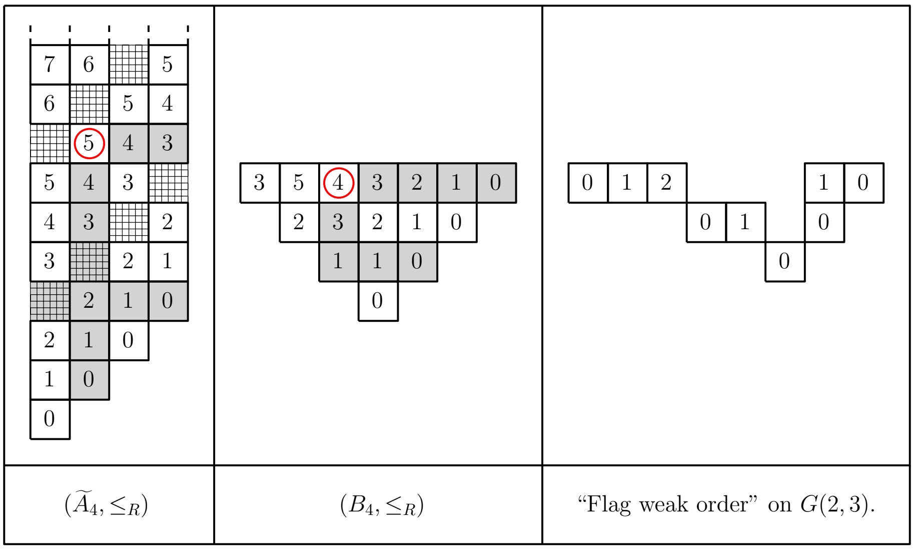


FIGURE 1.6. Valued digraphs associated with some posets. In each case, the digraph structure is given by an adapted notion of hook (see Chapter 3).

the notion of standard and semi-standard tableaux (some of these tableaux are depicted in Figure 1.7)

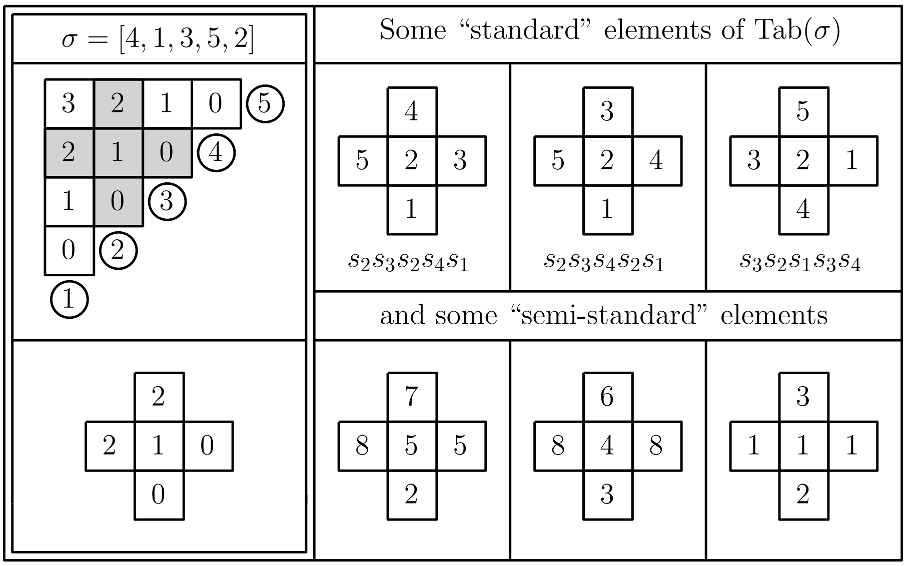


FIGURE 1.7. Some tableaux associated with the permutation $[4, 1, 3, 5, 2]$ and their associated monomials.

It is natural to ask if these previous results can be generalized to any Coxeter group. However, in this larger arena many difficulties appear, and to overcome these we have to change our point of view on the considered objects. Let us first explain the method we followed to construct the valued digraphs associated with Coxeter groups of type B and $\widetilde{A}$. In each case, the method can be split into two steps: first, we find a good notion of inversion set associated with types B and $\widetilde{A}$ (*i.e.* which generalizes the properties of inversion sets of permutations) in order to construct a candidate of valued digraph; then, we use the interpretation of the elements of B_n (resp. $\widetilde{A_n}$) as signed permutations (resp. affine permutations) in order to prove that these valued digraphs provide a description of the weak order by using combinatorial techniques.

In general, we do not have an interpretation of Coxeter groups as a group of permutations (or similar objects), but we have a general notion of inversion set thanks to root systems. Root systems are geometrical objects allowing us to see any Coxeter group as a reflection group, but generally not in an Euclidean space. More precisely, a root system of a Coxeter group W

of rank n is a subset Φ of the vector space $\mathbb{R}^n$ on which W acts and allowing us to interpret geometrically many properties of W. In particular, there exists a partition of Φ into two subsets Φ^+ and Φ^-, respectively called the sets of positive and negative roots, from which we can define a notion of inversion set: for all $\omega \in W$, we set

$$\mathrm{Inv}(\omega) := \Phi^+ \cap \omega(\Phi^-).$$

Then, it is shown in [**BB**] that for all $\tau \in W$, ω is smaller than τ for the weak order if and only if $\mathrm{Inv}(\omega) \subseteq \mathrm{Inv}(\tau)$.

Then, the idea is to construct a valued digraph $\mathcal{G}$ whose vertices are the elements of Φ^+ and such that $IS(\mathcal{G})$ is made exactly of the inversion sets of the elements of W. However, two major difficulties appear: there is no canonical choice for a valued digraph in the case of dihedral groups (many valued digraphs work); when we try to generalize the construction to the case of any Coxeter group, the resulting digraph is generally not acyclic, so that the theory developed in Chapter 3 does not apply. These two remarks lead us to the content of Chapter 4.

In Chapter 4, we begin with generalizing the construction made in the previous chapter, allowing us to study the case where the valued digraph is not acyclic. This new construction is a natural consequence of the content of Chapter 3. Indeed, in order to simplify some proofs we introduced a characterization of the elements of $IS(\mathcal{G})$, which does not rely on peeling sequences and does not rely on the properties of the digraph (such as acyclicity). Therefore, we can take this characterization as a definition of $IS(\mathcal{G})$ and remove the conditions on the digraph. More precisely, we have the following definition.

Definition 2. Let $\mathcal{G} = (G, \theta)$ be a couple of a digraph G together with a function $\theta : \mathcal{V}(G) \longrightarrow \mathbb{N}$ such that

$$0 \le \theta(z) \le d^+(z) \text{ pour tout } z \in \mathcal{V}(\mathcal{G}).$$

We denote by $IS(\mathcal{G})$ the set defined by

$$IS(\mathcal{G}) := \left\{ A \subseteq \mathcal{V}(\mathcal{G}) \;\middle|\; \begin{array}{l} \theta(z) \le d^+_A(z) \text{ for all } z \in A \\ \theta(z) \ge d^+_A(z) \text{ for all } z \in \mathcal{V}(\mathcal{G}) \setminus A \end{array} \right\},$$

where $d^+_A(z)$ is the number of arcs in G having z as starting point and an element of A as ending point.

This definition generalizes the one proposed in Chapter 3, but note that a subtlety appears. When the digraph is infinite, simple and acyclic, then with this definition $IS(\mathcal{G})$ contains sets being infinite (for example, $\mathcal{V}(\mathcal{G})$ is always in $IS(\mathcal{G})$), and these sets can not be defined using initial sections of peeling sequences. In this more general framework, we have the following theorem.

Theorem 3. For all valued digraph $\mathcal{G}$, $(IS(\mathcal{G}), \subseteq)$ is a complete lattice.

In particular, this provides a natural embedding of posets of Chapter 3 into a complete lattice (when the underlying digraph is infinite).

This last point is particularly important for the sequel of this chapter. Indeed, Dyer conjectured in [**D1**] and [**D4**] that it is possible to extend the weak order on any Coxeter group into a complete lattice. This conjecture uses the notion of *bi-closed sets* of Φ^+. We say that a set $A \subseteq \Phi^+$ is *closed* if and only if for all $\alpha, \beta, \gamma \in \Phi^+$ such that $\gamma = a\alpha + b\beta$, with $a > 0$ and $b > 0$ we have that

$$\text{if } \alpha, \beta \in A, \text{ then } \gamma \in A.$$

We say that A is bi-closed if and only if both A and $\Phi^+ \setminus A$ are closed, and we denote by $\mathcal{B}(\Phi^+)$ the set of bi-closed sets of Φ^+. It is known that finite bi-closed sets are exactly the inversion sets of the elements of W (see, for instance, [**P**]). Thus, the poset $(\mathcal{B}(\Phi^+), \subseteq)$ is an extension of the weak order on W, and Dyer's conjecture says that this poset is a complete lattice. We represent in Figure 1.8 the poset $(\mathcal{B}(\Phi^+), \subseteq)$ obtained considering bi-closed sets of a root system of the infinite dihedral group.

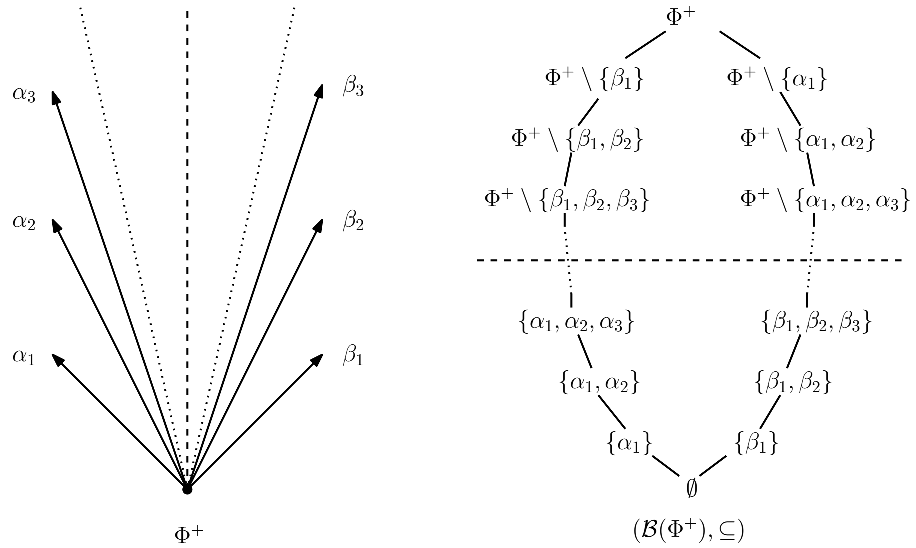


FIGURE 1.8. On the right, the poset $(\mathcal{B}(\Phi^+), \subseteq)$ obtained considering a root system of the infinite dihedral group (on the left).

Therefore, the idea is to look for a valued digraph $\mathcal{G}$ having Φ^+ as set of vertices and such that $IS(\mathcal{G}) = \mathcal{B}(\Phi^+)$. In the case of the dihedral group, this new constraint restricts considerably the available choices and leads us to introduce a new family of valued digraph that we call *scaffoldings*. On Figure 1.9 we depict the lattice obtained considering an infinite scaffolding. We see that the lattice obtained with this scaffolding and the poset of the bi-closed

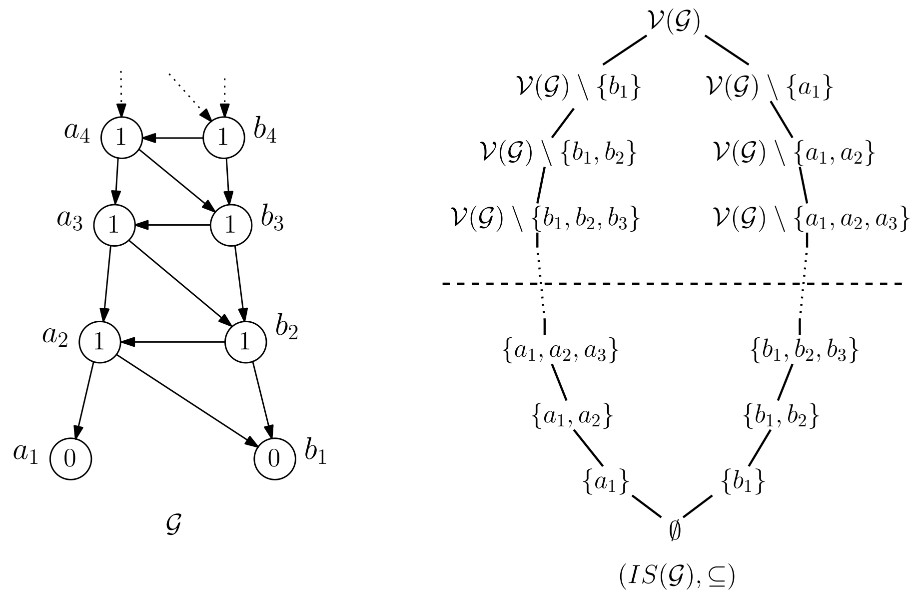


FIGURE 1.9. On the left, an infinite scaffolding $\mathcal{G}$; on the right, the resulting lattice $(IS(\mathcal{G}), \subseteq)$.

sets of the infinite dihedral group are isomorphic, identifying α_i with a_i and β_i with b_i. Using this information, we use scaffoldings (more precisely, we glue scaffoldings together with respect to some constraint given by the geometry of the root system) in order to construct a family of valued digraph that we call *well-assembled on* Φ^+. We then show that for all well-assembled on Φ^+ valued digraph $\mathcal{G}$, we have $\mathcal{B}(\Phi^+) \subseteq IS(\mathcal{G})$ and for all $A \in IS(\mathcal{G})$, we have $\Phi^+ \setminus A \in IS(\mathcal{G})$. Thus, we have an embedding of $(\mathcal{B}(\Phi^+), \subseteq)$ into a complete ortho-lattice.

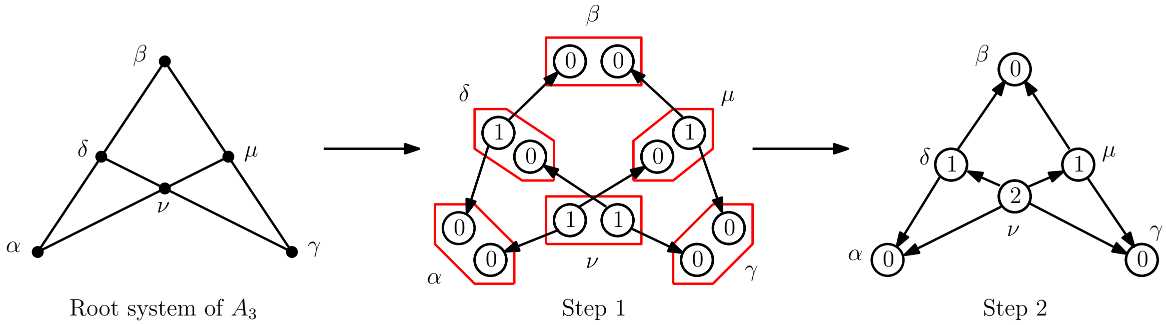


FIGURE 1.10. Construction of a valued digraph which is well-assembled on a root system of A_3. One can check that the resulting lattice is isomorphic to $(A_3, \leq_R)$. See Chapter 4 for more details about this construction.

One of the main difficulties appearing at this stage is the infinite character of the considered valued digraphs. Indeed, it is quite technical to make tests in order to identify the elements of $IS(\mathcal{G})$, and this problem leads us to the definition of a new family of valued digraph, namely *projective valued digraphs*. These valued digraphs (denoted by $\mathcal{G}_\infty$) are valued digraphs that can be seen as a limit (in a certain sense) of a sequence $(\mathcal{G}_i)_{i\geq 1}$ of finite acyclic valued digraphs. In this case, we prove that the family $(IS(\mathcal{G}_i))_{i\geq 1}$ is a projective system (for a well-chosen set of projections), and that $IS(\mathcal{G}_\infty)$ can be seen as the projective limit of this projective system. This allows us to study the lattice structure of $(IS(\mathcal{G}_\infty), \subseteq)$ through the finite lattices $(IS(\mathcal{G}_i), \subseteq)$ and to make effective tests.

Another interesting point about projective valued digraphs is their similarity with the finite acyclic case. Indeed, we can associate each projective valued digraph $\mathcal{G}_\infty$ with a set of total orderings of the vertices of $\mathcal{G}_\infty$, which we denote by $PS(\mathcal{G}_\infty)$, generalizing the notion of peeling sequences. That is, each element of $IS(\mathcal{G}_\infty)$ is an initial section of an element of $PS(\mathcal{G}_\infty)$. Furthermore, $PS(\mathcal{G}_\infty)$ totally encodes maximal chains of $(IS(\mathcal{G}_\infty), \subseteq)$: for all chain $\mathcal{C}$ of $(IS(\mathcal{G}_\infty), \subseteq)$, we have that $\mathcal{C}$ is maximal if and only if there exists $I \in PS(\mathcal{G}_\infty)$ such that $\mathcal{C}$ equals the set of the initial sections of I. This property leads us to consider another conjecture of Dyer (see [**D4**]) about the structure of maximal chains of $(\mathcal{B}(\Phi^+), \subseteq)$. Indeed, Dyer conjectures that for all chain $\mathcal{C}$ of $(\mathcal{B}(\Phi^+), \subseteq)$, we have that $\mathcal{C}$ is maximal if and only if there exists a reflection orderings L of Φ^+ such that $\mathcal{C}$ equals the set of the initial sections of L. Thus, in the sequel of Chapter 4 we prove that there exists a projective well-assembled on Φ^+ valued digraph, and we prove that for each such valued digraph $\mathcal{G}_\infty$ we have that each reflection orderings of Φ^+ is in $PS(\mathcal{G}_\infty)$. Then, we show that in this context the two conjectures of Dyer are equivalent: we have that $\mathcal{B}(\Phi^+) = IS(\mathcal{G}_\infty)$ if and only if $PS(\mathcal{G}_\infty)$ is the set of the reflections orderings of Φ^+. Moreover, the projective structure allows us to perform many tests, and we conjecture that there exists at least one projective well-assembled on Φ^+ valued digraph $\mathcal{G}_\infty$ such that $\mathcal{B}(\Phi^+) = IS(\mathcal{G}_\infty)$.

The end of this chapter is devoted to the study of some applications of our construction. We begin with a quick examinations of connections existing between our theory and abstract convex geometry. In particular, we introduce several closure operators, and we show some of their properties.

Using one of these closure operator, we then introduce some results in the direction of the extension of Cambrian semi-lattices mentioned earlier in the introduction. More precisely, we show how one can associate to each Cambrian semi-lattice a valued digraph, and we prove that the Cambrian semi-lattice is a sub-poset of the resulting lattice. We observe on various examples on type A that this lattice is isomorphic to the associated Cambrian lattice, but we do not have an explanation of this fact yet.

In Chapter 5, we study the particular case of the Tamari lattice, introducing a valued digraph denoted by $\mathcal{A}_n^{\updownarrow}$ and showing that the lattice $(IS(\mathcal{A}_n^{\updownarrow}), \subseteq)$ is isomorphic to the $(n+1)$-th Tamari lattice. Our proof is based on the construction of a one-to-one correspondence between $IS(\mathcal{A}_n^{\uparrow})$ and the set of Dyck paths of size $n+1$, allowing us to provide a similar description of m-Tamari lattices.

CHAPTER 2

Preliminaries and notations

In this chapter we give some definitions and basic properties we will use in this thesis.

2.1. Tableaux, digraphs and lattices

For further informations about the objects introduced in this section, we refer the reader to [**S3**].

2.1.1. Partitions and tableaux. A *partition* λ of a nonnegative integer $n \in \mathbb{N}$, is a nonincreasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ such that $\sum \lambda_i = n$. The integers $\lambda_i \neq 0$ are called *parts* of the partition λ. The *Ferrers diagram* of λ is a finite collection of boxes, or cells, arranged in left-justified rows of lengths given by the parts of λ. By flipping this diagram over its main diagonal we obtain the diagram of the *conjugate partition* of λ denoted λ'. We usually identify a partition with its Ferrers diagram.

More generally, in this thesis we work with diagrams of arbitrary shape, namely finite subsets of $\mathbb{N} \times \mathbb{N}$, without any constraint: let $S \subset \mathbb{N} \times \mathbb{N}$ such that $|S| = n$ (where $|S|$ denotes the cardinal of S). A *tableau* T of shape S is a bijective filling of S (seen as a set of boxes) with entries in $[n] := \{1, 2, \ldots, n\}$. Given a tableau we denote its shape by $\mathrm{Sh}(T)$. The set of all tableaux of shape S will be denoted by Tab_S. If we require $\mathrm{Sh}(T)$ to be a partition λ, then T will be what is usually called a *Young tableau*. Moreover, if we consider Young tableaux satisfying the conditions that the filling is

(1) increasing from left to right across each row;
(2) increasing down each column;

we obtain the set of *standard Young tableaux* of shape λ, denoted by $\mathrm{SYT}(\lambda)$.

Definition 2.1.1. Let S be a diagram and $\mathfrak{c} = (a, b)$ be a box of S. We define the following sets,

$$L_{a,b}(S) = \{\ (k, b) \mid k \geq a, (k, b) \in S\},\ \ A_{a,b}(S) = \{\ (a, k) \mid k > b, (a, k) \in S\}, \tag{2.1}$$

$$H_{a,b}(S) = A_{a,b} \biguplus L_{a,b}, \tag{2.2}$$

respectively called the *leg*, the *arm*, and the *hook* based on (a, b). We will denote by $l_{a,b}(S)$, $\mathsf{a}_{a,b}(S)$, and $h_{a,b}(S)$ their respective cardinalities.

This notion of hook allows us to enumerate standard Young tableaux, thanks to the well-known *hook-length formula* (see [**S3**] for more details about this formula).

Theorem 2.1.2. *Let λ be a partition of the integer n, seen as a diagram. Then, we have*

$$|\mathrm{SYT}(\lambda)| = \frac{n!}{\prod_{(a,b)\in\lambda} h_{a,b}(\lambda)}.$$

In [**EG**], Edelman and Greene introduced the concept of balanced tableaux, defined as follows.

Definition 2.1.3. Let λ be a partition of the integer n. A *balanced tableau* $T = (t_{a,b})_{(a,b)\in\lambda}$ of shape λ is a Young tableau satisfying the following condition:

$$\text{for all } (a, b) \in \lambda,\ \mathsf{a}_{a,b} = |\{(x, y) \in H_{a,b}(\lambda) \mid t_{x,y} > t_{a,b}\}|.$$

We denote by $\mathrm{Bal}(\lambda)$ the set of all balanced tableaux of shape λ

In the same paper, Edelman and Greene proved the following result, which is the main motivation for the content of Chapter 6.

THEOREM 2.1.4. *For all $\lambda \vdash n$, we have* $|\mathrm{Bal}(\lambda)| = |\mathrm{SYT}(\lambda)|$.

2.1.2. Digraphs.

DEFINITION 2.1.5. A *digraph* is a couple $G = (V, E)$, where V is the set of *vertices* of G (not necessarily finite), and E is a multi-set E of elements of $V \times V$, called the set of *arcs* of G.

In this thesis, we are particularly interested in a special family of digraphs, that we define below.

DEFINITION 2.1.6. We say that a digraph $G = (V, E)$ is *simple* if and only if E is a set and for all $z \in V$, $(z, z) \notin E$. An *oriented cycle* (also called a cycle) of G is a sequence of vertices $z_1, z_2, \ldots, z_n$ such that $(z_i, z_{i+1}) \in E$ for all i, where the indices are taken modulo n. A digraph which does not have any cycle is called *acyclic*.

We finish with defining a useful statistic on the vertices of a digraph.

DEFINITION 2.1.7. Set $G = (V, E)$ a digraph, $A \subseteq V$, and $z \in V$. We define the following statistic

$$d^+_A(G, z) := \sum_{\substack{y \in A \\ (z,y) \in E}} M_{(z,y)},$$

where $M_{(z,y)}$ denotes the *multiplicity* of the arc (z, y) in E. Note that its value can be infinite when the underlying digraph is infinite. In the special case where $A = V$, the obtained statistic is called the *out-degree* of z, denoted $d^+(z)$.

2.1.3. Posets and lattices.

DEFINITION 2.1.8. A *poset* is a pair $(P, \leq)$ where P is a set and $\leq$ is a binary relation on P being:

- reflexive, *i.e.* for all $x \in P$, $x \leq x$;
- transitive, *i.e.* for all x, y, z in P, if $x \leq y$ and $y \leq z$, then $x \leq z$;
- anti-symmetric, *i.e.* for all x, y in P, if $x \leq y$ and $y \leq x$, then $x = y$.

We say that a poset $(P, \leq)$ is *locally finite* if and only if for all $x, y \in P$, the set $\{z \in P \mid x \leq z \leq y\}$ is finite. For any locally finite poset, one can define a function $\mu : P \times P \to \mathbb{N}$, called its *Möbius function*, to be the unique application satisfying the following conditions:

- for all $x \in P$, $\mu(x, x) = 1$;
- for all $x, y \in P$, $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$.

Let $(P, \leq)$ be a poset and X be a subset of P, a *lower bound* (resp. *upper bound*) of X in P is an element $z \in P$ such that for all $x \in X$, we have $z \leq x$ (resp. $x \leq z$). If there exists $z \in P$ such that for all lower bound y of X we have $y \leq z$, then z is necessarily unique and is called the *infimum* (or the *meet*) of X, usually denoted by $\wedge X$. Similarly we have the dual notion of *supremum* (also called the *join*), that is an element $x \in P$ such that for all upper bound y of X we have $x \leq y$, which is usually denoted by $\vee X$. We say that $(P, \leq)$ is a *complete meet semi-lattice* (resp. *complete join semi-lattice*) if and only if any subset of P admits an infimum (resp. a supremum). Finally, we say that a poset which is both a complete meet and join semi-lattice is a *complete lattice*.

In Chapter 4, we will mainly focus on some specific complete lattices, namely complete *ortho-lattices*. We say that a complete lattice $(P, \leq)$ is a complete ortho-lattice if and only if there exists an application $x \mapsto x^{\perp}$ from P to itself, called the *ortho-complement* of P, such that for all $x, y \in P$ we have that:

- the supremum of $\{x, x^{\perp}\}$ (usually denoted by $x \vee x^{\perp}$) is the maximal element of P;
- the infimum of $\{x, x^{\perp}\}$ (usually denoted by $x \wedge x^{\perp}$) is the minimal element of P;
- if $x \leq y$, then $y^{\perp} \leq x^{\perp}$;
- $x^{\perp\perp} = x$.

We say that $(P, \leq)$ is bounded if and only if there are a maximal and a minimal element in P, *i.e.* P itself has a supremum and an infimum. This notion generally allows us to simplify the proof of the fact that a given poset is a complete lattice, thanks to the following classical lemma.

LEMMA 2.1.9. *If $(P, \leq)$ is a bounded complete meet (or join) semi-lattice, then it is a complete lattice.*

PROOF. Let $X \subseteq P$ and C be the set of the upper bounds of X, which is non empty since $(P, \leq)$ is bounded. We denote by z the infimum of C. By definition, any element x of X is a lower bound of C, so that $x \leq z$ by definition of the infimum. Thus, z is the join of X, so that P is a complete lattice. The proof in the case where $(P, \leq)$ is a join semi-lattice is similar. $\square$

We now give two classical notions of poset theory, which will be useful in Chapters 3 and 4.

DEFINITION 2.1.10. A subset A of P is called a *lower set* of $(P, \leq)$ if and only if for all x, y in P, if $y \in A$ and $x \leq y$, then $x \in A$.

DEFINITION 2.1.11. A poset $(L, \preceq)$ is called a *total order* if and only if for all x and y in L, we have either $x \preceq y$, or $y \preceq x$. The lower sets of a total order are called *initial sections* of L. Classical examples of total orders are given by the *linear extensions* of any finite poset $(P, \leq)$, which we recall are total orders $(P, \preceq)$ such that $x \preceq y$ whenever $x \leq y$.

We conclude with a classical lemma connecting lower sets of a finite posets to its linear extensions (see [**S3**]).

LEMMA 2.1.12. *The lower sets of any finite poset are exactly the initial sections of its linear extensions.*

PROOF. This can be easily proved by induction on the cardinality of P. $\square$

2.2. Coxeter groups and weak order

In this section we recall some basic properties and definitions about Coxeter groups and their associated root systems. We refer the reader to [**BB**] and [**H**] for more details about these topics.

2.2.1. Coxeter groups and weak order. Let S be a set (usually finite) and $M = (m_{s,t})_{s,t \in S}$ a *Coxeter matrix*, that is a matrix which takes values into $\mathbb{Z}_{\geq 1} \cup \{\infty\}$, which is symmetrical and such that $m_{s,s} = 1$ for all $s \in S$. The *Coxeter group* W associated with S and M is the group generated by S and subject only to the relations $(st)^{m_{s,t}} = Id$, for any s and t in S. By convention, when $m_{s,t} = \infty$ the previous relation just means that st is not of finite order. The couple (W, S) is called a *Coxeter system*. The cardinality of S is called the *rank* of W. It is convenient to represent such a Coxeter system (W, S) by its *Coxeter graph*, that is the non-oriented graph whose vertices are the elements of S, with an edge between s and t if and only if $m_{s,t} \geq 3$, and such that edges are labelled by $m_{s,t}$ whenever $m_{s,t} > 3$. Note that the finite Coxeter groups are completely classified, and we recall in Figure 2.1 the complete list of the Coxeter graph of each finite irreducible Coxeter group.

To each element ω in W we can associate its *length*, denoted by $\ell(\omega)$, and defined by

$$\ell(\omega) := \min\{k \in \mathbb{N} \mid \omega = s_1 \cdots s_k,\ s_i \in S\}.$$

This statistic is the rank function of a graded poset structure on W, namely the *(right) weak order* on W. That is, the reflexive and transitive closure of the relation

$$\forall \omega \in W,\ s \in S,\ \omega \leq_R \omega.s \text{ if and only if } \ell(\omega) < \ell(\omega.s).$$

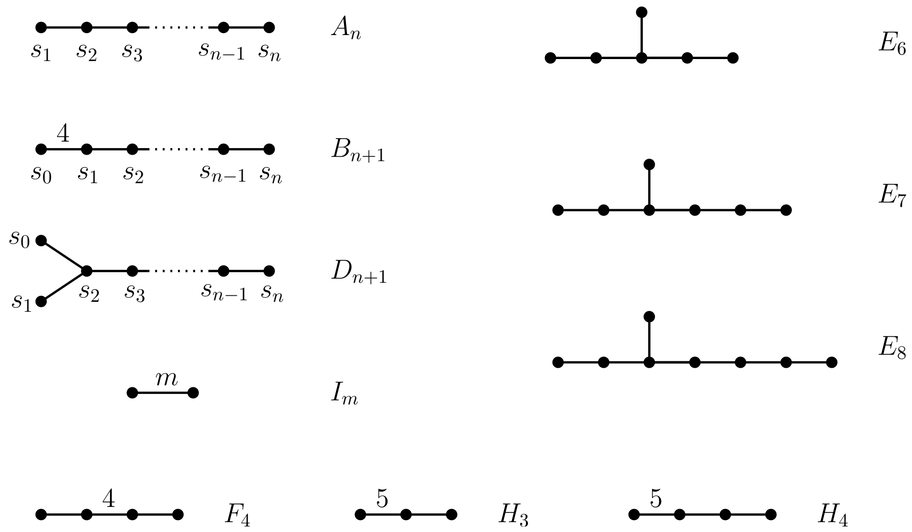


FIGURE 2.1. Coxeter graphs of all finite Coxeter groups.

In [**B**], Björner proved the following general result about this poset.

THEOREM 2.2.1. *The poset $(W, \leq_R)$ is a graded meet semi-lattice in general, and a graded complete lattice when W is finite.*

Finally, we define the *reduced decompositions* of any $\omega \in W$ to be the elements of the set

$$\mathrm{Red}(\omega) := \{(i_1, \ldots, i_{\ell(\omega)}) \mid s_j \in S \text{ and } s_{i_1} \cdots s_{\ell(\omega)} = \omega\}.$$

As usual, most of the time we will say that $s_{i_1} \cdots s_{\ell(\omega)}$ is a reduced decomposition, instead of $(i_1, \ldots, i_{\ell(\omega)})$.

CHAPTER 3

A family of posets defined from simple acyclic digraphs

This chapter comes from the article **[V]** *"A new family of posets generalizing the weak order on some Coxeter groups" available at http://arxiv.org/pdf/1508.06141v2.pdf.*

Introduction

The weak order on a Coxeter group W is a partial order on W which plays a significant role in many areas of algebra and algebraic combinatorics as Grassmannian geometry, and Schubert polynomials (see [**FGRS**]). Moreover, it is closely related to the geometry of the root system associated with a Coxeter group (see [**D4**], [**DH**], [**P**], or [**DHR**]), and to the theory of quasi-symmetric functions (see [**BM**] for a general survey) thanks to the Stanley symmetric functions. These functions were introduced by Stanley in [**S2**], in order to enumerate the reduced decompositions of any permutation σ in the symmetric group S_n, equivalently enumerating the maximal chains from the identity to σ in the weak order on S_n, and turned out to be of fundamental importance in many areas of algebra (see [**BJS**]). In [**L2**], Lam generalized Stanley's work to the affine Coxeter group of type A.

In this chapter we introduce a new family of posets, defined from a digraph together with a valuation on its vertices. Here, we focus exclusively on the case where the digraph is simple and acyclic, in which case the corresponding poset has a rich combinatorial structure. It appears that many well-known posets can be described within this theory, and after a careful case by case study, we show that the weak order on Coxeter groups of type A, $\widetilde{A}$ and B, the flag weak order on wreath product $\mathbb{Z}_r \wr S_n$ (see [**ABR**]), and the up-set (resp. down-set) lattice of any finite poset, admit such a description.

The study of this family of posets will be further developed in Chapter 4, in which we will show how they can be used to study two long-standing conjectures of Matthew Dyer on the geometry of root systems in infinite Coxeter groups (see [**D1**] and [**D4**]). Moreover, in another subsequent publication we will highlight connections which exist between our theory and Tamari and Cambrian lattices. Note that except for the case of Coxeter groups of type A, the content of Chapter 4 will not overlap the content of the current chapter. Indeed, Chapter 4 is mainly centred on algebraic and geometric aspects of this construction, while here we develop the combinatorial ones: we give a new formula for the values of the Möbius function and we provide a new combinatorial model for the maximal chains in the weak order on Coxeter groups of type A, B and $\widetilde{A}$.

Our construction relies on a generalization of the notion of linear extension of a finite poset to simple acyclic digraph, and leads us to associate quasi-symmetric functions with each element of our posets, as it is the case for linear extensions in the context of P-partitions (see [**S1**]). It seems that most of the functions associated with an element do not give much insights about the underlying poset structure. However, the form of the underlying digraph sometimes leads to a canonical choice among these quasi-symmetric functions, which occurs when considering the digraphs associated with types A and $\widetilde{A}$. In these two cases, we show (following a similar method as in [**FGRS**] and [**YY**]) that the canonical quasi-symmetric functions which arise are exactly the Stanley and Lam's symmetric functions.

In the author's opinion, Section 3.4 just scratch the surface of the connections existing between our construction and quasi-symmetric functions presented in and would require a more exhaustive study. Moreover, our results suggest that the construction presented in this chapter could be generalized so as to obtain the weak order on any Coxeter group (see Chapter 4), and thus may lead to a generalization of Stanley symmetric functions to a wider class of Coxeter groups. A good starting point would be to look for a combinatorial description of type B Stanley symmetric functions, using the digraph introduced in this chapter.

The Chapter is organised as follows: in Section 3.1 we define the family of posets from *valued digraphs*, which are couples of a simple acyclic digraph together with a valuation on its vertices. We exhibit some general properties of these posets, namely they are *graded complete meet semi-lattices* in the general case, *graded complete lattices* when the underlying digraph is finite, and we give a simple and explicit formula to compute the values of their *Möbius function*. In Section 3.3, we show that the (right) weak order on Coxeter groups A_{n-1}, B_n and $\widetilde{A}_n$, the flag weak order on $\mathbb{Z}_r \wr S_n$, and the up-set (resp. down-set) lattice of any finite poset can be described thanks to this theory. In Section 3.4, we exhibit a link between these posets and the theory of quasi-symmetric functions. More precisely, we explain how the series associated with any *P-partition* (see **[BM]**), the Stanley symmetric functions, and Lam's generalization naturally arise from this description.

3.1. Definition of a new family of posets

We present here a method to obtain all lower sets of a finite poset $\mathcal{P}$. Since $\mathcal{P}$ is finite, there exists a_1 in $\mathcal{P}$ which is a minimum, that is if $x \leq a_1$ in $\mathcal{P}$, then $x = a_1$. Let $\mathcal{P}_2 = (P \setminus \{a_1\}, \leq)$ be the finite poset obtained by removing a_1 from $\mathcal{P}$. Then, there exists a_2 in $\mathcal{P}_2$ which is a minimum and we can define the poset $\mathcal{P}_3$ obtained by removing a_2 in $\mathcal{P}_2$, and so on. Finally, we end with an injective sequence $[a_1, \ldots, a_n]$ of elements of $\mathcal{P}$. This sequence is a linear extension of $\mathcal{P}$ by construction. Furthermore, one can easily prove by induction that all the linear extensions of $\mathcal{P}$ can be obtained by this way.

In a certain sense, this method "peels" a finite poset element by element, in order to obtain a family of sequences which give rise to an interesting family of sets (here, the lower sets). Here, we propose to apply a similar principle to a simple acyclic digraph. Namely, we will peel the digraph vertex by vertex, with respect to a constraint given by a valuation on its vertices. It will give rise to a family of sequences of vertices of the graph, then to a family of subsets of vertices having an interesting poset structure once ordered by inclusion.

We start with the definition of the valuation on the vertices of a simple acyclic digraph.

DEFINITION 3.1.1. Let $G = (V, E)$ be a simple acyclic digraph. A valuation $\theta : V \to \mathbb{N}$ is called an *out-degree compatible valuation* on G (OCV) if and only if for all $x \in V$, we have

$$0 \leq \theta(x) \leq d^+(x).$$

A pair $\mathcal{G} = (G, \theta)$, where G is a simple acyclic digraph and θ is an OCV, is called a *valued digraph*.

In what follows, $\mathcal{G} = (G, \theta)$ will denote a valued digraph. Recall that our aim is to generalize the method which peels finite posets to valued digraphs. Thus, we first need to specify which vertices of a valued digraph can be peeled. This is the point of the following definition.

DEFINITION 3.1.2 (Erasable vertex). A vertex x of G is called *erasable* in $\mathcal{G}$ if and only if:

- $\theta(x) = 0$;
- for all $z \in V$ such that $(z, x) \in E$, we have $\theta(z) \neq 0$.

We now introduce the *peeling process*, which is indeed a generalisation of the process on finite posets presented in the introduction of this section.

DEFINITION 3.1.3 (Peeling process and peeling sequences). Given $\mathcal{G} = (G, \theta)$ a valued digraph, we construct recursively two sequences: a sequence $L = [x_1, x_2, \ldots]$ of elements of V, and a sequence $(\mathcal{G}_i = (G_i, \theta_i))_{1 \leq i}$ of valued digraphs as follows.

(1) Let $\mathcal{G}_1 = \mathcal{G}$.

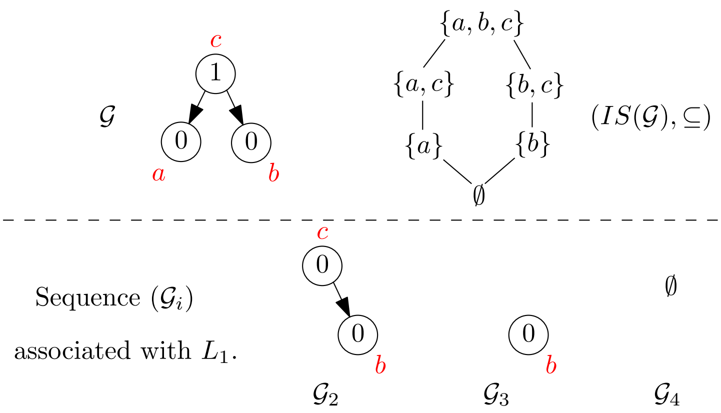


FIGURE 3.1.

(2) If there is not any erasable vertex in $\mathcal{G}_i$, the process stop. Otherwise, choose x a vertex of G_i which is erasable in $\mathcal{G}_i$, and set $x_i = x$.
   (a) Let G_{i+1} be the simple acyclic directed graph obtained by removing the vertex x_i in G_i and all the arcs of the form (z, x_i) or (x_i, z) in G_i.
   (b) Let θ_{i+1} be the OCV on G_{i+1} such that $\theta_{i+1}(y) = \theta_i(y) - 1$ if (y, x_i) is an arc of G_i, and $\theta_{i+1}(y) = \theta_i(y)$ otherwise. Then set $\mathcal{G}_{i+1} = (G_{i+1}, \theta_{i+1})$ and iterate Step 2.

A sequence L arising from this process is called a *peeling sequence* of $\mathcal{G}$, and we denote $PS(\mathcal{G})$ the set of all peeling sequences of $\mathcal{G}$.

Recall that the lower sets of any finite poset $\mathcal{P}$ are the initial sections of some linear extension $\mathcal{P}$, and we can extends this notion to valued digraph in the natural way.

DEFINITION 3.1.4. Let $L = [x_1, x_2, \ldots]$ be a peeling sequence of $\mathcal{G}$. The *initial sections* of L are the sets of the form $\{x_1, x_2 \ldots, x_k\}$, $k \in \mathbb{N}^*$. By convention, $\emptyset$ is an initial section of L. The set of the initial sections of all the peeling sequences of $\mathcal{G}$ will be denoted by $IS(\mathcal{G})$.

Finally, recall that the lower sets of any finite poset gives rise, once ordered by inclusion, to a classical lattice called its down-set lattice. Once again, this concept naturally generalizes to valued digraphs, and the posets we will consider all along this Chapter are the posets $(IS(\mathcal{G}), \subseteq)$ for some valued digraphs $\mathcal{G}$.

EXAMPLE 3.1.0.1. Consider $\mathcal{G}$ as depicted in the upper left corner of Figure 3.1. The peeling sequences of $\mathcal{G}$ are $L_1 = [a, c, b]$ and $L_2 = [b, c, a]$, thus $IS(\mathcal{G}) = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

We finish this section with stating our main results concerning the properties of $(IS(\mathcal{G}), \subseteq)$. The proofs are given in Section 3.2.

THEOREM 3.1.5. *Let $\mathcal{G} = (G, \theta)$ be a valued digraph, the poset $(IS(\mathcal{G}), \subseteq)$ is a graded complete meet semi-lattice, and its rank function is $\rho : A \to |A|$. Moreover, if G is finite, then $(IS(\mathcal{G}), \subseteq)$ is a complete lattice.*

We also have an explicit formula for the values of the Möbius function of $(IS(\mathcal{G}), \subseteq)$. For the sake of clarity, we give the formula only for the couples of the form $(\emptyset, A)$, $A \in IS(\mathcal{G})$, but a similar one can be stated for all couples in $IS(\mathcal{G})$.

THEOREM 3.1.6. *Let $A \in IS(\mathcal{G})$, $\mathcal{N}(A) = \{x \in A \mid \theta(x) = 0\}$ and $\mathcal{F}(A) = \{x \in A \mid A \setminus \{x\} \in IS(\mathcal{G})\}$, we have two cases.*

(1) If $\mathcal{F}(A) = \mathcal{N}(A)$, then $\mu(\emptyset, A) = (-1)^{|\mathcal{N}(A)|}$.
(2) Otherwise, $\mu(\emptyset, A) = 0$.

Let us briefly explain how one can use Theorem 3.1.6 to compute all the values of the Möbius function. Let $A, B \in IS(\mathcal{G})$ such that $A \subseteq B$, and denote by $\mathcal{G}'$ the valued digraph obtained after we peeled all the vertices of A using the peeling process. It is straightforward to check using Proposition 3.2.1 (see Section 6.3.3 below) that $B \setminus A \in IS(\mathcal{G}')$, and that the intervals $[A, B]_{\mathcal{G}}$ and $[\emptyset, B \setminus A]$ are isomorphic. Therefore, if we denote by μ' the Möbius function of $(IS(\mathcal{G}'), \subseteq)$ then we have

$$\mu(A, B) = \mu'(\emptyset, B \setminus A).$$

3.2. Proofs

In this section, we provide the proofs of Theorems 3.1.5 and 3.1.6. Both proofs rely on an intrinsic characterization of the elements of $IS(\mathcal{G})$ (Proposition 3.2.1) and on a technical lemma (Lemma 3.2.3), which we give here.

PROPOSITION 3.2.1. *Let $\mathcal{G} = (G, \theta)$ be a valued digraph and A be a finite subset of vertices of G, then $A \in IS(\mathcal{G})$ if and only if:*

(1) for all $x \in A$, $\theta(x) \leq |\{\ y \mid y \in A \text{ and } (x, y) \in E\}|$;
(2) for all $x \in V \setminus A$, $\theta(x) \geq |\{\ y \mid y \in A \text{ and } (x, y) \in E\}|$.

PROOF. Assume that $A \in IS(\mathcal{G})$, *i.e.* there exists $L = [x_1, x_2 \ldots] \in PS(\mathcal{G})$ and $k \in \mathbb{N}^*$ such that $\{x_1, \ldots, x_k\} = A$. Let x be a vertex of G, then we divide our study into two cases.

- If $x \in A$, then there exists $i \leq k$ such that $x_i = x$, and we obviously have
$$\{\ x_j \mid j < i \text{ and } (x_i, x_j) \in E\} \subseteq \{\ y \mid y \in A \text{ and } (x_i, y) \in E\}.$$
Furthermore, by definition of the peeling process, we have $\theta(x_i) = |\{\ x_j \mid j < i \text{ and } (x_i, x_j) \in E\}|$. Hence, x satisfies point (1).
- If $x \notin A$, then set $\mathcal{G}_i = (G_i, \theta_i)$ the sequence associated with L. By definition of the peeling process, we have $\theta_{k+1}(x) = \theta(x) - |\{\ y \mid y \in A \text{ and } (x, y) \in E\}| \geq 0$. Hence, x satisfies Point (2).

Conversely, assume that A satisfies both points (1) and (2). We prove that $A \in IS(\mathcal{G})$ recursively on $k := |A|$. If $k = 1$, then set x the vertex of G such that $A = \{x\}$. We have $\theta(x) = 0$ thanks to Point (1), and Point (2) implies that for all $y \neq x$ such that $(y, x) \in E$ we have $\theta(y) \geq 1 > 0$. Hence, x is erasable in $\mathcal{G}$, so that $A \in IS(\mathcal{G})$. Let k be such that the property is true and assume that $|A| = k + 1$. We first prove that there exists a vertex in A which is erasable. Since G is acyclic and A is finite, there exists $z \in A$ such that for all $y \in A$, $(z, y) \notin E$. Then, by Point (1), we have $\theta(z) = 0$. Again, since G is acyclic and A is finite, there exists x in A such that $\theta(x) = 0$ and for all $y \in A$, if $(y, x) \in E$ then $\theta(y) > 0$. Furthermore, for all $z \in A$ such that $\theta(z) = 0$, if there exists $y \in V \setminus A$ such that $(y, z) \in E$, then $\theta(y) > 0$ by Point (2). Consequently, this vertex x is erasable and can be peeled at the first step of the peeling process. Thus, if we set $\mathcal{G}'$ the valued digraph obtained with the peeling process after we peeled the vertex x, then $|A \setminus \{x\}| = k$ and $A \setminus \{x\}$ clearly satisfies Points (1) and (2) in $\mathcal{G}'$. Therefore, by induction we have $A \in IS(\mathcal{G})$. □

REMARK 3.2.2. We stress that Proposition 3.2.1 is fundamental, not only because it simplifies the proofs of Theorems 3.1.5 and 3.1.6, but also because it leads to the content of Chapter 4.

We finish with a technical lemma.

LEMMA 3.2.3. *Let $S \subseteq IS(\mathcal{G})$ and denote by X the set $\bigcap_{A \in S} A$. If there exists $x \in X$ such that $\theta(x) = 0$, then there exists $z \in X$ which is erasable in $\mathcal{G}$.*

PROOF. We first recall that the underlying digraph of a valued digraph is acyclic. Let $z \in X$ be such that $\theta(z) = 0$. For all $y \notin X$, there exists $B \in S$ such that $y \notin B$. Therefore, if $(y, z) \in E$, then by Proposition 3.2.1 we have $\theta(y) \geq 1$. Assume by contradiction that for all $x \in X$ such that $\theta(x) = 0$, there exists $y \in X$ such that $\theta(y) = 0$ and $(y, x) \in E$. Since X is finite, this implies that there is a cycle in G, which is absurd. Hence, there exists an erasable vertex of $\mathcal{G}$ in X, and this ends the proof. $\square$

3.2.1. Proof of Theorem 3.1.5. We divide the proof of Theorem 3.1.5 into two distinct steps. First, we prove that $(IS(\mathcal{G}), \subseteq)$ is a graded poset (Proposition 3.2.4). Then, we prove that it is a meet semi-lattice (Corollary 3.2.6), constructing explicitly the infimum of any subset of $IS(\mathcal{G})$.

Let us begin with a proposition, which immediately implies that $(IS(\mathcal{G}), \subseteq)$ is graded.

PROPOSITION 3.2.4. *Let A and B be two elements of $IS(\mathcal{G})$, and denote by k and q the cardinality of A and B, respectively. If $A \subseteq B$, then there exists $L = [x_1, x_2, \ldots] \in PS(\mathcal{G})$ such that $A = \{x_1, \ldots, x_k\}$ and $B = \{x_1, \ldots, x_q\}$. Consequently, $(IS(\mathcal{G}), \subseteq)$ is graded with rank function $A \mapsto |A|$.*

PROOF. Since the case $k = q$ is obvious, we assume that $k < q$. Let us now perform the peeling process on $\mathcal{G}$, in order to construct the claimed peeling sequence. Since $A \in IS(\mathcal{G})$, we begin with constructing the sequence L by peeling the first k elements $x_1, \ldots, x_k$ in A. Consequently, $A = \{x_1, \ldots, x_k\}$.

Let us now consider the valued digraph $\mathcal{G}_{k+1} = (G_{k+1}, \theta_{k+1})$ coming from the peeling process after we peeled $x_1, \ldots, x_k$. By definition of $\mathcal{G}_{k+1}$, $C = B \setminus A$ is a subset of vertices of G_{k+1}. We prove that C is in $IS(\mathcal{G}_{k+1})$ checking that C satisfies both Points (1) and (2) of Proposition 3.2.1. By construction, for all $z \in C$ we have $\theta_{k+1}(z) = \theta(z) - |\{y \in A \mid (z, y) \in E\}|$, and

$$\{y \in C \mid (z, y) \in E\} = \{y \in B \mid (z, y) \in E\} \setminus \{y \in A \mid (z, y) \in E\}.$$

Since $B \in IS(\mathcal{G})$, we have by Proposition 3.2.1 that $\theta(z) \leq |\{y \in B \mid (z, y) \in E\}|$, so that

$$\theta_{k+1}(z) \leq |\{y \in B \mid (z, y) \in E\}| - |\{y \in A \mid (z, y) \in E\}| = |\{y \in C \mid (z, y) \in E\}|.$$

Then, C satisfies Point (1) of Proposition 3.2.1. Using similar arguments, we show that C also satisfies Point (2) of Proposition 3.2.1. Hence, there exists a peeling sequence $[x_{k+1}, x_{k+2}, \ldots]$ of $\mathcal{G}_{k+1}$ such that $C = \{x_{k+1}, \ldots, x_q\}$, and finally, the sequence $L = [x_1, x_2, \ldots]$ is a peeling sequence of $\mathcal{G}$ such that $A = \{x_1, \ldots, x_k\}$ and $B = \{x_1, \ldots, x_q\}$. This ends the proof of the proposition. $\square$

We end the proof of Theorem 3.1.5 showing that $(IS(\mathcal{G}), \subseteq)$ is a meet semi-lattice. For that purpose, we construct explicitly the infimum (also called the *meet*) of a set $S \subseteq IS(\mathcal{G})$.

Construction of the meet. Let S be a subset of $IS(\mathcal{G})$ and X be the intersection of all the elements of S, we construct recursively a set $C \in IS(\mathcal{G})$ as follows.

If for all $x \in X$, $\theta(x) \neq 0$, we set $C = \emptyset$. Otherwise, let $z_1 \in X$ be an erasable vertex of $\mathcal{G}$ and start the peeling process by peeling this vertex. We denote by $\mathcal{G}_2 = (G_2, \theta_2)$ the obtained valued digraph. Then, for all $A \in S$, we have $A \setminus \{z_1\} \in IS(\mathcal{G}_2)$. Therefore, we can again apply Lemma 3.2.3 to $X \setminus \{z_1\}$ seen as a subset of vertices of G_2: if for all $x \in X \setminus \{z_1\}$ we have $\theta_2(x) \neq 0$, then we set $C = \{z_1\}$; otherwise, let $z_2 \in X \setminus \{z_1\}$ be an erasable vertex of $\mathcal{G}_2$ and perform the peeling process peeling this z_2 in $\mathcal{G}_2$. We repeat this procedure until there is not any erasable vertex left (this process always ends, since X is finite), and we set C the resulting set. By construction, $C \in IS(\mathcal{G})$.

At first glance, this set C does not appear to be well defined, and seems to depend heavily on the choices of vertices made at each step of its construction. The next proposition shows that is not the case.

PROPOSITION 3.2.5. *Let $A \in IS(\mathcal{G})$. If $A \subseteq X$, then $A \subseteq C$.*

PROOF. We split the proof into two cases.

- If for all $z \in X$ we have $\theta(z) \neq 0$, then, by definition of the peeling process, we have that $A = \emptyset = C$.

- If there exists $z \in X$ such that $\theta(z) = 0$, then $C \neq \emptyset$. Thus there exists $L = [x_1, x_2, \ldots] \in PS(\mathcal{G})$ and $k \in \mathbb{N}^*$ such that $C = \{x_1, \ldots, c_k\}$. Let us denote by $\mathcal{G}_i = (G_i, \theta_i)_i$ the sequence of valued digraphs associated with L and let $L' = [z_1, z_2, \ldots]$ be in $PS(\mathcal{G})$ such that $A = \{z_1, \ldots, z_{|A|}\}$.

  Assume by contradiction that $A \not\subseteq C$ and consider $j \leq |A|$ minimal such that $z_j \notin C$. We have that z_j is a vertex of G_{k+1} and, by minimality, for all $q < j$ there exists $1 \leq i_q \leq k$ such that $z_q = x_{i_q}$. Let us now compute the value of $\theta_{k+1}(z_j)$. By definition of the peeling process, we have

  $$\theta(z_j) = |\{q < j \mid (z_j, z_q) \in E\}| = |\{q < j \mid (z_j, x_{i_q}) \in E\}|.$$

  However, for all $q < j$ we have $i_q \leq k$, so that

  $$\theta_{k+1}(z_j) = \theta(z_j) - |\{p \leq k \mid (z_j, x_p) \in E\}| \leq 0.$$

  Thus, $\theta_{k+1}(z_j) = 0$ and this is absurd by construction of C, hence $A \subseteq C$.

This ends the proof □

As $C \subseteq X$ by construction, Proposition 3.2.5 implies the following corollary.

COROLLARY 3.2.6. *The set C is the infimum of S.*

Finally, note that if the underlying graph G is finite, then we obviously have that V, the set of all the vertices of G, satisfies the Points (1) and (2) of Proposition 3.2.1. Consequently, $V \in IS(\mathcal{G})$, and the poset $(IS(\mathcal{G}), \subseteq)$ is bounded. Thus, $(IS(\mathcal{G}), \subseteq)$ is a lattice since it is a meet semi-lattice (more precisely, it is a complete lattice), and this ends the proof of Theorem 3.1.5.

3.2.2. Proof of Theorem 3.1.6. The proof of this formula is purely combinatorial, and is based on the well-known *Inclusion-Exclusion Principle* (see §2 in **[S3]**). We first introduce some notations: for all $A \in IS(\mathcal{G})$, we denote by $[\emptyset, A]$ the set $\{B \in IS(\mathcal{G}) \mid \emptyset \subseteq B \subseteq A\}$. Let A, $\mathcal{N}(A)$ and $\mathcal{F}(A)$ be as defined in Theorem 3.1.6, for all $S \subseteq \mathcal{F}(A)$, let us denote by A_S the infimum of $\{A \setminus \{x\} \mid x \in S\} \subseteq IS(\mathcal{G})$.

We begin the proof with a technical lemma.

LEMMA 3.2.7. *Let $S \subseteq \mathcal{F}(A)$, we have $A_S \neq \emptyset$ if and only if $\mathcal{N}(A) \not\subseteq S$.*

PROOF. Obviously, $\bigcap_{x \in S}(A \setminus \{x\}) = A \setminus S$. Thanks to Lemma 3.2.3, if $\mathcal{N}(A) \not\subseteq S$, then there exists $z \in A \setminus S$ which is erasable in $\mathcal{G}$, so that the infimum of $\{A \setminus \{x\} \mid x \in S\}$ is not $\emptyset$. The proof of the converse implication is based on similar arguments. □

An immediate consequence of the meet semi-lattice structure of $\mathcal{P}(\mathcal{G})$ is that, for all A and B in $IS(\mathcal{G})$, $[\emptyset, A] \cap [\emptyset, B] = [\emptyset, A \wedge B]$ where $A \wedge B$ is the infimum of $\{A, B\}$. This basic remark leads to the claimed formula.

First, we have

$$[\emptyset, A] \setminus \{A\} = \bigcup_{x \in \mathcal{F}(A)} [\emptyset, A_{\{x\}}].$$

Then, by the Inclusion-Exclusion Principle, we have

$$|[\emptyset, A] \setminus \{A\}| = \sum_{\emptyset \neq S \subseteq \mathcal{F}(A)} (-1)^{|S|+1} \left| \bigcap_{x \in S} [\emptyset, A_{\{x\}}] \right| = \sum_{\emptyset \neq S \subseteq \mathcal{F}(A)} (-1)^{|S|+1} \, |[\emptyset, A_S]| \,.$$

Once applied to the Möbius function of $\mathcal{P}(\mathcal{G})$, this gives rise to the following identity:

$$\mu(\emptyset, A) = - \sum_{\emptyset \neq S \subseteq \mathcal{F}(A)} (-1)^{|S|+1} \sum_{B \in [\emptyset, A_S]} \mu(\emptyset, B). \tag{3.1}$$

By definition of the Möbius function, $\sum_{B\in[\emptyset,A_S]}\mu(\emptyset,B)=1$ if $A_S=\emptyset$, and 0 otherwise. Hence, thanks to Lemma 3.2.7, if $\mathcal{N}(A)\not\subset\mathcal{F}(A)$, then $\mu(\emptyset,A)=0$. Otherwise, Equation (3.1) becomes

$$\begin{aligned}\mu(\emptyset,A)&=-\sum_{\mathcal{N}(A)\subseteq S\subseteq\mathcal{F}(A)}(-1)^{|S|+1}=(-1)^{|\mathcal{N}(A)|}\sum_{S\subseteq\mathcal{F}(A)\setminus\mathcal{N}(A)}(-1)^{|S|}\\&=(-1)^{|\mathcal{N}(A)|}(1-1)^{|\mathcal{F}(A)\setminus\mathcal{N}(A)|}.\end{aligned}$$

Theorem 3.1.6 follows immediately.

3.3. Link with the weak order

In this section, we show several examples of classical posets which can be described using valued digraphs. We first recall the definition of weak order on a Coxeter groups.

Let W be a Coxeter group with generating set S, the weak order on W is the poset $(W,\leq_R)$, defined as follows: we say that $w\leq_R\tau$ if and only if there exists $s_1,\dots,s_k$ in S such that $\tau=\omega s_1\cdots s_k$ and $\ell(\tau)=\ell(w)+k$.

It is well-known that $(W,\leq_R)$ is a complete meet semi-lattice when W is infinite, a complete lattice when W is finite, and that its Möbius function takes values into $\{\pm1,0\}$ (see [**B**] and [**BB**]). Hence, it is natural to look for an interpretation of the weak order through the theory developed in the previous section. Indeed such an interpretation exists in some cases, and we give an explicit description for the following list of posets.

THEOREM 3.3.1. *For each poset $(\mathcal{P},\leq)$ in the following list, there exists an explicit valued digraph $\mathcal{G}$ such that $(\mathcal{P},\leq)$ is isomorphic to $(IS(\mathcal{G}),\subseteq)$:*

- *$(W,\leq_R)$ where $W=A_{n-1},B_n,\widetilde{A_n}$ and $\leq_R$ isthe (right) weak order on W;*
- *the flag weak order $(\mathbb{Z}_r\wr S_n,\leq_f)$;*
- *the up-set (resp. down-set) lattice of any finite poset.*

We prove Theorem 3.3.1 with a careful case-by-case study, which is done in the following sections. More precisely, in Section 3.3.1 we provide a candidate of valued digraph associated with the weak order on A_{n-1}, and we prove that this candidate indeed provides a description of the weak order on A_{n-1} in Section 3.3.2. Similarly, in Section 3.3.3 we construct a valued digraph associated with B_n, and we prove in Section 3.3.4 that this valued digraph describe the weak order on B_n. Section 3.3.5 is devoted to the study of the weak order on $\widetilde{A_n}$. Finally, in Sections 3.3.6 and 3.3.7 we study the cases of the flag weak order and the up-set lattice, respectively.

3.3.1. Weak order on A_{n-1}.

Recall that A_{n-1} is the Coxeter group with generating set $S=\{s_1,\dots,s_{n-1}\}$ and with Coxeter matrix $M=(m_{st})_{s,t\in S}$ given by $m_{s_is_{i+1}}=3$ for all $1\leq i\leq n-2$, $m_{ss}=1$ for all $s\in S$ and $m_{st}=2$ otherwise. As usual, we identify A_{n-1} with the symmetric group S_n, identifying the generator s_i with the *simple transposition* of S_n which exchanges the integers i and $i+1$.

When we try to find a valued digraph $\mathcal{G}=(G,\theta)$ such that $(A_{n-1},\leq_R)$ is isomorphic to $(IS(\mathcal{G}),\subseteq)$, the first problem arising is that, on the one hand we have a poset whose elements are permutations, and on the other hand we have a poset whose elements are sets. In order to overcome this difficulty, let us consider a canonical set associated with each permutation $\sigma\in S_n$, its *inversion set*:

$$\mathrm{Inv}(\sigma)=\{(a,b)\in[n]^2\mid a<b\text{ and }\sigma^{-1}(a)>\sigma^{-1}(b)\}.\tag{3.2}$$

Note that the definition we give here is not the usual one (see [**BB**, eq.(1.25)] for the classical definition). However, it is straightforward to check that the connection between inversion sets and the weak order on S_n is preserved. That is, we have the following property (see [**BB**, p.67]): for any $\sigma, \omega \in S_n$,

$$\sigma \leq_R \omega \text{ if and only if } \mathrm{Inv}(\sigma) \subseteq \mathrm{Inv}(\omega). \tag{3.3}$$

This property allows us to clarify our goal: we are looking for a valued digraph $\mathcal{G} = (G, \theta)$ such that,

(1) the vertices of the graph are indexed by couples of integers $(a, b) \in [n]^2$ such that $a < b$;
(2) the digraph structure of G, together with the valuation θ, imply that $IS(\mathcal{G})$ is constituted exactly of the sets of the form $\mathrm{Inv}(\sigma)$, $\sigma \in S_n$.

There is a convenient way to represent the set $\{(a, b) \in [n]^2 \mid a < b\}$, considering the *n-th staircase diagram*, namely the Ferrers diagram of the partition $\lambda_n = (n-1, n-2, \ldots, 1)$ of size $N = \binom{n}{2}$. On the left of Figure 3.2, the diagram associated to the case $n = 5$ is represented. The coordinates of each box can be read thanks to the circled integers on the diagonal. From now on, we identify λ_n with the set $\{(a, b) \in [n]^2 \mid a < b\}$. As shown in the middle of Figure 3.2,

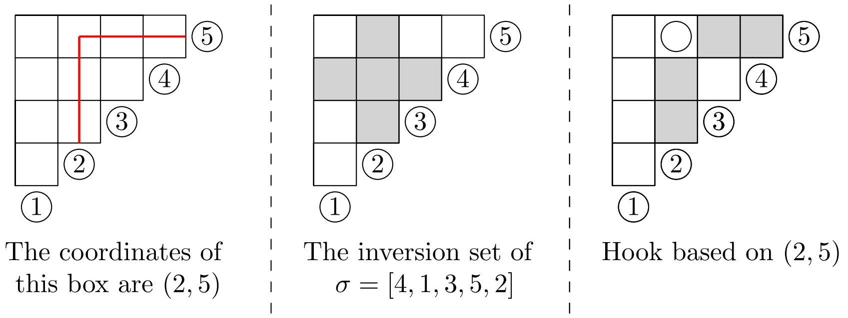


FIGURE 3.2.

one can easily visualize the inversion set of any element of S_n as a subset of boxes in λ_n. Note that the set made of all the boxes of the diagram corresponds to the inversion set of the *reverse permutation* $[n, n-1, \ldots, 1] \in S_n$, which is the maximal element in the weak order.

We can define a digraph structure G on the staircase diagram λ_n (where the vertices are the boxes of the diagram), thanks to a classical combinatorial object associated to each box $\mathfrak{c} \in \lambda_n$, *the hook based on* $\mathfrak{c}$, denoted $H(\mathfrak{c})$, consisting of $\mathfrak{c}$ and all the boxes which are on the right and below $\mathfrak{c}$ (see Figure 3.2, on the right): we say that there is an arc from $\mathfrak{c}$ to $\mathfrak{d}$ if and only if $\mathfrak{c} \neq \mathfrak{d}$ and $\mathfrak{d} \in H(\mathfrak{c})$. Obviously, the resulting digraph G is acyclic, and the out-degree of any box is an even number. Thus, if we set θ the function defined by $\theta(\mathfrak{c}) := \frac{d^+(\mathfrak{c})}{2}$, then the couple $\mathcal{A} = (G, \theta)$ is a valued digraph. Let us summarize this construction in a definition.

DEFINITION 3.3.2. Let $G = (V, E)$ be the digraph such that

$$V := \lambda_n = \{(a, b) \in [n]^2 \mid a < b\} \text{ and } E := \{(\mathfrak{c}, \mathfrak{d}) \in \lambda_n^2 \mid \mathfrak{c} \neq \mathfrak{d} \text{ and } \mathfrak{d} \in H(\mathfrak{c})\}.$$

We denote by $\mathcal{A} := (G, \theta)$ the valued digraph such that for all $\mathfrak{c} \in \lambda_n$

$$\theta(\mathfrak{c}) := \frac{d^+(\mathfrak{c})}{2}.$$

One can check that the posets $(IS(\mathcal{A}), \subseteq)$ obtained in the cases $n = 2$, 3 and 4 are isomorphic to the weak order on A_1, A_2 and A_3, respectively. As stated in the following theorem, this situation is in fact general.

THEOREM 3.3.3. *The posets $(S_n, \leq_R)$ and $(IS(\mathcal{A}), \subseteq)$ are isomorphic.*

The next section is dedicated to the proof of Theorem 3.3.3.

3.3.2. Proof of Theorem 3.3.3. In this section, we show that

$$IS(\mathcal{A}) = \{\mathrm{Inv}(\sigma) \mid \sigma \in S_n\},$$

which immediately implies Theorem 3.3.3 (thanks to (3.3)). We divide our proof into three steps. First, for any permutation $\sigma \in S_n$ we define a statistic d_σ on $\lambda_n \backslash \mathrm{Inv}(\sigma)$ (Definition 3.3.4), which we characterize using the notion of *adjacency* (see Lemma 3.3.6). We then use this to give a combinatorial interpretation of the valuations θ_i appearing when we perform the peeling process on $\mathcal{A}$ (see Proposition 3.3.8). Finally, using this combinatorial interpretation, we prove that $IS(\mathcal{A}) = \{\mathrm{Inv}(\sigma) \mid \sigma \in S_n\}$ (see Proposition 3.3.9 and Corollary 3.3.10).

We begin with the definition of the statistic d_σ.

DEFINITION 3.3.4. Let $\sigma \in S_n$ and $(a, b) \in \lambda_n \setminus \mathrm{Inv}(\sigma)$. Then, we set:

$$d_\sigma(a, b) := |\{a < k < b \mid \sigma^{-1}(a) < \sigma^{-1}(k) < \sigma^{-1}(b)\}|.$$

Let $\sigma \in S_n$ and $(a, b) \in \lambda_n$, we say that a and b are *adjacent* in σ if and only if $\sigma^{-1}(b) = \sigma^{-1}(a) + 1$. One can clearly visualize two adjacent entries of a permutation $\sigma \in S_n$, using the *window notation* of σ. That is, a and b are adjacent in σ if and only if $(a, b) \in \lambda_n$ and

$$\sigma = [\sigma(1), \sigma(2), \ldots, a, b, \ldots, \sigma(n-1), \sigma(n)].$$

This notion is linked to the weak order thanks to the following well-known property: for all $\sigma, \omega \in S_n$ we have $\sigma \leq_R \omega$ and $\ell(\omega) = \ell(\sigma) + 1$ if and only if ω can be obtained from σ by swapping positions of two adjacent entries of σ and we say that ω *covers* σ, denoted by $\sigma \lhd_R \omega$. This can be transposed to the context of inversion sets using Equation (3.3) as follows: for all $\sigma, \omega \in S_n$, we have

$$\sigma \lhd_R \omega \text{ if and only if } \mathrm{Inv}(\omega) = \mathrm{Inv}(\sigma) \cup \{(a, b)\}$$

for a and b two adjacent entries of σ.

REMARK 3.3.5. Note that if a and b are two adjacent entries of σ, then $d_\sigma(a, b) = 0$, but the converse is not true.

We now provide a characterization of the statistic d_σ.

LEMMA 3.3.6. *Let $\sigma \in S_n$ and $(a, b) \in \lambda_n \setminus \mathrm{Inv}(\sigma)$. Then, we have*

$$d_\sigma(a, b) = |\{a < k < b \mid k \in \mathbb{N}\}| - |\{a < k < b \mid (a, k) \in \mathrm{Inv}(\sigma) \text{ or } (k, b) \in \mathrm{Inv}(\sigma)\}|.$$

PROOF. Let $s_1 \cdots s_q$ be a reduced decomposition of σ and denote by σ_i the permutation $s_1 \cdots s_i$, $0 \leq i \leq q$ (with the convention that $\sigma_0 = Id$). We prove by induction on i that the lemma is true for σ_i.

Note that the property is obviously true for σ_0. Let $i \geq 0$ be such that the property is true. For the sake of clarity, let us denote by δ_j the integer $d_{\sigma_j}(a, b)$. Since $s_1 \cdots s_q$ is reduced, we have $\sigma_i \lhd_R \sigma_{i+1}$, thus there exists a unique (a_{i+1}, b_{i+1}) in $\mathrm{Inv}(\sigma_{i+1}) \setminus \mathrm{Inv}(\sigma_i)$. We now show how one can deduce the value of δ_{i+1} from both δ_i and (a_{i+1}, b_{i+1}).

- (case $(a_{i+1}, b_{i+1}) = (a, k)$ with $a < k < b$) the permutation σ_{i+1} is obtained from σ_i by exchanging the position of the integer a with the position of the integer k. Moreover, since $\sigma_i \lhd_R \sigma_{i+1}$, a and k are adjacent in σ_i. However, $(a, b) \notin \mathrm{Inv}(\sigma_i)$, thus k lies strictly between a and b in the window notation of σ_i, *i.e.* we have

$$\sigma_i = [\sigma(1), \ldots, a, k, \ldots, b, \ldots, \sigma(n)].$$

  Hence, it is no longer the case in σ_{i+1}, so that $\delta_{i+1} = \delta_i - 1$.
- If $(a_{i+1}, b_{i+1}) = (k, b)$ with $a < k < b$, then with similar arguments we show that $\delta_{i+1} = \delta_i - 1$.
- In all other cases, both a_{i+1} and b_{i+1} either lie between a and b in σ_i, or they do note, and this is also true for σ_{i+1}. Therefore, we have $\delta_{i+1} = \delta_i$.

Finally, by induction hypothesis, σ_{i+1} satisfies the property, and this ends the proof. □

For the sake of clarity, we introduce the following useful notation.

Definition 3.3.7. Let $\mathcal{G} = (G, \theta)$ be a valued digraph and $A \in IS(\mathcal{G})$, we denote by $\mathcal{G}_A = (G_A, \theta_A)$ the valued digraph obtained after removing all the elements of A in $\mathcal{G}$ using the peeling process.

We are now able to provide a combinatorial interpretation of θ_A for some $A \in IS(\mathcal{A})$.

Proposition 3.3.8. *Let $A \in IS(\mathcal{A})$, if there exists $\sigma \in S_n$ such that $A = \mathrm{Inv}(\sigma)$, then for all $(a,b) \in \lambda_n \setminus \mathrm{Inv}(\sigma)$, we have $\theta_A(a,b) = d_\sigma(a,b)$.*

Proof. Let $(a,b) \in \lambda_n \setminus \mathrm{Inv}(\sigma)$, by construction of $\mathcal{A} = (G, \theta)$, there is an arc from (a,b) to (c,d) if and only if $(c,d) = (a,k)$ or (k,b) with $a < k < b$. Thus, by definition of the peeling process, we have

$$\theta_A(a,b) = \theta(a,b) - |\{a < k < b \mid (a,k) \in \mathrm{Inv}(\sigma) \text{ or } (k,b) \in \mathrm{Inv}(\sigma)\}|.$$

Moreover, we obviously have $\theta(a,b) = b - a - 1 = |\{a < k < b \mid k \in \mathbb{N}\}|$. Consequently, thanks to Lemma 3.3.6, we have

$$\theta_A(a,b) = |\{a < k < b \mid k \in \mathbb{N}\}| - |\{a < k < b \mid (a,k) \in \mathrm{Inv}(\sigma) \text{ or } (k,b) \in \mathrm{Inv}(\sigma)\}| = d_\sigma(a,b),$$

which ends the proof. □

Finally, we are now able to prove the main property of this section, which immediately leads to the proof of Theorem 3.3.3 (see Corollary 3.3.10).

Proposition 3.3.9. *Let $A \in IS(\mathcal{A})$, $\sigma \in S_n$ such that $A = \mathrm{Inv}(\sigma)$ and $(a,b) \in \lambda_n \setminus A$. Then, a and b are adjacent in σ if and only if (a,b) is erasable in $\mathcal{A}_A$.*

Proof. Assume that a and b are adjacent in σ, then $d_\sigma(a,b) = 0$. Let $(c,d) \in \lambda_n \setminus A$ be such that there is an arc from (c,d) to (a,b), thus we have $(c,d) = (a,p)$ with $p > b$ or $(c,d) = (q,b)$ with $q < a$. Since a and b are adjacent in σ, we have in the first case that b is between a and p in the window notation of σ, *i.e* we have

$$\sigma = [\sigma(1), \ldots, a, b, \ldots, p, \ldots, \sigma(n)],$$

and we have in the second case that a is between q and b in σ. In both cases, $\theta_A(c,d) = d_\sigma(c,d) \geq 1$. Consequently, (a,b) is erasable in $\mathcal{A}_A$.

We now prove the converse implication. Let $(a,b) \in \lambda_n \setminus A$ be erasable in $\mathcal{A}_A$, and assume by contradiction that a and b are not adjacent in σ. Then, there exists $1 \leq c \leq n$ which is between a and b in σ and since $\theta_A(a,b) = d_\sigma(a,b) = 0$, we have $c < a$ or $c > b$.

- Case $c < a$. Let d be maximal such that $d < a$ and d is between a and b in σ, and let k be an integer which is between d and b in σ (if such a k exists), we have:
  - by maximality of d, $k \notin [\![d, a]\!]$;
  - since $d_\sigma(a,b) = 0$, $k \notin [\![a, b]\!]$.

  Thus, $d_\sigma(d,b) = 0 = \theta(d,b)$, which is absurd since (a,b) is erasable and there is an arc from (d,b) to (a,b).
- Case $c > b$ leads to a similar contradiction.

This proves that a and b are adjacent in σ, and this ends the proof. □

Corollary 3.3.10. *$IS(\mathcal{A}) = \{\mathrm{Inv}(\sigma) \mid \sigma \in S_n\}$.*

Proof. Let $L = [(a_1, b_1), \ldots, (a_N, b_N)] \in PS(\mathcal{A})$, since $\mathrm{Inv}(Id) = \emptyset$, a_1 and b_1 are adjacent in Id by Proposition 3.3.9. Let σ_1 be the permutation which has $\{(a_1, b_1)\}$ as inversion set, then, using recursively Proposition 3.3.9, we show that for all $1 \leq k \leq N$, there exists a permutation σ_k which has $\{(a_1, b_1), \ldots, (a_k, b_k)\}$ as inversion set. This is enough to prove the corollary. □

This concludes the proof of Theorem 3.3.3.

3.3.3. Weak order on B_n. Recall that B_n is the Coxeter group with generating set $S = \{s_0, s_1, \ldots, s_{n-1}\}$, and with Coxeter matrix $M = (m_{st})_{s,t\in S}$ given by $m_{s_i s_{i+1}} = 3$ for all $1 \leq i \leq n-1$, $m_{ss} = 1$ for all $s \in S$, $m_{s_0 s_1} = 4$ and $m_{st} = 2$ otherwise. This group can be seen as the group of the *signed permutations* ω of the set $[\pm n] := \{-n, \ldots, -1, 1, \ldots, n\}$ satisfying $\omega(-m) = -\omega(m)$ for all m. Within this interpretation, s_0 is the signed permutation such that $s_0(1) = -1$ and $s_0(j) = j$ for all $j > 1$, and s_i is the permutation which exchange the positions of i and $i+1$ (and also the positions of $-i$ and $-i-1$). In what follows, we sometimes represent an element ω in B_n by its *full window notation*, that is:

$$[\omega(-n), \omega(-(n-1)), \ldots, \omega(-1), \omega(1), \ldots, \omega(n-1), \omega(n)].$$

Our aim in this section is to provide an interpretation of $(B_n, \leq_R)$ using our theory. First, we need to find a candidate of valued digraph. For that purpose, we follow the same method as in Section 3.3.1, using a good notion of inversion set.

REMARK 3.3.11. It is important to note that we will not use the notion of inversion set coming from root systems in this chapter. Indeed, the combinatorial techniques we use here heavily depend on the interpretation of B_n as a set of permutations, and not as a set of reflections. The drawback of this approach is that we have to relate by ourselves these inversion sets to the weak order on B_n. Fortunately, most of the technical points have already been accomplished in [**BB**].

We begin with associating to each element ω of B_n a B-inversion set, defined by:

$$\mathrm{Inv}_B(\omega) = \{(a,b) \in [\pm n] \times [n] \mid a < b,\ |a| \leq b \text{ and } \omega^{-1}(a) > \omega^{-1}(b)\}. \tag{3.4}$$

Let us now relate B-inversion set to the weak order on B_n. For that purpose, we need a definition and a result coming from [**BB**], which we now give.

DEFINITION 3.3.12 (see [**BB**], eq. (8.2) p.247). Let $\omega \in B_n$, the B-inversion number of ω is the quantity

$$\begin{aligned} \mathrm{inv}_B(\omega) :=& |\{(a,b) \in [n]^2 \mid a < b \text{ and } \omega(a) > \omega(b)\}| \\ &+ \ |\{(a,b) \in [n]^2 \mid a \leq b \text{ and } \omega(-a) > \omega(b)\}.| \end{aligned}$$

LEMMA 3.3.13 (see [**BB**], Eq. (8.6) and (8.7) p. 247). *Let $\omega \in B_n$ and $i \in [n-1]$, we have*

$$\mathrm{inv}_B(\omega s_i) = \begin{cases} \mathrm{inv}_B(\omega) + 1, & \text{if } \omega(i) < \omega(i+1), \\ \mathrm{inv}_B(\omega) - 1, & \text{if } \omega(i) > \omega(i+1). \end{cases}$$

We also have $\mathrm{inv}_B(\omega s_0) = \mathrm{inv}_b(\omega) + \mathit{sign}\,(\omega(1))$.

The statistic inv_B is related to B-inversion sets, thanks to the following lemma.

LEMMA 3.3.14. *For all $\omega \in B_n$, we have* $\mathrm{inv}_B(\omega) = |\mathrm{Inv}_B(\omega)|$.

PROOF. We have

$$\begin{aligned} \mathrm{inv}_B(\omega) &= |\{(a,b) \in [n]^2 \mid a < b \text{ and } \omega(a) > \omega(b)\}| \\ &\qquad + \ |\{(a,b) \in [n]^2 \mid a \leq b \text{ and } \omega(-a) > \omega(b)\}.| \\ &= |\{(a,b) \in [\pm n] \times [n] \mid a < b,\ |a| \leq b \text{ and } \omega(a) > \omega(b)\} \\ &= |\mathrm{Inv}_B(\omega)|, \end{aligned}$$

which is the expected result. □

We now begin to prove that B-inversion sets can be used to study $(B_n, \leq_R)$. That is, we show that for all $\omega, \tau \in B_n$ we have

$$\omega \leq_R \tau \text{ if and only if } \mathrm{Inv}_B(\omega) \subseteq \mathrm{Inv}_B(\tau). \tag{3.5}$$

We start with defining the equivalent of the notion of adjacency in type B.

DEFINITION 3.3.15. Let $\omega \in B_n$ and $a < b$ be in $[\pm n]$, we say that a and b are *B-adjacent* in ω if and only if the following two conditions are true:

(1) $|a| \leq b$,
(2) a and b are adjacent in ω (seen as a permutation of $[\pm n]$, *i.e.* the full window notation of ω is of the form

$$\omega = [\omega(-n), \dots, a, b, \dots, \omega(n)].$$

It appears that the notion of B-adjacency plays the same role in type B as the usual notion of adjacency do in type A, as shown in the next proposition.

PROPOSITION 3.3.16. *Let $\omega \in B_n$ and $0 \leq j \leq n-1$, there exists (a, b) such that $|a| \leq b$, and such that ωs_j is obtained from ω by swapping the positions of a and b and the positions of $-b$ and $-a$ in ω. Then, we have two possibilities:*

- *if a and b are B-adjacent in ω, then $\ell(\omega s_j) = \ell(\omega) + 1$ and $\mathrm{Inv}_B(\omega s_j) = \mathrm{Inv}_B(\omega) \cup \{(a, b)\}$;*
- *if a and b are not B-adjacent in ω, then $\ell(\omega s_j) = \ell(\omega) - 1$ and $\mathrm{Inv}_B(\omega s_j) = \mathrm{Inv}_B(\omega) \setminus \{(a, b)\}$.*

PROOF. This is a consequence of Lemma 3.3.14 together with Lemma 3.3.13 and Definition 3.3.15. □

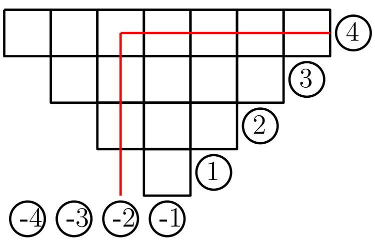
An immediate consequence of Proposition 3.3.16 is the following proposition.

PROPOSITION 3.3.17. *Let $\omega, \tau \in B_n$. Then, $\omega \lhd_R \tau$ if and only if there exists $a, b \in [\pm n]$ B-adjacent in ω such that $\mathrm{Inv}_B(\tau) = \mathrm{Inv}_B(\omega) \cup \{(a, b)\}$.*

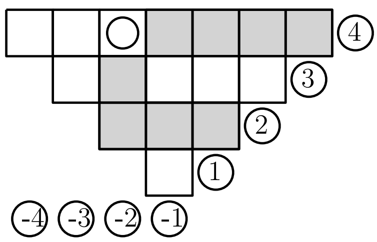
Proposition 3.3.17 implies the direct implication ($\Rightarrow$) of (3.5). Note that a proof of the converse implication of (3.5), which is of fundamental importance for our purpose, will follow from the results of Section 3.3.4 (see Corollary 3.3.25), and we will postpone till there.

Let us now introduce a way to visualize B-inversion sets. First, note that the B-inversion set of any element of B_n is a subset of $\{(a, b) \in [\pm n] \times [n] \mid |a| < b\}$. One can easily represent the set $\{(a, b) \in [\pm n] \times [n] \mid |a| < b\}$ considering the *shifted diagram* λ_n^s of the partition $(2n-1, 2n-3, \dots, 1)$, as depicted on Figure 3.3. The coordinates of each box can be read thanks to the circled integers.

From now on, we identify λ_n^s with the set $\{(a, b) \in [\pm n] \times [n] \mid |a| < b\}$.



The coordinates of this box are $(-2, 4)$.



Shifted hook based on $(-2, 4)$.

FIGURE 3.3.

We now define a digraph structure G on λ_n^s (where the vertices are the boxes of the diagram), using the equivalent of hooks of Ferrers diagrams in the shifted case, namely *shifted hooks* (as depicted on the right of Figure 3.3). The shifted hook based on (a, b) in λ_n^s is formally defined by

$$\widetilde{H}(a, b) := \{(a, b)\} \cup \left\{ (x, y) \in \lambda_n^s \mid \exists k \in \mathbb{N} \text{ such that } a < k < b \text{ and } (x, y) = \left| \begin{array}{c} (k, b) \\ \text{or } (a, k) \\ \text{or } (-k, -a) \end{array} \right. \right\}.$$

Following the same methods of Section 3.3.1, we define a digraph structure G on λ_n^s by saying that there is an arc from $\mathfrak{c}$ to $\mathfrak{d}$ in G if and only if $\mathfrak{c} \neq \mathfrak{d}$ and $\mathfrak{d}$ is in the shifted hook based on $\mathfrak{c}$. It appears that G is acyclic and the out-degree of any box is an even number, so that the valuation $\theta(\mathfrak{c}) = \frac{d^+(\mathfrak{c})}{2}$ is an OCV on G. Thus, $\mathcal{B} := (G, \theta)$ is a valued digraph. Let us summarize this construction in a definition.

Definition 3.3.18. Let $G = (V, E)$ be the digraph such that

$$V := \lambda_n^s \text{ and } E := \{(\mathfrak{c}, \mathfrak{d}) \in (\lambda_n^s)^2 \mid \mathfrak{c} \neq \mathfrak{d} \text{ and } \mathfrak{d} \in \widetilde{H}(\mathfrak{c})\}.$$

We denote by $\mathcal{B} = (G, \theta)$ the valued digraph such that for any $\mathfrak{c} \in \lambda_n^s$

$$\theta(\mathfrak{c}) := \frac{d^+(\mathfrak{c})}{2}.$$

One can easily check that the poset $(IS(\mathcal{B}), \subseteq)$ is isomorphic to the weak order on $(B_n, \leq_R)$ when $n = 2$ or 3. This situation is in fact general, as stated in the following theorem.

Theorem 3.3.19. *The posets $(B_n, \leq_R)$ and $(IS(\mathcal{B}), \subseteq)$ are isomorphic.*

The proof of this theorem follows the exact same pattern as the one of Theorem 3.3.3. However, many technical difficulties appear in the B_n case, so that we detail completely the proofs in the following section.

3.3.4. Proof of Theorem 3.3.19.

In this section, we show that

$$IS(\mathcal{B}) = \{\mathrm{Inv}_B(\omega) \mid \omega \in B_n\},$$

which implies Theorem 3.3.19. We follow the same method as in Section 3.3.2 and we divide our proof into three steps. First, for any $\omega \in B_n$ we define a statistic d_ω on $\lambda_n^s \setminus \mathrm{Inv}_B(\omega)$ (Definition 3.3.20). Then by using the notion of adjacency (see 3.3.15) in B_n we provide an alternative definition of d_ω (Lemma 3.3.21), leading to a combinatorial interpretation of the valuations appearing when one performs the peeling process on $\mathcal{B}$ (Proposition 3.3.22). Finally, we prove that $IS(\mathcal{B}) = \{\mathrm{Inv}_B(\omega) \mid \omega \in B_n\}$ by using this combinatorial interpretation (Corollary 3.3.24). Moreover, as a consequence we obtain the converse implication of (3.5) (Corollary 3.3.25), so that $(IS(\mathcal{B}), \subseteq)$ and $(B_n, \leq_R)$ are isomorphic.

We begin with the definition of the statistic d_ω.

Definition 3.3.20. Let $\omega \in B_n$ and $(a, b) \in \lambda_n^s \setminus \mathrm{Inv}_B(\omega)$. We define the statistic $d_\omega(a, b)$ as follows:

- if $|a| < b$, then $d_\omega(a, b) := |\{a < k < b \mid \omega^{-1}(a) < \omega^{-1}(k) < \omega^{-1}(b)\}|$;
- if $-a = b$, then $d_\omega(a, -a) := |\{1 \leq k < -a \mid \omega^{-1}(a) < \omega^{-1}(k) < \omega^{-1}(-a)\}|$.

The statistic d_ω admits the following characterization.

Lemma 3.3.21. *Let $\omega \in B_n$, and $(a, b) \in \lambda_n^s \setminus \mathrm{Inv}_B(\omega)$, then we have*

$$d_\omega(a, b) = d_{Id}(a, b) - |\widetilde{H}(a, b) \cap \mathrm{Inv}_B(\omega)|.$$

Proof. Let $s_1 \cdots s_q$ be a reduced decomposition of ω and denote by ω_i the signed permutation $s_1 \cdots s_i$, $0 \leq i \leq q$. We prove by induction on i that the lemma is true for ω_i.

The property is obviously true for Id. Let $i \geq 0$ be such that the property is true. For the sake of clarity, let us denote by δ_j the integer $d_{\omega_j}(a, b)$. Since $s_1 \cdots s_q$ is reduced, we have $\sigma_i \lhd_R \sigma_{i+1}$, hence there exists a unique (a_{i+1}, b_{i+1}) in $\mathrm{Inv}_B(\omega_{i+1}) \setminus \mathrm{Inv}_B(\omega_i)$. We now show how one can deduce the value of δ_{i+1} from both δ_i and (a_{i+1}, b_{i+1}). We split our study into three cases

- (Case $(a_{i+1}, b_{i+1}) = (a, k)$ with $a < k < b$ and $|a| \leq k$) ω_{i+1} is obtained from ω_i by swapping the positions of a and k and the positions of $-k$ and $-a$. Furthermore, a and k are adjacent in ω_i and $(a, b) \notin \mathrm{Inv}_B(\omega_i)$, so that we have
$$\omega_i = [\ldots,\ a\ ,\ k\ ,\ldots,\ b\ ,\ldots].$$
We now distinguish two sub-cases.
  - If $|a| < b$, then the full window notation of ω has one of the three following forms:
$$\begin{aligned} &[\ldots,\ a\ ,\ k\ ,\ldots,\ -k\ ,\ -a\ ,\ldots,\ b\ ,\ldots], \\ \text{or } &[\ldots,\ a\ ,\ k\ ,\ldots,\ b\ ,\ldots,\ -k\ ,\ -a\ ,\ldots], \\ \text{or } &[\ldots,\ -k\ ,\ -a\ ,\ldots,\ a\ ,\ k\ ,\ldots,\ b\ ,\ldots]. \end{aligned}$$
Therefore, in all cases either both $-k$ and $-a$ are between a and b in ω_i, or both $-k$ and $-a$ are not between a and b. Hence, it is again the case in ω_{i+1}, so we have $\delta_{i+1} = \delta_i - 1$. Moreover, we also have
$$|\widetilde{H}(a, b) \cap \mathrm{Inv}_B(\omega_{i+1})| = |\widetilde{H}(a, b) \cap \mathrm{Inv}_B(\omega_i)| + 1.$$
  - If $b = -a$, then we have
$$\omega_i = [\ldots,\ a\ ,\ k\ ,\ldots,\ -k\ ,\ -a\ ,\ldots],$$
so that both k and $-k$ lie between a and $-a$ in ω_i, and it is no longer the case in ω_{i+1}. Hence, by definition of $d_\omega(a, -a)$, we have $\delta_{i+1} = \delta_i - 1$. Furthermore, we also have
$$|\widetilde{H}(a, b) \cap \mathrm{Inv}_B(\omega_{i+1})| = |\widetilde{H}(a, b) \cap \mathrm{Inv}_B(\omega_i)| + 1.$$
- (case $(a_{i+1}, b_{i+1}) = (k, b)$ or $(-k, -a)$ with $a < k < b$) using similar arguments as in the previous case, we show that $\delta_{i+1} = \delta_i - 1$ (notice that the case $(a_{i+1}, b_{i+1}) = (-b, -k)$ cannot occur thanks to the condition $|a| \leq b$) and
$$|\widetilde{H}(a, b) \cap \mathrm{Inv}_B(\omega_{i+1})| = |\widetilde{H}(a, b) \cap \mathrm{Inv}_B(\omega_i)| + 1.$$
- Otherwise, we have that both a_{i+1} and b_{i+1} either lie between a and b in ω_i, or do not lie between a and b (and similarly for $-b_{i+1}$ and $-a_{i+1}$). Thus, it is still true in ω_{i+1}, so that $\delta_{i+1} = \delta_i$ and
$$|\widetilde{H}(a, b) \cap \mathrm{Inv}_B(\omega_{i+1})| = |\widetilde{H}(a, b) \cap \mathrm{Inv}_B(\omega_i)|.$$

By induction hypothesis ω_{i+1} satisfy the property, so that the lemma is proved. ☐

We now give a combinatorial interpretation of θ_A for some $A \in IS(\mathcal{B})$

Proposition 3.3.22. *Let $A \in IS(\mathcal{A})$, if there exists $\omega \in B_n$ such that $A = \mathrm{Inv}_B(\omega)$, then for all $(a, b) \in \lambda_n^s \setminus \mathrm{Inv}_B(\omega)$, we have $\theta_A(a, b) = d_\omega(a, b)$.*

Proof. Note that $\theta(a, b) = d_{Id}(a, b)$. Thus, by the definitions of the underlying digraph of $\mathcal{B}$ using shifted hooks and of the peeling process, and thanks to Lemma 3.3.21, the property follows. ☐

Proposition 3.3.22 allows us to link the weak order on B_n and the poset $(IS(\mathcal{B}), \subseteq)$, as it is shown in the next proposition.

Proposition 3.3.23. *Let $A \in IS(\mathcal{B})$. If there exists $\omega \in B_n$ such that $A = \mathrm{Inv}_B(\omega)$, then for all $(a, b) \in \lambda_n^s \setminus A$, we have that (a, b) is erasable in $\mathcal{B}_A$ if and only if a and b are adjacent in ω.*

Proof. Let $(a, b) \in \lambda_n^s \setminus A$ and assume that a and b are adjacent in ω. Our aim is to prove that (a, b) is erasable in $\mathcal{B}_A$.

First, note that $\theta_A(a, b) = d_\omega(a, b) = 0$. Let $(c, d) \in \lambda_n^s \setminus A$ be such that there is an arc from (c, d) to (a, b). We prove that $\theta_A(c, d) \neq 0$. Equivalently, we show that $d_\omega(c, d) \neq 0$. By definition of the underlying digraph of $\mathcal{B}$, we have only three cases which we now detail.

- ($(a, b) = (c, p)$ such that $|c| \leq p < d$). Since a and b are adjacent in ω, we have that c and p are adjacent in ω. Moreover, we have $(c, d) \notin A = \mathrm{Inv}_B(\omega)$, so that c is on the left of d in the window notation of ω. It follows that we have
$$\omega = [\ldots,\ c\ ,\ p\ ,\ldots,\ d\ ,\ldots].$$
However, we have $c < p < d$ by hypothesis, hence $d_\omega(c, d) \geq 1$.
- ($(a, b) = (q, d)$ with $c < q < d$). We have that q and d are adjacent in ω. Moreover, we have $(c, d) \notin \mathrm{Inv}_B(\omega)$, so that we have
$$\omega = [\ldots,\ c\ ,\ldots,\ q\ ,\ d\ ,\ldots].$$
Nevertheless, we have $c < q < d$ by hypothesis, hence $d_\omega(c, d) \geq 1$.
- ($(a, b) = (k, -c)$ with $c \leq k < -c$). First, note that we have $c < -k \leq -c$. Moreover, we have $(c, d) \in \lambda_n^s$, so that $-c \leq |c| \leq d$. We thus have $c < -k \leq -c \leq d$. Assume by contradiction that $-k = d$, then $-c = d$, hence we have
$$(a, b) = (-d, d) = (k, -c) = (c, d).$$
Consequently, there is an arc from (a, b) to (a, b) in the underlying graph of $\mathcal{B}$, and this is absurd. Therefore, we have $c < -k < d$.

  Let us now show that $-k$ lies between c and d in ω. By hypothesis, we have
$$\omega = [\ldots,\ k\ ,\ -c\ ,\ldots],$$
but ω is a signed permutation, so that we have
$$\omega = [\ldots,\ c\ ,\ -k\ ,\ldots].$$
However, $(c, d) \notin \mathrm{Inv}_B(\omega)$, hence we have
$$\omega = [\ldots,\ c\ ,\ -k\ ,\ldots,\ d\ ,\ldots].$$
Therefore, if $-c \neq d$, then we have $d_\omega(c, d) \geq 1$. If $-c = d$, then we have
$$\omega = [\ldots,\ c\ ,\ -k\ ,\ldots,\ k\ ,\ -c\ ,\ldots],$$
and we also have $c < -k < k < -c$, so that $d_\omega(c, d) \geq 1$

In all cases, we have $d_\omega(c, d) \geq 1$, but $\theta(c, d) = d_\omega(c, d)$ by Proposition 3.3.22, hence $\theta(c, d) \geq 1$. Thus, we just proved that for all box $\mathfrak{c} \in \lambda_n^s \setminus A$, if there is an arc from $\mathfrak{c}$ to (a, b), then $\theta(\mathfrak{c}) \geq 1$. Consequently, (a, b) is erasable in $\mathcal{B}_A$.

Let us now prove the converse. Let $(a, b) \in \lambda_n^s \setminus A$ be erasable in $\mathcal{B}_A$ and assume by contradiction that a and b are not adjacent in ω. We divide the study into two cases.

- (Case $a = -b$) Since $-b$ and b are not adjacent in ω, there exists k lying between $-b$ and b in ω. By symmetry, both k and $-k$ lie between $-b$ and b, thus we can suppose that $k > 0$. Furthermore, $d_\omega(-b, b) = 0$, so that $k > b$. Let us consider $p > b$ minimal lying between a and b and let q be an integer lying between $-b$ and p in ω (if such a q exists). Then, we have
$$\omega = [\ldots,\ -b\ ,\ldots,\ q\ ,\ldots,\ p\ ,\ldots,\ b\ ,\ldots],$$
so that q is between $-b$ and b in ω. Moreover, we have the following facts:
  - by minimality of p, $q \notin [\![b, p]\!]$;
  - since $d_\omega(-b, b) = 0$, $q \notin [\![-b, b]\!]$.

  Consequently, $q \notin [\![-b, p]\!]$, hence $d_\omega(-b, p) = 0 = \theta_A(-b, p)$. However, there is an arc from $(-b, p)$ to $(-b, b)$, and this is a contradiction since $(-b, b)$ is erasable.
- (Case $|a| < b$) There exists k lying between a and b such that either $k > b$ or $k < a$. In the first case, similar arguments as in the previous case lead to a contradiction with the fact that (a, b) is erasable. In the second case, we consider $p < a$ maximal lying between a and b. We have the following two sub-cases.

– If $-b \leq p < a$, then for each q between p and b in ω, we have either $q < p$ by maximality, or $q > b$ because $d_\omega(a, b) = 0$. Thus, $d_\omega(p, b) = 0$, so that $\theta(p, b) = 0$. but there is an arc from (p, b) to (a, b), hence it contradicts the fact that (a, b) is erasable.

– If $p < -b$, then we prove that $d_\omega(-b, -p) = 0$. For that purpose, assume by contradiction that $d_\omega(-b, -p) = 0$. Thus, there exists q between $-b$ and $-p$ in ω such that $-b < q < -p$. Then, we have $p < -q < b$. Moreover, since ω is a signed permutation we have

$$\omega = [\ldots,\ p\ ,\ldots,\ -q\ ,\ldots,\ b\ ,\ldots], \tag{3.6}$$

but p is between a and b in ω, hence we have

$$\omega = [\ldots,\ a\ ,\ldots,\ -q\ ,\ldots,\ b\ ,\ldots].$$

Therefore, we have $p < -q < b$ and by maximality of p, we have $a \leq -q$. Since p is between a and b in ω, thanks to 3.6, we have $a \neq -q$. Eventually, we have $a < -q < b$, and this is absurd since $d_\omega(a, b) = 0$. Consequently, we have $d_\omega(-b, -p) = 0$, so that $\theta(-b, -p) = 0$. But there is an arc from $(-b, -p)$ to (a, b), hence it contradicts the fact that (a, b) is erasable.

Finally, in all cases we obtain a contradiction. Thus, a and b are adjacent in ω and this concludes the proof. □

With Proposition 3.3.23, one can prove the following result using exactly the same method as in the proof of Corollary 3.3.10.

COROLLARY 3.3.24. $IS(\mathcal{B}) = \{\mathrm{Inv}_B(\omega) \mid \omega \in B_n\}$.

This result has the following important consequence (which gives the converse direction of (3.5)).

COROLLARY 3.3.25. *Let $\sigma, \omega \in B_n$, then $\sigma \leq_R \omega$ if and only if $\mathrm{Inv}_B(\sigma) \subseteq \mathrm{Inv}_B(\omega)$.*

PROOF. The direct direction is given by (3.5), and we now prove the converse. Assume that $\mathrm{Inv}_B(\sigma) \subseteq \mathrm{Inv}_B(\omega)$, then there exist $L = [z_1, \ldots, z_{n^2}] \in PS(\mathcal{B})$ and $p \leq q$ two integers such that $\mathrm{Inv}(\sigma) = \{z_1, \ldots, z_p\}$ and $\mathrm{Inv}(\omega) = \{z_1, \ldots, z_q\}$. Then thanks to Corollary 3.3.24 and Proposition 3.3.17, there exist $\sigma_1, \ldots, \sigma_k$ such that $\sigma = \sigma_1 \lhd_R \sigma_2 \lhd_R \ldots \lhd_R \sigma_k = \omega$, and this ends the proof. □

Consequently, thanks to Corollary 3.3.24 the posets $(IS(\mathcal{B}), \subseteq)$ and $(B_n, \leq_R)$ are isomorphic. This concludes the proof of Theorem 3.3.19.

3.3.5. Weak order on $\widetilde{A}_n$. Recall that $\widetilde{A}_n$ is the Coxeter Group with generating set $S = \{s_1, \ldots, s_n\}$, and with Coxeter matrix given by $m_{s_i s_j} = 3$ if $j = i+1$ (where the indices are taken modulo n), and $m_{s_i s_j} = 2$ otherwise. This group can be seen as the group of the *affine permutations*, that is, the group of all the bijections $\sigma : \mathbb{Z} \mapsto \mathbb{Z}$ such that:

(1) for all k and q in $\mathbb{Z}$, $\sigma(q + kn) = \sigma(q) + kn$;
(2) $\sigma(1) + \sigma(2) + \cdots + \sigma(n) = \frac{n(n+1)}{2}$.

Thanks to this interpretation, we identify s_i with the affine permutation swapping positions of the integers $i + kn$ and $i + 1 + kn$, for all $k \in \mathbb{Z}$.

We are going to follow the same method of the previous sections. In order to find a candidate of valued digraph, we consider a notion of $\widetilde{A}$-inversion set adapted to the case of $\widetilde{A_n}$ (see Definition 3.3.29). After we checked that this notion is effectively related to the weak order on $\widetilde{A_n}$ (see Property 3.8), we propose a graphical interpretation of $\widetilde{A}$-inversion sets using *cylindrical diagrams*. Once again, this representation carries a natural notion of hooks, called *cylindrical hooks*, which leads us to define a digraph structure on a cylindrical diagram as in Sections 3.3.1 and 3.3.3. Then, we define a valued digraph using the resulting digraph, and we check that the obtained lattice is indeed isomorphic to $(\widetilde{A_n}, \leq_R)$.

REMARK 3.3.26. We point out that, as in Section 3.3.3, the notion of $\widetilde{A}$-inversion set we use here does not come from a root system of $\widetilde{A_n}$. This choice gives the same benefits (a "permutation point of view" on the weak order) and disadvantages (we have to relate $\widetilde{A}$-inversion sets to the weak order by ourselves) as in Section 3.3.3. Fortunately, once again most of technical points have already been studied in [**BB**].

We associate to each affine permutation σ an $\widetilde{A}$-inversion set, define as follows:

$$\mathrm{Inv}_{\widetilde{A}}(\sigma) := \{(a,b) \in [n] \times \mathbb{N}^* \mid a < b, \text{ and } \sigma^{-1}(a) > \sigma^{-1}(b)\}, \tag{3.7}$$

DEFINITION 3.3.27 (see [**BB**], eq. (8.30) p.261). Let $\sigma \in \widetilde{A_n}$, the $\widetilde{A}$-inversion number of σ is the quantity

$$\mathrm{inv}_{\widetilde{A}}(\sigma) := |\{(a,b) \in [n] \times \mathbb{N}^* \mid a < b \text{ and } \sigma(a) > \sigma(b)\}|$$

Note that we clearly have $\mathrm{inv}_{\widetilde{A}}(\sigma) = |\mathrm{Inv}_{\widetilde{A}}(\sigma)|$ for all $\sigma \in \widetilde{A_n}$.

LEMMA 3.3.28 (see [**BB**], Eq. (8.34) p. 262). *Let $\sigma \in \widetilde{A_n}$ and $i \in [n]$, we have*

$$\mathrm{inv}_{\widetilde{A}}(\sigma s_i) = \begin{cases} \mathrm{inv}_{\widetilde{A}}(\sigma) + 1, & \text{if } \sigma(i) < \sigma(i+1), \\ \mathrm{inv}_{\widetilde{A}}(\sigma) - 1, & \text{if } \sigma(i) > \sigma(i+1). \end{cases}$$

We now begin to prove that $\widetilde{A}$-inversion sets can be used to study $(\widetilde{A_n}, \leq_R)$. That is, we show that we have

$$\text{for all } \sigma, \omega \in \widetilde{A_n}, \text{ if } \sigma \leq_R \omega, \text{ then } \mathrm{Inv}_{\widetilde{A}}(\sigma) \subseteq \mathrm{Inv}_{\widetilde{A}}(\omega). \tag{3.8}$$

We start with defining the equivalent of the notion of adjacency in type $\widetilde{A}$.

DEFINITION 3.3.29. Let $\sigma \in \widetilde{A_n}$ and $(a,b) \in [n] \times \mathbb{N}^*$. We say that a and b are $\widetilde{A}$-*adjacent* in σ if and only if $a < b$ and $\sigma^{-1}(a) = \sigma^{-1}(b) - 1$.

We are now able to state the lemma which connects $\widetilde{A}$-inversion sets to $(\widetilde{A_n}, \leq_R)$.

LEMMA 3.3.30. *Let $\sigma \in \widetilde{A_n}$ and $1 \leq j \leq n$. Then, there exists (a,b) such that $1 \leq a \leq n$, $a < b$ and σs_j is obtained from σ by swapping positions of the integers $a + kn$ and $b + kn$ for all $k \in \mathbb{Z}$, and we have two possibilities:*

- *if a and b are adjacent in σ, then $\ell(\sigma s_j) = \ell(\sigma) + 1$ and $\mathrm{Inv}_{\widetilde{A}}(\sigma s_j) = \mathrm{Inv}_{\widetilde{A}}(\sigma) \cup \{(a,b)\}$;*
- *if a and b are not adjacent in σ, then $\ell(\sigma s_j) = \ell(\sigma) - 1$ and $\mathrm{Inv}_{\widetilde{A}}(\sigma s_j) = \mathrm{Inv}_{\widetilde{A}}(\sigma) \setminus \{(a,b)\}$.*

PROOF. This is an immediate translation of the results in Lemma 3.3.28 in terms of $\widetilde{A}$-adjacency. □

An immediate consequence of Lemma 3.3.30 is that (3.8) holds.

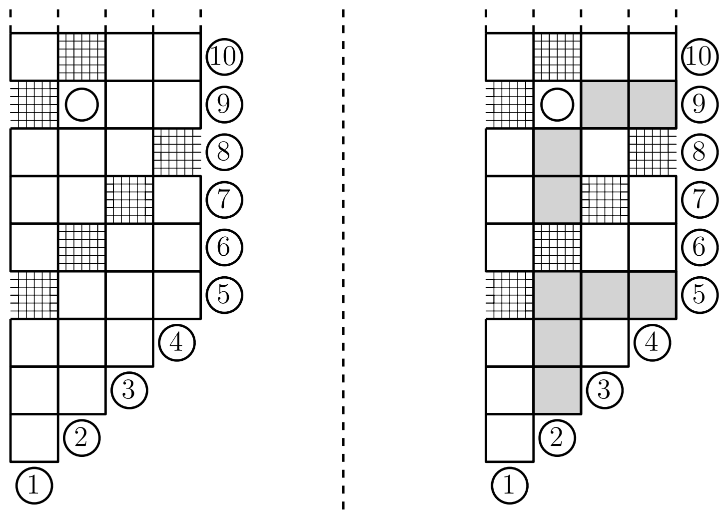
REMARK 3.3.31. As in Section 3.3.3, note that the converse implication holds, and it is also a by-product of the following results.

We now introduce a convenient way to represent $\widetilde{A}$-inversion sets. First, note that for all $\sigma \in \widetilde{A_n}$ and for all $a, b \in \mathbb{Z}$ such that $1 \leq a \leq n$ and $b \equiv a \pmod n$, since σ is an affine permutation we have $(a,b) \notin \mathrm{Inv}_{\widetilde{A}}(\sigma)$. Thus, each $\widetilde{A}$-inversion is a subset of

$$\{(a,b) \in \mathbb{N}^2 \mid 1 \leq a \leq n,\ b \not\equiv a \pmod n,\ a < b\}.$$

This set can be represented by a diagram, which we denote by λ_n^{cyl}, as depicted in Figure 3.4.

From now on, we identify λ_n^{cyl} with the set $\{(a,b) \in \mathbb{N}^2 \mid 1 \leq a \leq n,\ b \not\equiv a \pmod n,\ a < b\}$. This diagram λ_n^{cyl} can be thought as an infinite version of the diagram associated with the symmetric group rolled around a cylinder. With this point of view, λ_n^{cyl} naturally carries a notion of hooks, which we call *cylindrical hooks*, as depicted on the right of Figure 3.4. More



The coordinates of this box are $(2, 9)$.

Cylindric hook based on $(2, 9)$.

FIGURE 3.4. Diagram λ_4^{cyl}

formally, for all $(a, b) \in \lambda_n^{cyl}$, the cylindrical hook based on (a, b) is the subset $H^{cyl}(a, b)$ of λ_n^{cyl} defined by:

$$H^{cyl}(a,b) := \ \{(a,k) \in \lambda_n^{cyl} \mid a < k < b\} \ \cup \left(\bigcup_{\substack{q \equiv b \pmod n \\ q \leq b}} \{(k,q) \in \lambda_n^{cyl} \mid a < k < b\} \right)$$

Consequently, we can define a digraph structure G on λ_n^{cyl} using cylindrical hooks. That is, for all $\mathfrak{c}, \mathfrak{d} \in \lambda_n^{cyl}$, there is an arc from $\mathfrak{c}$ to $\mathfrak{d}$ in G if and only if $\mathfrak{c} \neq \mathfrak{d}$ and $\mathfrak{d} \in H^{cyl}(\mathfrak{c})$. Notice that the out-degree of a box of λ_n^{cyl} is generally not an even number, so that we cannot define the valuation as in the previous sections. Nevertheless, after some tests it appears that the valuation θ defined for all $(a, b) \in \lambda_n^{cyl}$ by

$$\theta(a,b) := |\{(a,k) \in \lambda_n^{cyl} \mid a < k < b\}|,$$

which is just the number of boxes which are below (a, b) in the graphical representation of λ_n^{cyl}, seems to lead to the expected description of the weak order on $\widetilde{A}_n$. Before moving to the proof that this is indeed the case, let us summarize this construction in a definition.

DEFINITION 3.3.32. Let $G = (V, E)$ be the digraph defined by

$$V := \lambda_n^{cyl} \text{ and } E := \{(\mathfrak{c}, \mathfrak{d}) \in (\lambda_n^{cyl})^2 \mid \mathfrak{c} \neq \mathfrak{d} \text{ and } \mathfrak{d} \in H^{cyl}(\mathfrak{c})\}.$$

We denote by $\widetilde{\mathcal{A}} = (G, \theta)$ the valued digraph such that for all $(a, b) \in \lambda_n^{cyl}$,

$$\theta(a,b) = |\{(a,k) \in \lambda_n^{cyl} \mid a < k < b\}|.$$

Our aim is now to prove that $(IS(\widetilde{\mathcal{A}}), \subseteq)$ is isomorphic to $(\widetilde{A}_n, \leq_R)$. This can be done following exactly the same method as in Section 3.3.4, and we refer the reader to the introduction of Section 3.3.4 for the detail of the different steps.

We first define the statistic on the affine permutations, which leads us to the combinatorial interpretation of the valuations appearing when one perform the peeling process on $\widetilde{\mathcal{A}}$.

DEFINITION 3.3.33. Let $\sigma \in \widetilde{A}_n$ and $(a, b) \in \lambda_n^{cyl} \setminus \mathrm{Inv}_{\widetilde{A}}(\sigma)$. We set

$$d_\sigma(a,b) := |\{a < k < b \mid k \not\equiv a \pmod n,\ \sigma^{-1}(a) \leq \sigma^{-1}(k) \leq \sigma^{-1}(b)\}|.$$

We then have the following characterization of the statistic d_σ.

LEMMA 3.3.34. *For all $\sigma \in \widetilde{A_n}$ and $(a,b) \in \lambda_n^{cyl} \setminus \mathrm{Inv}_{\widetilde{A}}(\sigma)$, we have*

$$d_\sigma(a,b) = d_{Id}(a,b) - |H^{cyl}(a,b) \cap \mathrm{Inv}_{\widetilde{A}}(\sigma)|.$$

PROOF. The proof is similar as the one of Lemma 3.3.6. □

Thanks to Lemma 3.3.34, we have the following proposition.

PROPOSITION 3.3.35. *Let $A \in IS(\widetilde{A})$, if there exists $\sigma \in \widetilde{A_n}$ such that $A = \mathrm{Inv}_{\widetilde{A}}(\sigma)$, then for all $(a,b) \in \lambda_n^{cyl} \setminus A$ we have $\theta_A(a,b) = d_\sigma(a,b)$.*

PROOF. The proof is similar as the one of Proposition 3.3.8. □

We can now state and prove the main proposition of this section.

PROPOSITION 3.3.36. *Let $A \in IS(\widetilde{A})$, if there exists $\sigma \in \widetilde{A_n}$ such that $A = \mathrm{Inv}_{\widetilde{A}}(\sigma)$, then (a,b) is erasable in $\widetilde{A}_A$ if and only if a and b are $\widetilde{A}$-adjacent in σ.*

PROOF. Once again, the proof is similar to that of Proposition 3.3.9. □

Eventually, we have the following three corollaries that conclude this section.

COROLLARY 3.3.37. *$IS(\widetilde{A}) = \{\mathrm{Inv}_{\widetilde{A}}(\sigma) \mid \sigma \in \widetilde{A_n}\}$.*

COROLLARY 3.3.38. *Let $\sigma, \omega \in \widetilde{A_n}$. Then, $\sigma \leq_R \omega$, if and only if $\mathrm{Inv}_{\widetilde{A}}(\sigma) \subseteq \mathrm{Inv}_{\widetilde{A}}(\omega)$.*

COROLLARY 3.3.39. *The two posets $(IS(\widetilde{\mathcal{A}}), \subseteq)$ and $(\widetilde{A_n}, \leq_R)$ are isomorphic.*

3.3.6. Flag Weak Order on $\mathbb{Z}_r \wr S_n$. In this section, we consider an order on $G(r,n) := \mathbb{Z}_r \wr S_n$ (introduced by Adin, Brenti and Roichman in [**ABR**]), called the *flag weak order*, that generalizes the weak order on the symmetric group. In order to define this new poset, let us first introduce some notations and definitions. We denote by $\mathbb{Z}_r$ the (additive) cyclic group of order r and by $G(r,n)$ the group

$$G(r,n) := \{((c_1,\dots,c_n),\sigma) \mid c_i \in [r],\ \sigma \in S_n\}$$

with the group operation given by

$$((c_1,\dots,c_n),\sigma).((d_1,\dots,d_n),\omega) = ((c_{\omega(1)} + d_1,\dots,c_{\omega(n)} + d_n),\sigma\omega),$$

where the sums $c_{\omega(i)} + d_i$ are taken modulo r. This group is usually called the group of *r-colored permutations*, i.e. bijections g of the set $\mathbb{Z}_r \times \{1,\dots,n\}$ onto itself such that:

$$g(c,i) = (d,j) \Longrightarrow g(c+c',i) = (d+c',j).$$

Note that the group $G(r,n)$ can also be viewed as a *complex reflection group*. However, once again it is the "permutation point of view" on $G(r,n)$ that allows us to apply our theory here.

Before moving to the definition of the flag weak order, we introduce some useful notations taken from [**ABR**].

DEFINITION 3.3.40. Let $\pi = ((c_1,\dots,c_n),\sigma)$ be in $G(r,n)$, we define:

(1) $|\pi| = \sigma$;
(2) $\mathrm{Inv}(\pi) = \mathrm{Inv}(|\pi|)$;
(3) $n(\pi) = \sum_i c_i$;
(4) $\mathrm{finv}(\pi) = r.|\mathrm{Inv}(\pi)| + n(\pi)$, called the *flag inversion number* of π.

Let us now present the philosophy behind the definition of the flag weak order, by first recalling the definition of the weak order on the symmetric group. The definition of $(S_n, \leq_R)$ can be decomposed into two distinct steps:

- first, we consider a specific set S of generator of S_n (here the simple transpositions);
- then, we consider a statistic ℓ on S_n (here the length) and we define the weak order to be the reflexive and transitive closure of the relation

$$\text{for all } \sigma,\omega \in S_n,\ \sigma \lhd_R \omega \Longleftrightarrow \exists s \in S \text{ such that } \ \omega = \sigma s \ \text{ and } \ \ell(\tau) < \ell(\pi).$$

The flag weak order is defined by following a similar pattern:

- first, we define a special generating set of $G(r,n)$, denoted by $A \cup B$;
- then, we define the flag weak order to be the reflexive and transitive closure of the relation: for all $\pi, \tau \in G(r,n)$,

$$\pi \lhd_f \tau \iff \exists s \in A \cup B \text{ such that } \tau = \pi s \ \text{ and } \ \mathrm{finv}(\tau) < \mathrm{finv}(\pi).$$

As one can notice, the only difference with the definition of $(S_n, \leq_R)$ is that we swapped the length with the flag inversion number (see Definition 3.3.40). Let us now formalize this construction in a definition.

Definition 3.3.41 (Flag weak order, see [**ABR**]). We denote by A and B the two subsets of $G(r,n)$ defined by

$$A = \{a_i \in G(r,n) \mid i \in [n-1],\ a_i = ((\delta_{i1}, \ldots, \delta_{ii}, \ldots, \delta_{in}), s_i)\}, \text{ and}$$
$$B = \{b_i \in G(r,n) \mid i \in [n],\ b_i = ((\delta_{i1}, \ldots, \delta_{ii}, \ldots, \delta_{in}), Id)\},$$

where s_i is the i-th elementary transposition of the symmetric group and $\delta_{ij} = 1$ if $i = j$, and 0 otherwise. The *flag weak order* $\leq_f$ on $G(r,n)$ is the reflexive and transitive closure of the relation $\lhd_f$ defined by:

$$\forall \pi, \tau \in G(r,n),\ \pi \lhd_f \tau \iff \exists s \in A \cup B \text{ such that } \tau = \pi s \ \text{ and } \ \mathrm{finv}(\tau) < \mathrm{finv}(\pi).$$

The following lemma provide a complete description of covering relations in the flag weak order.

Lemma 3.3.42 ([**ABR**], Prop. 7.4). *Let $\pi = ((c_1, \ldots, c_n), \sigma) \in G(r,n)$ and $s \in A \cup B$. Then, πs covers π in the flag weak order if and only if one of the two following situations occur:*

(1) there exists $1 \leq i \leq n$ such that $s = b_i \in B$ and $c_i \neq r-1$;
(2) there exists $1 \leq i \leq n-1$ such that $s = a_i \in A$, $c_{i+1} = r-1$ and $\sigma(i) < \sigma(i+1)$.

We now have enough general informations about the flag weak order to propose a valued digraph which describes the flag weak order. The key point leading us to the construction of this valued digraph is that the elements of A "look like" the simple transpositions of S_n. That is, they act on r-coloured permutations as simple transpositions act on permutation, by swapping the positions of two adjacent entries. Furthermore, one can note that the function

$$\begin{array}{cccc} \phi & : & S_n & \longrightarrow & G(r,n) \\ & & \sigma & \longmapsto & ((0,\ldots,0),\sigma) \end{array}$$

is an injective poset morphism from $(S_n, \leq_R)$ to $(G(r,n), \leq_f)$. Thus, the flag weak order contains a sub-poset isomorphic to the weak order on S_n. Using this facts as hints and after some "guess and try" tests on the example of $G(2,4)$, the author found out a candidate of valued digraph, which is depicted on Figure 3.5. Once again, the digraph structure of this diagram is

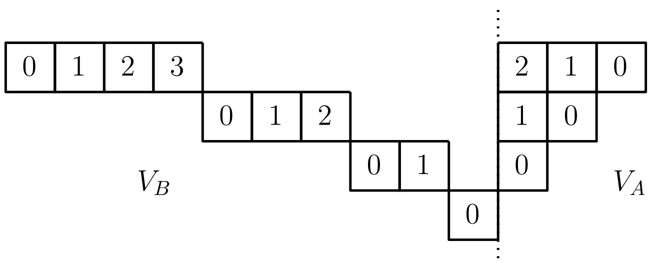


Figure 3.5. The candidate of valued digraph for $(G(2,4), \leq_f)$

given implicitly, using a suitable notion of hook. A bit more formally, we say that for all boxes $\mathfrak{c}$ and $\mathfrak{d}$ in this diagram, there is an arc from $\mathfrak{c}$ to $\mathfrak{d}$ if and only if $\mathfrak{c} \neq \mathfrak{d}$ and $\mathfrak{d}$ is either in the same row and on the right of $\mathfrak{c}$, or in the same column and below $\mathfrak{c}$. With this definition, one can check that the resulting poset is indeed isomorphic to $(G(2,4), \leq_f)$.

Let us now generalize and formalize this construction to the case of r and n arbitrary, by first defining the diagram.

Definition 3.3.43. We set $\lambda_{r,n} := V_{A,n} \cup V_{B,r,n}$ where

$$V_{A,n} := \left\{(a,b) \in [n]^2 \mid a < b\right\}, \text{ and } V_{B,r,n} := \{(a,b) \in \mathbb{Z} \times [n] \mid -b(r-1) \leq a \leq -1\}.$$

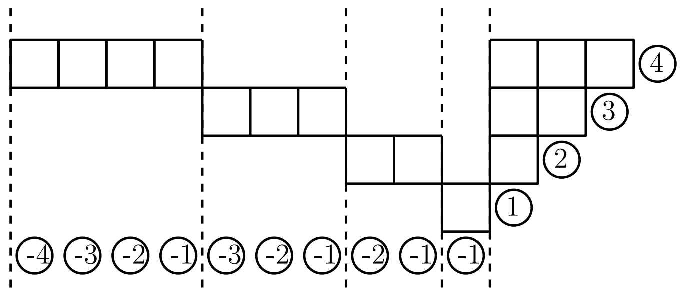


Figure 3.6. Graphical representation of $\lambda_{2,4}$ with its coordinates

Note that we have $V_{A,n} = \lambda_n$ (see Section 3.3.1), and we sometimes use this notation. Let us now define the notion of hook associated with the diagram $\lambda_{r,n}$, being suggested by its graphical representation (see Figure 3.6).

Definition 3.3.44. Let $(a,b) \in \lambda_{r,n}$, we denote by $H_f(a,b)$ the subset of $\lambda_{r,n}$ defined by

$$H_f(a,b) := \begin{cases} \{(x,y) \in \lambda_{r,n} \mid \exists k \in \mathbb{N},\ a < k < b,\ (x,y) = (a,k) \text{ or } (k,b)\} & \text{if } (a,b) \in V_{A,n}, \\ \{(x,y) \in \lambda_{r,n} \mid a < x \text{ and } y = b\} & \text{if } (a,b) \in V_{B,r,n}. \end{cases}$$

Eventually, we can now define the valued digraph.

Definition 3.3.45. Let $G = (V,E)$ be the digraph defined by

$$V := \lambda_{r,n} \text{ and } E := \{(\mathfrak{c}, \mathfrak{d}) \in \lambda_{r,n}^2 \mid \mathfrak{c} \neq \mathfrak{d} \text{ and } \mathfrak{d} \in H_f(\mathfrak{c})\}.$$

We denote by $\mathcal{G}(r,n) := (G,\theta)$ the valued digraph such that for all $(a,b) \in \lambda_{r,n}$

$$\theta(a,b) := \begin{cases} b - a - 1 & \text{if } (a,b) \in V_{A,n}, \\ b + \left\lfloor \dfrac{a}{r-1} \right\rfloor & \text{if } (a,b) \in V_{B,r,n}. \end{cases}$$

Example 3.3.6.1. We represent on Figure 3.7 the valued digraph $\mathcal{G}(2,4)$. As one can see, this is exactly the valued digraph depicted on Figure 3.5.

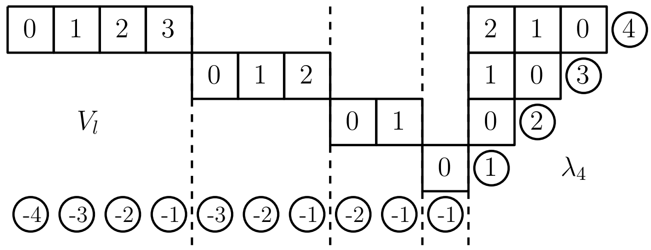


Figure 3.7.

Our aim is now to show that $(IS(\mathcal{G}(r,n)), \subseteq)$ and $(G(r,n), \leq_f)$ are isomorphic, by constructing an explicit poset isomorphism. For that purpose, we split our study into two distinct steps: we first construct a bijection between $IS(\mathcal{G}(r,n))$ and $G(r,n)$ (see Definition 3.3.49 and

Proposition 3.3.51), and then we show that this bijection is in fact a poset isomorphism (see Theorem 3.3.52). We begin with a lemma, which shows how we can associate a permutation with each element of $IS(\mathcal{G}(r,n))$.

LEMMA 3.3.46. *Let $U \in IS(\mathcal{G}(r,n)$, then $U \cap V_{A,n}$ is the inversion set of a permutation in A_{n-1}.*

PROOF. Let us denote by X the set $U \cap V_{A,n}$ and by E' the set of arcs of $\mathcal{A}$, where $\mathcal{A}$ is the valued digraph associated to $(S_n, \leq_R)$ defined in Section 3.3.1. Since $\lambda_n = V_{A,n}$, if $X \in IS(\mathcal{A})$, then X is the inversion set of a permutation thanks to Corollary 3.3.10. We still have to show that X is in $IS(\mathcal{A})$. First, notice that for all $\mathfrak{c}$ in $V_{A,n}$, we have $H(\mathfrak{c}) = H_f(\mathfrak{c})$, where $H(\mathfrak{c})$ is the hook based on $\mathfrak{c}$ in λ_n defined in Section 3.3.1. Thus, by definition of the underlying digraph of $\mathcal{G}(r,n)$, for all $\mathfrak{c} \in V_{A,n}$ we have

$$\begin{aligned}\{\ \mathfrak{d} \in \lambda_n \mid \mathfrak{d} \in U,\ (\mathfrak{c},\mathfrak{d}) \in E\} &= U \cap H_f(\mathfrak{c})\\ &= U \cap H(\mathfrak{c})\\ &= \{\ \mathfrak{d} \in \lambda_n \mid \mathfrak{d} \in U,\ (\mathfrak{c},\mathfrak{d}) \in E'\}.\end{aligned}$$

Consequently, for all $\mathfrak{c} \in \lambda_n$, if $\mathfrak{c} \in U$, then by Proposition 3.2.1 we have

$$\begin{aligned}\theta(\mathfrak{c}) &\leq |\{\mathfrak{d} \in \lambda_n \mid \mathfrak{d} \in U,\ (\mathfrak{c},\mathfrak{d}) \in E\}|\\ &\leq |\{\mathfrak{d} \in \lambda_n \mid \mathfrak{d} \in U,\ (\mathfrak{c},\mathfrak{d}) \in E'\}|,\end{aligned}$$

and the converse inequality holds when $\mathfrak{c} \notin U$. Thus, X is in $IS(\mathcal{A})$ by Proposition 3.2.1, and this ends the proof. □

Thanks to Lemma 3.3.46, one can associate to each element of $IS(\mathcal{G}(r,n))$ a permutation in S_n. What remains to understand is how to associate a color to each value of the permutation. For that purpose, we introduce a new notation.

DEFINITION 3.3.47. Let $U \in IS(\mathcal{G}(r,n))$ and $i \in [n]$, we define the following two quantities

$$R_i(U) := |\{(x,i) \in U \mid (x,i) \in V_{A,n}\}| \text{ and } L_i(U) := |\{(x,i) \in U \mid (x,i) \in V_{B,r,n}\}|.$$

LEMMA 3.3.48. *Let $U \in IS(\mathcal{G}(r,n))$ and $i \in [n]$. Then, we have*

$$0 \leq L_i(U) - (r-1)R_i(U) \leq r-1.$$

PROOF. By definition, for all $(x,i) \in V_{B,r,n}$ and $(y,i) \in V_{A,n}$, we have $((x,i),(y,i)) \in E$. Thus, we have for all $(x,i) \in V_{B,r,n}$

$$R_i(U) \leq |\{\mathfrak{d} \in U \mid ((x,i),\mathfrak{d}) \in E\}|.$$

Let us consider the following set

$$X = \{(x,i) \in V_{B,r,n} \mid -i(r-1) \leq x < -(i - R_i(U))(r-1)\}.$$

Clearly, we have $|X| = (r-1)R_i(U)$. Moreover, by definition we have for all $(x,i) \in X$ that

$$\begin{aligned}&\frac{x}{r-1} < R_i(U) - i \implies \left\lfloor \frac{x}{r-1} \right\rfloor < R_i(U) - i \implies i + \left\lfloor \frac{x}{r-1} \right\rfloor < R_i(U)\\ &\implies \theta(x,i) < R_i(U) \implies \theta(x,i) < |\{\mathfrak{d} \in U \mid ((x,i),\mathfrak{d}) \in E\}|.\end{aligned}$$

We thus have $X \subseteq U$ by Proposition 3.2.1. Therefore, we have

$$(r-1)R_i(U) = |X| \leq L_i(U) \implies 0 \leq L_i(U) - (r-1)R_i(U). \tag{3.9}$$

To prove the converse inequality, we consider the set

$$Y = \{(y,i) \in V_{B,r,n} \mid -(i - R_i(U) - 1)(r-1) \leq y \leq -1\}.$$

For all $(y,i) \in Y$, we have

$$R_i(U) - i + 1 \leq \frac{y}{r-1} \implies R_i(U) - i < \left\lfloor \frac{y}{r-1} \right\rfloor \implies R_i(U) < \theta(y,i).$$

Let us now fix (y,i) in Y. We show by backward induction on y that $(y,i) \notin U$. By definition, each arc having $(-1,i)$ as starting point has an element of $V_{A,n}$ as ending point. Moreover, such an arc has its ending point in row i, so that we have

$$|\{\mathfrak{d} \in U \mid ((-1,i),\mathfrak{d}) \in E\}| \ \leq \ R_i(U) \ < \ \theta(-1,i).$$

Thus, $(-1,i) \notin U$ by Proposition 3.2.1, and one can finish the induction using similar arguments. Consequently, $Y \cap U$ is empty. However, we have $|Y| = (i - R_i(U) - 1)(r-1)$, so that

$$L_i(U) \leq i(r-1) - |Y| \ = \ (R_i(U)+1)(r-1) \implies L_i(U) - R_i(U)(r-1) \leq r-1. \tag{3.10}$$

Combining 3.10 and 3.9, we have the expected result. ☐

Thanks to Lemma 3.3.48 and Lemma 3.3.46, we are now able to associate a r-colored permutation to each element of $IS(\mathcal{G}(r,n))$.

Definition 3.3.49. Let $U \in IS(\mathcal{G}(r,n))$. We denote by σ_U the unique permutation such that $\mathrm{Inv}(\sigma_U) = U \cap V_{A,n}$, and we denote by $(c_i(U))_{1\leq i\leq n}$ the sequence defined by

$$c_{\sigma^{-1}(i)}(U) := L_i(U) - (r-1)R_i(U).$$

We denote by Ψ the map from $IS(\mathcal{G}(r,n))$ to $G(r,n)$ defined by

$$\text{for all } U \in IS(\mathcal{G}(r,n)),\ \Psi(U) \ = \ ((c_i(U))_{1\leq i\leq n}, \sigma_U).$$

In what follows, we will show that the function Ψ is a poset isomorphism between $(IS(\mathcal{G}(r,n)), \subseteq)$ and $(G(r,n), \leq_f)$. We first give a technical lemma, which is useful for both step of our proof.

Lemma 3.3.50. *Let $U \in IS(\mathcal{G}(r,n))$ and $(a,b) \in V_{B,r,n}$. Then, for all $k \in \mathbb{Z}$ such that*

$$-b(r-1) \leq k < a,$$

if $(a,b) \in U$, then $(k,b) \in U$. On the representation of $\mathcal{G}(r,n)$ as a diagram, this means that if a box of $V_{B,r,n}$ is in U, then all the boxes which are strictly on its left and in the same row are also in U.

Proof. Let k in $\mathbb{Z}$ be such that $-b(r-1) \leq k < a$ and assume that $(a,b) \in U$. By Proposition 3.2.1, we have

$$\theta(a,b) \leq |\{\mathfrak{d} \in U \mid ((a,b),\mathfrak{d}) \in E\}|.$$

Let us consider $\mathfrak{d} \in U$ such that $((a,b),\mathfrak{d}) \in E$. By construction of the underlying digraph of $\mathcal{G}(r,n)$, we have $((k,b),\mathfrak{d}) \in E$. Moreover, $((k,b),(a,b))$ is also in E, so that

$$|\{\mathfrak{d} \in U \mid ((a,b),\mathfrak{d}) \in E\}| < |\{\mathfrak{d} \in U \mid ((k,b),\mathfrak{d}) \in E\}|.$$

Finally, by definition of θ we have $\theta(k,b) \leq \theta(a,b)$, hence we have

$$\theta(k,b) < |\{\mathfrak{d} \in U \mid ((k,b),\mathfrak{d}) \in E\}|.$$

Thus, $(k,b) \in U$ by Proposition 3.2.1, and this concludes the proof. ☐

Proposition 3.3.51. *The function Ψ is a bijection.*

Proof. Let us first prove that Ψ is injective. Let $U, U' \in \mathcal{G}(r,n)$ such that $\Psi(U) = \Psi(U')$. Then $\sigma_U = \sigma_{U'}$, hence $\mathrm{Inv}(\sigma_U) = \mathrm{Inv}(\sigma_{U'})$, so that $U \cap V_{A,n} = U' \cap V_{A,n}$. Therefore, we have $R_j(U) = R_j(U')$ for all $j \in [n]$. Let us now fix $j \in [n]$, by definition of $(c_i)_{1\leq i\leq n}$ we have

$$L_j(U) - (r-1)R_j(U) = L_j(U') - (r-1)R_j(U'),$$

so that $L_j(U) = L_j(U')$. Thus, the number of boxes that are in $V_{B,r,n} \cap U$ and in row j equals the number of the boxes that are in $V_{B,r,n} \cap U'$ and in row j. However, thanks to Lemma 3.3.50, these boxes are left-justified in $V_{B,r,n}$, hence we have

$$\{(a,j) \in V_{B,r,n} \mid (a,j) \in U\} = \{(a,j) \in V_{B,r,n} \mid (a,j) \in U'\}.$$

Thus, $U \cap V_{B,r,n} = U' \cap V_{B,r,n}$ so $U = U'$, and this proves that Ψ is injective.

We now prove that Ψ is surjective. Let $\pi = ((c_i)_i, \sigma) \in G(r,n)$, we denote by R_i the quantity defined by

$$R_i := |\{(x,i) \in V_{A,n} \mid (x,i) \in \mathrm{Inv}(\sigma)\}|,$$

and we define the following sets:

$$\text{for all } i \in [n],\ U_i = \{(x,i) \mid -i(r-1) \leq x < -(i-R_i)(r-1) + c_{\sigma^{-1}(i)}\}.$$

We prove that $U := \mathrm{Inv}(\sigma) \cup U_1 \cup U_2 \cup \ldots \cup U_n$ is in $IS(\mathcal{G}(r,n))$. For that purpose, let us consider $\mathfrak{c} \in \lambda_{r,n}$ and divide our study into four cases.

(1) (case $\mathfrak{c} \in U \cap V_{A,n}$). Following a similar method as in the proof of Lemma 3.3.46, one can show that
$$\theta(\mathfrak{c}) \leq |\{\mathfrak{d} \in U \mid (\mathfrak{c},\mathfrak{d}) \in E\}|$$
(2) (case $\mathfrak{c} \in V_{A,n} \setminus U$). A similar argument as in Case (1) shows that
$$\theta(\mathfrak{c}) \geq |\{\mathfrak{d} \in U \mid (\mathfrak{c},\mathfrak{d}) \in E\}|$$
(3) (case $\mathfrak{c} \in U \cap V_{B,r,n}$). We set $(x,i) = \mathfrak{c}$. By definition of U_i, we have
$$\begin{aligned} & x < -(i-R_i)(r-1) + c_{\sigma^{-1}}(i) \implies x < -(i-R_i)(r-1) + (r-1) \\ \implies & \frac{x}{r-1} < 1 + R_i - i \implies \left\lfloor \frac{x}{r-1} \right\rfloor \leq R_i - i \implies \theta(\mathfrak{c}) \leq R_i, \end{aligned}$$
but $R_i \leq |\{\mathfrak{d} \in U \mid (\mathfrak{c},\mathfrak{d}) \in E\}|$ by definition of the digraph, hence we have
$$\theta(\mathfrak{c}) \leq |\{\mathfrak{d} \in U \mid (\mathfrak{c},\mathfrak{d}) \in E\}|.$$
(4) (case $\mathfrak{c} \in V_{B,r,n} \setminus U$). A similar argument as in case (3) shows that
$$\theta(\mathfrak{c}) \geq |\{\mathfrak{d} \in U \mid (\mathfrak{c},\mathfrak{d}) \in E\}|.$$

Consequently, $U \in IS(\mathcal{G}(r,n))$ thanks to Proposition 3.2.1, and by construction we have $\Psi(U) = \pi$. Thus, Ψ is surjective, and this conclude the proof. □

We now prove that Ψ is a morphism of posets.

Proposition 3.3.52. *Let $U, U' \in IS(\mathcal{G}(r,n))$, we have that U covers U' in $(IS(\mathcal{G}(r,n)), \subseteq)$ if and only if $\Psi(U)$ covers $\Psi(U')$ in the flag weak order.*

Proof. We set

$$\Psi(U) := ((c_i)_i, \sigma) = \pi \text{ and } \Psi(U') := ((c_i')_i, \omega) = \pi'.$$

Assume that U covers U' in $(IS(\mathcal{G}(r,n)), \subseteq)$. Since $(IS(\mathcal{G}(r,n)), \subseteq)$ is graded, there exists $(x,y) \in \lambda_{r,n} \setminus U$ such that

$$U = U' \cup \{(x,y)\}.$$

We now prove that $\Psi(U)$ covers $\Psi(U')$ using Lemma 3.3.42. There are two cases.

- (Case $(x,y) \in V_{A,n}$). We have that ω is obtained from σ by swapping positions of x and y. Moreover, by definition of Ψ we have the following two facts:
$$\sigma^{-1}(x) = \sigma^{-1}(y) + 1 \text{ and } (x,y) \notin \mathrm{Inv}(\sigma),$$
so that x and y are adjacent in σ. It remains to show that $c_{\sigma^{-1}(y)}(U) = r-1$. For the sake of clarity, let us denote by i the integer $\sigma^{-1}(y)$, and assume by contradiction that $c_i(U) < r-1$. Since we have
$$c_i(U) = L_i(U) - (r-1)R_i(U), {}_i(U) = L_i(U') \text{ and } R_i(U') = R_i(U) + 1,$$
we thus have $L_i(U') - (r-1)R_i(U') < 0$, and this contradicts Lemma 3.3.48. Therefore, we have $\pi' = \pi.a_i$, $c_{i+1} = r-1$ and $\sigma(i) < \sigma(i+1)$, so that π' covers π in $G(r,n)$ by Lemma 3.3.42.

- (Case $(x, y) \in V_{B,r,n}$). As in the previous case, let us denote by i the integer $\sigma^{-1}(y)$, and assume by contradiction that $c_i(U) = r - 1$. By definition, we have
$$c_i(U) = L_i(U) - (r-1)R_i(U) = r - 1 \text{ and } L_i(U') = L_i(U) + 1,$$
so that $c_i(U') = r$, which is absurd. Thus, we have $c_i(U) < r - 1$ and it is clear that $\pi' = \pi.b_i$. Consequently, π' covers π by Lemma 3.3.42.

We now prove the converse. If $\pi' = \pi.b_i$, then $c_i(B) = c_i(A) + 1$, so that U' is obtained from U by adding just one box in the i-th line of $V_{B,r,n}$ by Lemma 3.3.50. Thus U' covers U. If $\pi' = \pi.a_i$, then a straightforward calculation using the definition of the function Ψ shows that $U' = U \cup \{(\sigma^{-1}(i), \sigma^{-1}(i+1)\}$, so that U' covers U. This concludes the proof. $\square$

As an immediate consequence of Propositions 3.3.51 and 3.3.52, we have the following corollary, which concludes this section.

Corollary 3.3.53. *The posets $(G(r, n), \leq_f)$ and $(IS(\mathcal{G}(r, n)), \subseteq)$ are isomorphic.*

3.3.7. Down-set (resp. up-set) lattice of a finite poset. In this section, we consider $\mathcal{P} = (P, \leq)$ a finite poset. Let us denote by $G = (V, E)$ the digraph defined by
$$V := P \text{ and } E := \{(x, y) \in P^2 \mid x \neq y \text{ and } x \leq y\}.$$
It is clear that G is a simple acyclic digraph, and we denote by $\mathcal{G}(P) = (G, \theta)$ the valued digraph such that for all $z \in P$, $\theta(z) = 0$.

Proposition 3.3.54. *The set $PS(\mathcal{G}(P))$ equals the set of the linear extensions of P.*

Proof. Let us perform the peeling process on $\mathcal{G}(P)$. By definition of θ, a vertex $z \in P$ is erasable in $\mathcal{G}(P)$ if and only if we have the following property: for all $y \in P$
$$\text{if } y \leq z, \text{ then } y = z,$$
i.e. z is a minimum of $(P, \leq)$. Let us denote by $\mathcal{G}(P)'$ the valued digraph obtained after we peeled a minimum element z of P, and denote by P' the poset $(P \setminus \{z\}, \leq)$. Clearly, we have
$$\mathcal{G}(P)' = \mathcal{G}(P').$$
Therefore, applying the peeling process on $\mathcal{G}(P)$ is equivalent to performing on P the process described in the introduction of Section 3.1. It follows that each peeling sequence of $\mathcal{G}(P)$ is a linear extension of P. The converse implication can be easily proved by induction on the cardinality of P. $\square$

We have the following immediate corollary.

Corollary 3.3.55. *The poset $(IS(\mathcal{G}(P)), \subseteq)$ is isomorphic to the down-set lattice of $(P, \leq)$.*

Remark 3.3.56. Note that one can obtain the up-set lattice of $(P, \leq)$ by the same method, considering the same digraph G endowed with the valuation η defined by
$$\text{for all } z \in P, \ \eta(z) := d^+(z).$$

3.4. Generalized columns and Quasi-symmetric functions

Symmetric functions can be defined as the homogeneous formal power series $F(x_1, x_2, \ldots)$ in countable infinitely many variables being invariant under the action of the symmetric group. That is, for any monomial $X = x_{i_1} \cdots x_{i_k}$ appearing in F and for any simple transposition $s_j \in S_n$, the monomials X and $s_j.X$ have the same coefficient, where $s_j.X$ denote the monomial obtained by permuting the variables x_j and x_{j+1} in X. Quasi-symmetric functions admits a similar definition. We say that $F(x_1, x_2, \ldots)$ is a *quasi-symmetric function* if and only if for any monomial $X = x_{i_1} \cdots x_{i_k}$ appearing in F and for any simple transposition s_j such that not both x_j and x_{j+1} appear in X, then X and $s_j.X$ have the same coefficient in F. In particular, symmetric functions are quasi-symmetric functions.

A useful basis of the space of quasi-symmetric functions (of degree n) is given by the *fundamental quasi-symmetric functions*, introduced by Gessel in **[G]** (see also **[S3]** (7.81)). They are defined as follows: for any $X \subseteq [n-1]$ we set

$$G_X^n(x_1, x_2, ...) := \sum_{\substack{i_1 \le i_2 \le ... \le i_n \\ i_j < i_{j+1} \text{ if } j \in X}} x_{i_1} \cdots x_{i_n},$$

that we generally denote by G_X when there is no ambiguity.

3.4.1. Linear extensions and quasi-symmetric functions. The first occurrence of quasi-symmetric function goes back to the thesis work of Stanley, via the notion of P-partition which generalizes the concept of classical partition of an integer. A P-partition is the couple of a finite poset P (with $|P| = n$), together with a given bijection γ from P to $[n]$. For any linear extension $L = [z_1, \dots, z_n]$ of P, let $\mathrm{Des}(L, \gamma)$ be the set of all the indices $j \in [n-1]$ such that $\gamma(z_j) > \gamma(z_{j+1})$, called the *descent set* of L. Stanley associates in **[S1]** a formal power series with the P-partition (P, γ) as follows:

$$\Gamma(P, \gamma) := \sum_L G_{\mathrm{Des}(L,\gamma)}, \tag{3.11}$$

where the sum is over all linear extensions of P. Note that this is a classical reformulation of the original definition, that we give in the following proposition.

PROPOSITION 3.4.1. *A (P, γ)-partition is a function f from P to $\mathbb{N}^*$ such that there exists a linear extension $L = [z_1, \dots, z_n]$ of P which satisfies:*

(1) for all $1 \le i < j \le n$, $f(z_i) \le f(z_j)$;
(2) for all $1 \le i < j \le n$, if $\gamma(z_i) > \gamma(z_j)$, then $f(z_i) < f(z_j)$.

We have $\Gamma(P, \gamma) = \sum_f \prod_{p \in P} x_{f(p)}$, where the sum is over all (P, γ)-partitions.

3.4.2. Definition of the formal power series. Thanks to Section 3.3.7, we have that the notion of peeling sequence is a generalization of linear extension of a finite poset to a valued digraph. Thus, it is natural to look for a generalization of (3.11) to the case of valued digraphs. This is the point of this section. We begin with introducing a useful notation.

DEFINITION 3.4.2. Let $\mathcal{G} = (G, \theta)$ be a valued digraph. For all $A \in (IS(\mathcal{G}), \subseteq)$ we denote by $PS(A)$ the set defined by

$$PS_A(\mathcal{G}) := \left\{ [z_1, \dots, z_{|A|}] \;\middle|\; \begin{array}{l} A = \{z_1, \dots, z_{|A|}\} \\ \exists [x_1, x_2, \dots] \in PS(\mathcal{G}) \text{ such that } x_i = z_i \text{ for all } i \le |A| \end{array} \right\}.$$

A straightforward way to generalize the series $\Gamma(P, \gamma)$ would be to consider a bijection μ from the vertices of $\mathcal{G}$ to $\{1, \dots, |V|\}$, and directly adapt (3.11) to this new context. However, in the sequel we will need a slightly more general definition, which is inspired by the *column-strictness conditions* introduced in **[FGRS]** and **[YY]**.

DEFINITION 3.4.3. A set of *generalized columns* of $\mathcal{G}$ is a family $\mathcal{U} = (\mathcal{U}_z)_{z \in V}$ of subsets of V. Let $A \in IS(\mathcal{G})$, $\mathcal{U}$ be a set of generalized columns and f be a function from A to $\mathbb{N}$. We say that f is a *$(A, \mathcal{U})$-semi-standard function* if and only if there exists $L = [z_1, \dots, z_n] \in PS_A(\mathcal{G})$ such that:

(1) for all $1 \le i < j \le n$, we have $f(z_i) \le f(z_j)$;
(2) for all $1 \le i < j \le n$, if $z_j \in \mathcal{U}_{z_i}$, then $f(z_i) < f(z_j)$.

Such a peeling sequence is called a *f-compatible peeling sequence* of A. We denote by $\mathrm{SSF}(A, \mathcal{U})$ the set of all the $(A, \mathcal{U})$-semi-standard functions (when there is no ambiguity, we simply denote it by $\mathrm{SSF}(A)$). Finally, we define the formal power series

$$\Gamma(A, \mathcal{U}) := \sum_{f \in \mathrm{SSF}(A)} \prod_{z \in A} x_{f(z)}.$$

PROPOSITION 3.4.4. *The series $\Gamma(A,\mathcal{U})$ is a quasi-symmetric function.*

PROOF. Let $n = |A|$, $i \in \mathbb{N}^*$ and $f \in \mathrm{SSF}(A)$ such that $f^{-1}(\{i\}) \neq \emptyset$ and $f^{-1}(\{i+1\}) = \emptyset$. Let $L = [z_1, \ldots, z_n]$ be a f-compatible peeling sequence of A and denote by $\widehat{f}$ the function from A to $\mathbb{N}^*$ defined by

$$\text{for all } z \in A,\ \widehat{f}(z) = \begin{cases} i+1 \text{ if } z \in f^{-1}(\{i\}), \\ f(z) \text{ otherwise.} \end{cases}$$

We prove that L is a $\widehat{f}$-compatible sequence. Since $f \in \mathrm{SSF}(A)$, then $\widehat{f}$ is weakly increasing along $[z_1, \ldots, z_n]$. Assume by contradiction that there exists $p < q$ such that $z_q \in \mathcal{U}_{z_p}$ and $\widehat{f}(z_p) = \widehat{f}(z_q)$. Then, we have $f(z_p) = f(z_q)$ and this contradicts the fact that L is f-compatible. Therefore, L is $\widehat{f}$-compatible, hence $\widehat{f}$ is an $(A,\mathcal{U})$-semi-standard function, and this is enough to prove that $\Gamma(A,\mathcal{U})$ is quasi-symmetric. This concludes the proof. $\square$

It appears that $\Gamma(A,\mathcal{U})$ is a generalization of the function associated with a P partition, thanks to the following immediate proposition.

PROPOSITION 3.4.5. *Let $(P, \leq)$ be a finite poset, (P, γ) be a P-partition and $\mathcal{G}(P)$ be the valued digraph defined in Section 3.3.7. We set $\mathcal{U}_z(\gamma) := \{y \in P \mid \gamma(z) > \gamma(y)\}$ and $\mathcal{U}(\gamma) = (\mathcal{U}_z(\gamma))_{z \in P}$ a set of generalized columns of $\mathcal{G}(P)$. Then*

$$\Gamma(P,\mathcal{U}(\gamma)) = \Gamma(P, \gamma).$$

QUESTION 3.4.6. As suggested before, we could have defined $\Gamma(A,\mathcal{U})$ associating a descent set to each element of $PS_A(\mathcal{G})$ in the obvious way and then summing all the associated fundamental quasi-symmetric functions. However, it seems not to be any particular reason to expect these two definitions to coincide in general. It should be interesting to investigate if there exists some valued digraphs with a choice of generalized columns for which this equality occurs.

We finish this section with an obvious lemma connecting this function to the enumeration of maximal chains in $(IS(\mathcal{G}), \subseteq)$.

LEMMA 3.4.7. *Let $\mathcal{G}$ be a valued digraph, $\mathcal{U}$ be a set of generalized columns of $\mathcal{G}$ and A be an element of $IS(\mathcal{G})$. Then, the coefficient of $x_1 x_2 \cdots x_n$ in $\Gamma(A,\mathcal{U})$ is equal to the number of maximal chain from $\emptyset$ to A in $(IS(\mathcal{G}), \subseteq)$.*

PROOF. This is clear by definition of $\mathrm{SSF}(A)$. $\square$

3.4.3. Type A and Stanley's symmetric function. In this section, we consider the valued digraph $\mathcal{A} = (G, \theta)$ associated with the weak order on A_{n-1} (see Section 3.3.1). Since $\mathcal{A}$ can be seen as the Ferrers diagram of the partition λ_n, we have a natural choice for a set of generalized columns, given by the columns of $\mathcal{A}$.

DEFINITION 3.4.8. The set of generalized columns $\mathcal{U}^{col} = (\mathcal{U}_{(a,b)})_{1 \leq a < b \leq n}$ of $\mathcal{A}$ is defined by:

$$\mathcal{U}_{(a,b)} := \{(a, k) \mid a < k \leq n\}.$$

Surprisingly, the series which arise from this choice of generalized columns are the well-known Stanley symmetric functions of type A (see [**S2**]). Let us first recall the definition of Stanley symmetric functions.

DEFINITION 3.4.9. Let $\sigma \in S_n$, the *Stanley symmetric function* associated with σ is the formal power series defined by:

$$F_\sigma(x_1, x_2, \ldots) := \sum_{(i_1,\ldots,i_{\ell(\sigma)}) \in \mathrm{Red}(\sigma)} \ \sum_{\substack{r_1 \leq r_2 \leq \ldots \leq r_{\ell(\sigma)} \\ r_j < r_{j+1} \text{ if } i_j < i_{j+1}}} x_{r_1} x_{r_2} \cdots x_{r_{\ell(\sigma)}},$$

where $\mathrm{Red}(\sigma)$ denote the set of the reduced decompositions of σ.

In [**FGRS**], the authors give a characterization of such function indexed by a permutation $\sigma \in S_n$, in terms of sums over a set of tableaux called *balanced labellings* of the *Rothe diagram* of σ. We will prove that the series arising from our description are exactly the Stanley symmetric functions. we follow the same method as them. Note that another method consists in constructing an explicit bijection between these balanced labellings and the elements of $\mathrm{SSF}(\mathrm{Inv}(\sigma), \mathcal{U}^{col})$, but we do not detail this here.

THEOREM 3.4.10. *For all $\sigma \in S_n$, we have:*

$$\Gamma(\mathrm{Inv}(\sigma), \mathcal{U}^{col}) = F_\sigma.$$

Before giving the proof, we need the following technical lemma.

DEFINITION 3.4.11. Let $\sigma \in S_n$, $f \in \mathrm{SSF}(\mathrm{Inv}(\sigma), \mathcal{U}^{col})$ and $M = \max\{f(c) \mid c \in \mathrm{Inv}(\sigma)\}$. The *leading cell* of f is the unique element $(a, b) \in \mathrm{Inv}(\sigma)$ such that:

(1) $f(a, b) = M$;
(2) the integer $\sigma^{-1}(a)$ is minimal such that (1) is true.

LEMMA 3.4.12. *Let $\sigma \in S_n$, $f \in \mathrm{SSF}(\mathrm{Inv}(\sigma))$ and (a, b) be the leading cell of f. Then, there exists $\omega \in S_n$ such that* $\mathrm{Inv}(\omega) = \mathrm{Inv}(\sigma) \setminus \{(a, b)\}$.

PROOF. Let $M = f(a, b)$ and $L = [(a_1, b_1), \ldots, (a_{\ell(\sigma)}, b_{\ell(\sigma)})] \in PS_{\mathrm{Inv}(\sigma)}(\mathcal{A})$ be a f-compatible peeling sequence. Thanks to Corollary 3.3.10, there exists $\sigma_1, \sigma_2, \ldots, \sigma_{\ell(\sigma)} \in S_n$ such that:

(1) $Id \lhd_R \sigma_1 \lhd_R \ldots \lhd_R \sigma_{\ell(\sigma)} = \sigma$;
(2) $\mathrm{Inv}(\sigma_i) = \{(a_1, b_1), \ldots, (a_i, b_i)\}$.

There exists k such that $(a_k, b_k) = (a, b)$, hence a and b are adjacent in σ_{k-1} and σ_k is obtained from σ_{k-1} by swapping the positions of a and b. Let us assume that the position of a and b in σ_k is preserved in σ, that is we have

$$\sigma_k = [\sigma_k(1), \ldots, b, a, \ldots, \sigma_k(n)] \quad \text{and} \quad \sigma = [\sigma(1), \ldots, b, a, \ldots, \sigma(n)]. \tag{3.12}$$

In that case, the permutation ω obtained from σ by swapping the positions of b and a satisfy $\mathrm{Inv}(\omega) = \mathrm{Inv}(\sigma) \setminus \{(a, b)\}$, which is exactly the expected result. We still have to prove that (3.12) is true. For that purpose, we show that $a_p \neq a$ and $b_p \neq b$ for all $p > k$, implying that the positions of a an b stay the same in $\sigma_k, \sigma_{k+1}, \ldots, \sigma_{\ell(\sigma)} = \sigma$.

Assume by contradiction that there exists $p > k$ such that $a_p = a$. Then, we have $f(a, b_p) = M = f(a, b)$ because f is weakly increasing along L. But (a, b_p) and (a, b) are in the same column of λ_n, and this is absurd by definition of $\mathcal{U}^{col}$. Similarly, assume by contradiction that there exists $p > k$ such that $b_p = b$. Then, for all $q > p$ we have $(a_q, b_q) \neq (a_p, a)$ (otherwise (a_p, b_p) and (a_q, b_q) would be both in the column a_p, but it is impossible since f takes the value M on both of them), hence we have

$$\sigma = [\sigma(1), \ldots, a_p, \ldots, a, \ldots, \sigma(n)].$$

Thus, we have $\sigma^{-1}(a_p) < \sigma^{-1}(a)$, and this contradicts the minimality of $\sigma^{-1}(a)$.

Consequently, 3.12 is true, and this concludes the proof. □

We now have everything we need to prove Theorem 3.4.10.

PROOF OF THEOREM 3.4.10. We define a bijection Ψ associating a $(\mathrm{Inv}(\sigma), \mathcal{U}^{col})$-semi-standard function f with a pair of sequences, denoted by

$$\Psi(f) = (\ [i_1, \ldots, i_k]\ ,\ [r_1, \ldots, r_k]\ ), \tag{3.13}$$

such that

(1) $s_{i_1} \cdots s_{i_k}$ is a reduced decomposition of σ;
(2) $(r_j)_j$ is weakly increasing and $r_j < r_{j+1}$ whenever $i_j < i_{j+1}$;
(3) $\displaystyle\prod_{1 \leq j \leq k} x_{r_j} = \prod_{z \in \mathrm{Inv}(\sigma)} f(z)$.

Clearly, if such a bijection exists, then Theorem 3.4.10 is true. We split our proof into two step: first, we define a function satisfying all the required conditions; then, we prove that it is bijection, constructing its reverse function.

Step 1: Definition of Ψ. Let $f \in \mathrm{SSF}(\mathrm{Inv}(\sigma))$, we define by backward induction a pair of sequences $[i_1, \dots, i_{\ell(\sigma)}]$ and $[r_1 \leq \dots \leq r_{\ell(\sigma)}]$ using Lemma 3.4.12 as follows:

- let $(a, b) \in \mathrm{Inv}(\sigma)$ be the leading cell of f;
- let $\omega \in S_n$ be such that $\mathrm{Inv}(\omega) = \mathrm{Inv}(\sigma) \setminus \{(a, b)\}$ and set $i_{\ell(\sigma)}$ and $r_{\ell(\sigma)}$ the two integers such that $\sigma = \omega s_{i_{\ell(\sigma)}}$ and $r_{\ell(\sigma)} = f(a, b)$;
- repeat this procedure swapping σ with ω and f with $g := f|_{\mathrm{Inv}(\omega)}$, and so on.

Let us now check that this pair of sequences satisfies conditions (1), (2) and (3). Clearly, we have $r_1 \leq r_2 \leq \dots \leq r_{\ell(\sigma)}$, and thanks to Lemma 3.4.12 $s_{i_1} \cdots s_{i_{\ell(\sigma)}}$ is in $\mathrm{Red}(\sigma)$. We still have to prove that $r_j < r_{j+1}$ whenever $i_j < i_{j+1}$. We prove the contrapositive: let j be such that $r_j = r_{j+1}$, and denote by a and b the two integers such that

$$\begin{aligned} s_{i_1} \cdots s_{i_{j-1}} &= [\dots, a, b, \dots], \\ s_{i_1} \cdots s_{i_j} &= [\dots, b, a, \dots]. \end{aligned}$$

Our aim is now to prove that i_{j+1} is strictly smaller than i_j. Assume by contradiction that $i_j \leq i_{j+1}$, and consider the three following cases.

- If $i_j = i_{j+1}$, then $s_{i_1} \cdots s_{i_{\ell(\sigma)}}$ is not reduced, which is absurd.
- If $i_{j+1} = i_j + 1$, then there exists c such that we have

  $$\begin{aligned} s_{i_1} \cdots s_{i_j} &= [\dots, b, a, c, \dots], \\ s_{i_1} \cdots s_{i_{j+1}} &= [\dots, b, c, a, \dots]. \end{aligned}$$

  However, (a, b) and (a, c) are in the same column of λ_n, and by hypothesis we have $f(a, b) = r_j = r_{j+1} = f(a, c)$, and this contradicts the fact that $f \in \mathrm{SSF}(\mathrm{Inv}(\sigma), \mathcal{U}^{col})$.
- If $i_{j+1} > i_j + 1$, then there exists two integers c and d such that

  $$\begin{aligned} s_{i_1} \cdots s_{i_j} &= [\dots, b, a, \dots, c, d, \dots], \\ s_{i_1} \cdots s_{i_{j+1}} &= [\dots, b, a, \dots, d, c, \dots]. \end{aligned}$$

  And this contradicts the fact that (c, d) is a leading-cell (of $f|_{\mathrm{Inv}(s_{i_1} \cdots s_{ij+1})}$, see the iterative definition of the sequences above). Indeed, we have $f(a, b) = f(c, d)$ and $\sigma^{-1}(a) < \sigma^{-1}(c)$, which contradicts the minimality of $\sigma^{-1}(c)$.

In all cases, we have a contradictions. Thus, $i_{j+1} \leq i_j$, and this concludes the proof.

Step 2: construction of the reverse function. Let $s_{i_1} \cdots s_{i_{\ell(\sigma)}} \in \mathrm{Red}(\sigma)$ and $r_1 \leq \dots \leq r_{\ell(\sigma)}$ be a sequence of integers such that $r_j < r_{j+1}$ whenever $i_j < i_{j+1}$. It is easy to associate a function $f : \mathrm{Inv}(\sigma) \to \mathbb{N}^*$ to this pair of sequences: let $L = [(a_i, b_i)]_{1 \leq i \leq \ell(\sigma)} \in PS_{\mathrm{Inv}(\sigma)}(\mathcal{A})$ be the sequence such that for all j,

$$\begin{aligned} s_{i_1} \cdots s_{i_j} &= [\dots, a_j, b_j, \dots], \\ s_{i_1} \cdots s_{i_{j+1}} &= [\dots, b_j, a_j, \dots]. \end{aligned}$$

Then, we define $f : \mathrm{Inv}(\sigma) \to \mathbb{N}^*$ by $f(a_j, b_j) = r_j$ for all j. Let us first show that $f \in \mathrm{SSF}(\mathrm{Inv}(\sigma))$. In order to do so, let us consider $j < k$ such that $a_j = a_k$. We prove that $f(a_j, b_j) < f(a_k, b_k)$, implying that f is in $\mathrm{SSF}(\mathrm{Inv}(\sigma))$. Consider the sequence $i_j, i_{j+1}, \dots, i_k$, and assume by contradiction that this sequence is decreasing. Then, for all $i \leq q < k$ we have either

$$\begin{aligned} s_{i_1} \cdots s_{i_q} &= [\dots, c, a_j, \dots], \\ \text{and } s_{i_1} \cdots s_{i_{q+1}} &= [\dots, a_j, c, \dots], \end{aligned}$$

or we have

$$s_{i_1}\cdots s_{i_q} = [\dots, c, d, \dots, a_j, \dots],$$
$$\text{and } s_{i_1}\cdots s_{i_{q+1}} = [\dots, d, c, \dots, a_j, \dots].$$

In other words, we obtain $s_{i_1}\cdots s_{i_{q+1}}$ from $s_{i_1}\cdots s_{i_q}$ either by swapping positions of a_j with an integer just on its left, or by swapping positions of two integers being on the left of a_j. Therefore, we $a_k \neq a_j$, which is absurd. Thus, there exists $j \leq q < k$ such that $i_q \leq i_{q+1}$, but $s_{i_1}\cdots s_{i_{\ell(\sigma)}}$ is reduced, so that $i_q < i_{q+1}$. Consequently, we have $r_q < r_{q+1}$ by definition so $f(a_j, b_j) < f(a_k, b_k)$.

In order to complete the proof, we just have to show that this function is the inverse of Ψ, and this can be easily done recursively: assume by contradiction that $(a_{\ell(\sigma)}, b_{\ell(\sigma)})$ is not the leading cell of f. Then, there exists $k < \ell(\sigma)$ such that $f(a_k, b_k) = f(a_{\ell(\sigma)}, b_{\ell(\sigma)})$ and $\sigma^{-1}(a_k) < \sigma^{-1}(a_{\ell(\sigma)})$. Therefore, we have

$$\sigma = [\dots, a_k, \dots, b_{\ell(\sigma)}, a_{\ell(\sigma)}, \dots],$$

so that there is an integer q such that $k \leq q < \ell(\sigma)$ and $i_q < i_{\ell(\sigma)}$, so that we have

$$f(a_k, b_k) = r_k \leq r_q < r_{\ell(\sigma)} = f(a_{\ell(\sigma)}, b_{\ell(\sigma)}),$$

and this is absurd. Repeating this argument, we have the expected property by induction, and this ends the proof. $\square$

This construction leads to a combinatorial interpretation of F_σ as a sum over a set of tableaux (which are depicted on Figure 3.8): let us consider a permutation $\sigma \in S_n$ and denote by A its inversion set $\mathrm{Inv}(\sigma)$ seen as as subset of boxes of λ_n. Clearly, A inherits the digraph structure and the valuation of $\mathcal{A}$. Moreover, A defines a valued digraph because $A \in IS(\mathcal{A})$ (in a sense, A define a "sub-valued digraph of $\mathcal{A}$"), so that we can perform the peeling process on it. Obviously, the arising sequences are precisely the elements of $PS_A(\mathcal{A})$, and we can represent each element of $PS_A(\mathcal{A})$ as a tableau of shape A. That is, let $L = [z_1, \dots, z_k] \in PS_A(\mathcal{A})$, then L can be represented as a tableau of shape A where the box z_i is filled by the integer i. These tableaux can be seen as the equivalent counterpart of standard tableaux within our theory.

Similarly, we can define a family of "semi-standard" tableaux by the following way: let $L = [z_1, \dots, z_k] \in PS_A(\mathcal{A})$, we construct a tableau of shape A by putting an integer t_i in the box z_i, satisfying the following two conditions:

(1) the sequence (t_i) is weakly increasing along L;
(2) a given integer cannot appear twice in the same column (equivalently, if $i < j$ and z_i and z_j are in the same column, then $t_i < t_j$).

Clearly, these tableaux are in bijection with the elements of $\mathrm{SSF}(A)$. Therefore, if we denote by x^T the monomial $x_1^{T(1)} x_2^{T(2)} \cdots$ where $T(i)$ is the number of occurrences of i in a tableau T obtained by the previous method, then the Stanley symmetric function F_σ is the sum over all the tableaux of these monomials x^T.

3.4.4. Type $\widetilde{A}$ and affine Stanley series. In this section we apply the same method used in the previous section to the valued digraph $\widetilde{\mathcal{A}} = (G, \theta)$, associated with the weak order on $\widetilde{A}_n$ (see Section 3.3.5). Once again, the graphical representation of G as a diagram leads to a natural choice for a set of generalized columns, given by the usual columns of G.

DEFINITION 3.4.13. We denote by $\mathcal{U}^{col} = (\mathcal{U}_{(a,b)})_{1\leq a\leq n,\ a<b}$ the set of generalized columns of $\widetilde{A}$ defined by:

$$\mathcal{U}_{(a,b)} = \{(a,k) \mid a < k \text{ and } k \not\equiv a \pmod{n}\}.$$

As in the A_{n-1} case, the series arising from this choice of generalized columns is known, namely the affine Stanley symmetric function, introduced by Lam in [**L2**]. Note that a combinatorial interpretation in terms of tableaux of this series has already been provided by Yun

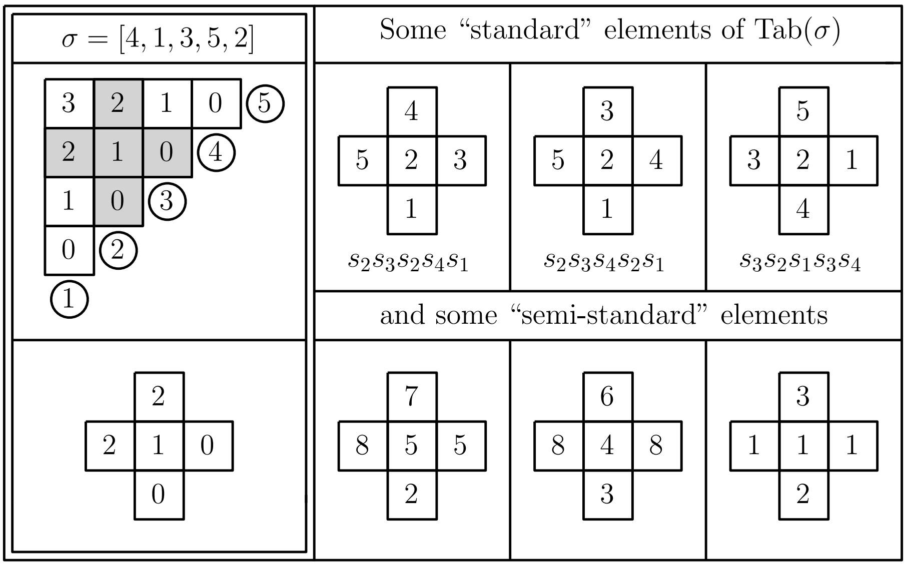


FIGURE 3.8.

and Yoo in **[YY]**, and the one arising from our model is very similar. Therefore, the proofs are similar.

Let us begin with the definition of the affine Stanley symmetric function.

DEFINITION 3.4.14. A sequence $(i_1,\ldots,i_k)$ of elements of $\mathbb{Z}/n\mathbb{Z}$ is called *cyclically decreasing* if and only if:

- each element of $\mathbb{Z}/n\mathbb{Z}$ appears at most once in the sequence;
- if there exists p and q such that $i_p = j$ and $i_q = j+1$, then $q < p$.

An affine permutation $\sigma \in \widetilde{A_n}$ is called cyclically decreasing if there exists a cyclically decreasing sequence $(i_1,\ldots,i_k)$ such that $s_{i_1}\cdots s_{i_k}$ is a reduced decomposition of σ (note that Id is cyclically decreasing by convention). For any $\omega \in \widetilde{A_n}$, a *cyclically decreasing factorization* of ω is an expression of ω as a product $\omega = v_1v_2\cdots v_r$ such that:

- each $v_i \in \widetilde{A_n}$ is cyclically decreasing;
- $\ell(\omega) = \ell(v_1) + \cdots + \ell(v_r)$.

Finally, the *affine Stanley symmetric function* F_ω associated with ω is defined as

$$\widetilde{F}_\omega(x_1,x_2,\ldots) := \sum_{\omega=v_1\cdots v_r} x_1^{\ell(v_1)}\cdots x_r^{\ell(v_r)},$$

where the sum is over all cyclically decreasing factorisations of ω (see **[L2]**).

DEFINITION 3.4.15. Let $\omega \in \widetilde{A_n}$, $f \in \mathrm{SSF}(\mathrm{Inv}(\omega))$, $L = [(a_i,b_i)] \in PS_{\mathrm{Inv}(\omega)}(\widetilde{A})$ be a f-compatible peeling sequence and $s_{i_1}\ldots s_{i_{\ell(\omega)}}$ be the reduced decomposition of ω associated with L. We define the following factorization of ω:

$$\Psi(f,L) := v_1v_2\cdots v_r,$$

where v_k is defined as follows: for all j,

- if $f(a_j,b_j) \neq k$, then $v_k = Id$;
- if there exists $p \leq q$ such that $f(a_j,b_j) = k$ if and only if $p \leq j \leq q$, then $v_k = s_{i_p}s_{i_{p+1}}\cdots s_{i_q}$.

PROPOSITION 3.4.16. *The function Ψ does not depend on the choice of the f-compatible peeling sequence L, and $\Psi(f,L)$ is a cyclically decreasing factorization of ω.*

PROOF. Since v_k is uniquely determined by the inversions (a_j,b_j) for which $f(a_j,b_j) = k$, the first statement of the proposition is clear.

It remains to show that v_k is cyclically decreasing for all k. If $v_k = Id$, then it is clear. Let k be such that $v_k \neq Id$, hence there exists $p \leq q$ such that $v_k = s_{i_p} \cdots s_{i_q}$. If $p = q$, v_k is obviously cyclically decreasing.

We now focus on the case $p < q$. Assume by contradiction that there exists $p \leq u < v \leq q$ such that $i_u = i_v + 1$. Without loss of generality, we can suppose that v is minimal with this property. As usual, we denote by ω_m the affine permutation $s_{i_1} \cdots s_{i_m}$. We have that ω_{m+1} is obtained from ω_m by swapping the positions of all the pair of integers $a_m + rn$ and $b_m + rn$, $r \in \mathbb{Z}$. For the sake of clarity, we just say that the positions of a_m and b_m are swapped. Let us now split the study into two cases.

- If for all $u \leq j < v$ we have $b_j \neq a_u$, then the position of a_u remains unchanged in each ω_j. Thus, by minimality of v, $(a_v, b_v) = (a_u, b_v)$, which is absurd since (a_u, b_u) and (a_u, b_v) are in the same column and $f \in \mathrm{SSF}(\mathrm{Inv}(\omega), \mathcal{U}^{col})$.
- If there exists $u \leq j < v$ such that $b_j = a_u$, then there exists an integer m such that we have
$$\omega_j = [\ldots, m, a_u, \ldots] \ \text{ and } \ \omega_{j+1} = [\ldots, a_u, m, \ldots].$$
Since we have $\omega_u = [\ldots, b_u, a_u, \ldots]$, there exists $u \leq j' < j$ such that
$$\omega_{j'} = [\ldots, m, b_u, \ldots] \ \text{ and } \ \omega_{j'+1} = [\ldots, b_u, m, \ldots].$$
Therefore, we have $f(m, b_u) = f(m, a_u)$ which is absurd since (m, b_u) and (m, a_u) are in the same column.

In all cases we have a contradiction, so that v_k is cyclically decreasing, and this concludes the proof. □

We are now able to prove the main theorem of this section, using this function Ψ (the proof is similar to that of the A_{n-1} case).

THEOREM 3.4.17. *For all $\omega \in \widetilde{A_n}$, we have*
$$\widetilde{F}_\omega = \Gamma(\mathrm{Inv}_{\widetilde{A}}(\omega), \mathcal{U}^{col}).$$

PROOF. Thanks to Proposition 3.4.16, we have a map which associate to each $f \in \mathrm{SSF}(\mathrm{Inv}(\omega))$ a cyclically decreasing factorization $\omega = v_1 \cdots v_r$. Moreover, we clearly have that
$$x_1^{\ell(v_1)} \cdots x_r^{\ell(v_r)} = \prod_{c \in \mathrm{Inv}_{\widetilde{A}}(\omega)} x_{f(c)}.$$
Consequently, we just have to show that this function is a bijection to prove the theorem.

Let us construct the reverse function: let $\omega = v_1 \cdots v_r$ be a cyclically decreasing factorization of ω. Consider any cyclically decreasing reduced decomposition of v_i and concatenate them to get a reduced decomposition of ω, *i.e.* $\omega = (s_{i_1} \cdots s_{i_{\ell(v_1)}})(s_{i_{\ell(v_1)+1}} \cdots s_{i_{\ell(v_1)+\ell(v_2)}}) \cdots$. Let $L = [(a_i, b_i)]_i \in PS_{\mathrm{Inv}(\omega)}(\widetilde{A})$ be the peeling sequences canonically associated with this reduced decomposition, and we define a function f from $\mathrm{Inv}_{\widetilde{A}}(\omega)$ to $\mathbb{N}^*$ by the following way:
$$\text{if } \ell(v_1) + \cdots + \ell(v_k) + 1 \leq j \leq \ell(v_1) + \cdots + \ell(v_k) + \ell(v_{k+1}), \text{ then } f(a_j, b_j) = k + 1.$$
Clearly this function f does not depends on the cyclically decreasing reduced decomposition chosen for each v_i, but depends only on the cyclically decreasing factorization of ω.

Let us prove that f is in $\mathrm{SSF}(\mathrm{Inv}_{\widetilde{A}}(\omega))$ As usual, we denote $\omega_j = s_{i_1} \cdots s_{i_j}$. Set k such that $\ell(v_k) \geq 2$, $p = \ell(v_1) + \cdots + \ell(v_{k-1}) + 1$, and $q = \ell(v_1) + \cdots + \ell(v_k)$. Now assume that there exists $p \leq u < v \leq q$ such that $f(a_u, b_u) = f(a_v, b_v) = k$ with $a_u = a_v$. Without loss of generality, we can suppose that u is maximal with this property, and that v is minimal with this property. Once again, there are two cases.

- If for all $u < j < v$, $b_j \neq a_u$, then the position of a_u remains unchanged in $\omega_{u+1}, \ldots, \omega_{v-1}$. Hence by minimality of v we have $i_v = i_u + 1$, which is absurd since v_k is cyclically decreasing.

- If there exists $u < j < v$ such that $b_j = a_u$, then there exists $u < j' < j$ such that $(a_{j'}, b_{j'}) = (a_j, b_u)$ since b_u is just on the left of a_u in ω_u. Hence we found $j' > u$ such that there exists $j > j'$ with $a_{j'} = a_j$, which is absurd by maximality of u.

Hence $f \in \mathrm{SSF}(\mathrm{Inv}(\omega))$, and this function clearly invert the one defined earlier, and this achieves the proof of the theorem. $\square$

CHAPTER 4

Extending the weak order and Cambrian semi-lattices

Introduction

When we deal with the *weak order* on a Coxeter group W of finite rank, we generally have two opposite configurations depending on whether W is finite or not. On the one hand, when W is finite the weak order defines a graded *complete ortho-lattice* on W, with the identity as bottom element, and with ortho-complement given by the map $\omega \mapsto \omega_0\omega$, where ω_0 is the maximal element of W. Moreover, the maximal chains of this poset are totally encoded by a well studied combinatorial object: the *reduced decompositions* of ω_0, that are, the expressions of ω_0 as a product of a minimal number of generators. On the other hand, when W is infinite the situation is less enjoyable: the weak order only defines a *complete meet semi-lattice* on W, and there are no clear equivalents of ω_0 and its reduced decompositions.

This difference of behaviour is particularly evident in the context of *Cambrian lattices*, which are well studied generalisations of the *Tamari lattices* to any finite Coxeter groups. They were introduced by Nathan Reading in [**R1**], who subsequently studied several of their aspects in [**R2**], [**R3**]. In particular, he showed that they are complete lattices. They are, for instance, connected to Coxeter-Catalan combinatorics and to *cluster algebras* (see, for instance, [**RS2**]), for example the *Hasse diagram* of a Cambrian lattice is isomorphic to the *exchange graph* of the corresponding Cluster algebra.

As shown by Reading and Speyer in [**RS3**], this construction can be extended to infinite Coxeter groups. Unsurprisingly, in this much more general case, the resulting poset inherits the "insufficiencies" of the weak order in the infinite case: we do not obtain a complete lattice, but a semi-lattice called the *Cambrian semi-lattice*. Furthermore, Reading and Speyer showed in [**RS2**] that this Cambrian semi-lattice does not give all the expected information about the associated Cluster algebra (for example, the Hasse diagram of a Cambrian semi-lattice corresponds to a strict sub-graph of the exchange graph). Nevertheless, they showed in [**RS1**] how this construction can be "completed" in the case of affine Coxeter groups, but in general this problem remains open.

This problem can also be viewed from another angle: instead of defining a Cambrian semi-lattice and then trying to "complete" it, one can try to "complete" the weak order first, and then to construct directly a Cambrian lattice from this completion. A little bit more formally, this method consists in two distinct steps. The first one consists in finding for each infinite Coxeter group W a poset $\mathcal{P}_W$ such that the weak order on W is isomorphic to a sub-poset of $\mathcal{P}_W$ (in the ideal situation, the weak order would be a *lower set* of $\mathcal{P}_W$), and such that $\mathcal{P}_W$ has enough "nice" properties generalizing those of the finite case. The second step consists in a generalisation of Cambrian lattices using $\mathcal{P}_W$ instead of the weak order.

In this chapter, we will first deeply study the first step, providing a theory in which many extensions of the weak order can be constructed, and we will finish with an approach of the second step within our theory, providing also an extension of Cambrian semi-lattices. More precisely, we introduce here a new family of complete lattices, which deeply generalises the construction of Chapter 3. The fundamental objects of this chapter are, once again, what we call *valued digraphs*. Namely, couples of the form $\mathcal{G} = (G, \theta)$, where $\mathcal{G}$ is a simple digraph (note that the difference with the previous chapter lies in the fact that we allow valued digraphs to contain cycles) and θ a function from $\mathcal{V}(G)$ the vertices of G to the nonnegative integers such that for all vertices z of G, $\theta(z)$ is weakly smaller than the *out-degree* of z (Definition 4.1.1). To each valued digraph $\mathcal{G} = (G, \theta)$, we associate a subset of the power set of $\mathcal{V}(G)$, denoted by $IS(\mathcal{G})$ (Definition 4.1.5), and defined by simple inequalities. We prove that in general $(IS(\mathcal{G}), \subseteq)$ is a complete lattice (Theorem 4.1.6), and we exhibit a family of valued digraphs, called *balanced valued digraphs* (Definition 4.1.9), such that $(IS(\mathcal{G}), \subseteq)$ is a complete ortho-lattice (Proposition 4.1.10), with ortho-complement given by the usual set complement of $\mathcal{V}(G)$.

We continue our study defining the notion of *projective* valued digraphs (Definition 4.2.10). That are, valued digraphs $\mathcal{G} = (G, \theta)$ that can be seen as the limit (in a certain sense) of a sequence $(\mathcal{G}_i)_{i\geq 1}$ of finite acyclic valued digraphs. This gives rise to a projective system, where the structure of $(IS(\mathcal{G}), \subseteq)$ can be studied completely using the sequence of finite complete lattices $(IS(\mathcal{G}_i), \subseteq)_{i\geq 1}$ (Theorem 4.2.17). Still in the case where $\mathcal{G}$ is projective, we define a set $PS(\mathcal{G})$ (Definition 4.2.4) of total orderings of $\mathcal{V}(G)$, which totally encodes the maximal chains of $(IS(\mathcal{G}), \subseteq)$. That is, for each chain $\mathcal{C}$ of $(IS(\mathcal{G}), \subseteq)$, $\mathcal{C}$ is maximal if and only if there exists $I \in PS(\mathcal{G})$ such that the elements of $\mathcal{C}$ are exactly the initial sections of I (Theorem 4.2.24). We finish this theoretic construction with showing that for all projective valued digraph $\mathcal{G}$, $(IS(\mathcal{G}), \subseteq)$ is an *algebraic* complete lattice (Theorem 4.2.30).

We then apply, in Section 4.3, these theoretic construction to the study of two conjectures of Dyer about an extension of the weak order on a Coxeter group W using the so-called bi-closed sets and reflection orderings of a root system of W (see **[D4]** and **[D1]**). More precisely, we introduce the notion of well-assembled on Φ^+ valued digraph, and we prove that each such valued digraph provides an extension of the weak order having most of the properties that are conjecturally attached to Dyer's extensions.

In Section 4.4, we provide a quick study of some connections there exist between our construction and convex geometry, associating to each acyclic valued digraph $\mathcal{G}$ a non-trivial *abstract convex geometry* and its associated *closure operator* extending the lattice introduced in this paper.

We finish, in Section 4.5, with giving an extension of Cambrian semi-lattices into complete lattices. More precisely, we explain how one can associate to each well-assembled on Φ^+ valued digraph $\mathcal{G}$ and each Cambrian semi-lattice of W, a sub-poset of $(IS(\mathcal{G}), \subseteq)$ extending this Cambrian semi-lattice, and we prove that this sub-poset is a complete lattice.

4.1. General construction

Let us first recall the following notation, introduced in Chapter 2.

DEFINITION 4.1.1. Let G be a digraph, $A \subseteq \mathcal{V}(G)$ and $z \in \mathcal{V}(G)$, we set

$$d_A^+(G, z) := |\{y \in A \mid (z, y) \in E(G)\}|,$$

which we simply denote by $d_A^+(z)$ when there is no ambiguity.

REMARK 4.1.2. Note that if $A \subseteq B \subseteq \mathcal{V}(G)$, then $d_A^+(z) \leq d_B^+(z)$ for all $z \in \mathcal{V}(G)$.

4.1.1. Definition of a family of posets.

We begin with the definition of the fundamental object of this paper, which is the straightforward generalization of Definition 3.1.1 to the case of any digraph (without the acyclicity condition).

DEFINITION 4.1.3. A *valued digraph* $\mathcal{G}$ is a couple (G, θ) of a simple digraph G, together with a valuation $\theta : \mathcal{V}(G) \longrightarrow \mathbb{N}$, such that for all $z \in \mathcal{V}(G)$, we have

$$0 \leq \theta(z) \leq d^+(z).$$

Throughout this section, $\mathcal{G}$ will denote a valued digraph with underlying digraph G and valuation θ.

REMARK 4.1.4. In this paper we essentially work with valued digraphs, hence we use the notations $\mathcal{V}(\mathcal{G}) := \mathcal{V}(G)$, $E(\mathcal{G}) := E(G)$, and $d_A^+(\mathcal{G}, z) := d_A^+(G, z)$ for the sake of clarity. Similarly, when we say that a valued digraph has a property commonly associated with ordinary digraphs (such as "$\mathcal{G}$ is acyclic"), we actually mean that the underlying digraph has this property.

We now define a family of subsets of $\mathcal{V}(\mathcal{G})$, which has a rich poset structure once ordered by inclusion.

DEFINITION 4.1.5. We denote by $IS(\mathcal{G})$ the family of subsets of $\mathcal{V}(\mathcal{G})$ defined as follows: for all $A \subseteq \mathcal{V}(\mathcal{G})$, we have $A \in IS(\mathcal{G})$ if and only if the following two conditions are true:
(1) $\forall z \in A$, $\theta(z) \leq d^+_A(z)$;
(2) $\forall z \in \mathcal{V}(\mathcal{G}) \setminus A$, $\theta(z) \geq d^+_A(z)$.

Notice that this definition generalize that of Chapter 3 thanks to Proposition 3.2.1. However, note that in the case of and infinite simple acyclic valued digraph, the set $IS(\mathcal{G})$ coming from Definition 4.1.5 is bigger than the one obtained using the peeling process. Indeed, Definition 4.1.5 allows us to consider infinite subsets of $\mathcal{V}(\mathcal{G})$. The following theorem gives the general structure of the poset $(IS(\mathcal{G}), \subseteq)$. However, notice that it does note replace the proof of Theorem 3.1.5 since it does not give us information about the graded structure of $(IS(\mathcal{G}), \subseteq)$ when $\mathcal{G}$ is finite and acyclic.

THEOREM 4.1.6. *For all valued digraph $\mathcal{G}$, the poset $(IS(\mathcal{G}), \subseteq)$ is a complete lattice.*

PROOF. First, note that both $\mathcal{V}(\mathcal{G})$ and $\emptyset$ are in $IS(\mathcal{G})$. Thus, $(IS(\mathcal{G}), \subseteq)$ is bounded, and it is enough to prove that it is a complete join semi-lattice.

We now construct explicitly the join of any subset of $IS(\mathcal{G})$. For that purpose, fix $X \subseteq IS(\mathcal{G})$ and denote by J_0 the set $\bigcup_{B \in X} B$, we then define recursively a sequence $(J_i)_{i \geq 0}$ as follows:

$$\forall i \geq 0,\ J_{i+1} = J_i \cup \{z \in \mathcal{V}(\mathcal{G}) \setminus J_i \mid \theta(z) < d^+_{J_i}(z)\}.$$

Obviously, we have $J_0 \subseteq J_1 \subseteq J_2 \subseteq \ldots$, and we denote by $J_\infty = \cup_{i \geq 0} J_i$ the limit of these sets. We now show that J_∞ is in $IS(\mathcal{G})$. Let $z \in \mathcal{V}(\mathcal{G})$, we have three cases.

- If $z \in J_0$, then there exists $B \in X$ such that $z \in B$, so that $\theta(z) \leq d^+_B(z)$, but $B \subseteq J_0 \subseteq J_\infty$, hence $\theta(z) \leq d^+_{J_\infty}(z)$ thanks to Remark 4.1.2.
- If $z \in J_\infty \setminus J_0$, then there exists $i \geq 0$ such that $z \in J_{i+1} \setminus J_i$, hence $\theta(z) < d^+_{J_i}(z) \leq d^+_{J_\infty}(z)$.
- If $z \in \mathcal{V}(\mathcal{G}) \setminus J_\infty$, then for all $i \geq 0$ we have $\theta(z) \geq d^+_{J_i}(z)$. Thus, if we denote by C_i the set $\{y \in J_i \mid (z, y) \in E(\mathcal{G})\}$, then we have $C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots$ and $|C_i| = d^+_{J_i}(z) \leq \theta(z)$ for all $i \geq 0$. Therefore, the sequence $(C_i)_{i \geq 0}$ is stationary at a given rank, *i.e.* there exists $N \geq 0$ such that for all $m \geq N$, $C_m = C_N$. Thus, we have $d^+_{J_\infty}(z) = d^+_{J_N}(z) \leq \theta(z)$.

It follows that J_∞ is in $IS(\mathcal{G})$.

Finally, we show that J_∞ is the join of X. Let A be an upper bound of X, we show by induction on i that $J_i \subseteq A$ for all $i \geq 0$. Since A is an upper bound of X, we have $J_0 \subseteq A$. Let $i \geq 0$ be such that the property is true and consider $z \in J_{i+1} \setminus J_i$, we have $\theta(z) < d^+_{J_i}(z) \leq d^+_A(z)$ by induction hypothesis. Therefore, $z \in A$, so that $J_{i+1} \subseteq A$. Thus, we have $J_\infty \subseteq A$, which implies that J_∞ is the join of X in $(IS(\mathcal{G}), \subseteq)$, and this ends the proof. □

We conclude this section with the study of a special class of valued digraphs, which is fundamental for our study of Dyer's conjectures, namely *balanced valued digraphs* (Definition 4.1.9). Before giving their definition, we briefly study a natural case of valued digraph such that the dual (as a poset) of their associated lattice can also be described using a valued digraph (Proposition 4.1.8).

DEFINITION 4.1.7. We say that a valued digraph $\mathcal{G}$ is of *finite out-degree* if and only if for all $z \in \mathcal{V}(\mathcal{G})$, $d^+_{\mathcal{V}(\mathcal{G})}(z) < +\infty$. For all valued digraph $\mathcal{G} = (G, \theta)$ of finite out-degree, we denote by $\widetilde{\mathcal{G}} = (G, \widetilde{\theta})$ the valued digraph such that for all $z \in \mathcal{V}(G)$, $\widetilde{\theta}(z) = d^+_{\mathcal{V}(\mathcal{G})}(z) - \theta(z)$. We call $\widetilde{\mathcal{G}}$ the *dual* valued digraph of $\mathcal{G}$.

PROPOSITION 4.1.8. *Let $\mathcal{G}$ be a valued digraph of finite out-degree, we have*

$$IS(\widetilde{\mathcal{G}}) = \{\mathcal{V}(\mathcal{G}) \setminus A \mid A \in IS(\mathcal{G})\}.$$

In particular, this implies that $(IS(\widetilde{\mathcal{G}}), \subseteq)$ *is isomorphic to the dual of* $(IS(\mathcal{G}), \subseteq)$.

PROOF. Let $A \in IS(\mathcal{G})$ and $z \in \mathcal{V}(\mathcal{G})$, we have two cases.

- If $z \in A$, then $\theta(z) \leq d^+_A(z)$, so that $-\theta(z) \geq -d^+_A(z)$ and finally we have

$$\widetilde{\theta}(z) = d^+_{\mathcal{V}(\mathcal{G})} - \theta(z) \geq d^+_{\mathcal{V}(\mathcal{G})} - d^+_A(z) = d^+_{\mathcal{V}(\mathcal{G})\setminus A}(z).$$

- If $z \in \mathcal{V}(\mathcal{G}) \setminus A$, then we prove with a similar argument that $\widetilde{\theta}(z) \leq d^+_{\mathcal{V}(\mathcal{G})\setminus A}(z)$.

This proves that $\mathcal{V}(\mathcal{G}) \setminus A \in IS(\widetilde{\mathcal{G}})$. With the same method, one can show that if B is in $IS(\widetilde{\mathcal{G}})$, then $\mathcal{V}(\mathcal{G}) \setminus B$ is in $IS(\mathcal{G})$, and this ends the proof. □

Among all valued digraphs of finite out-degree, some naturally lead to *complete ortho-lattices*, as shown in Proposition 4.1.10.

DEFINITION 4.1.9. A valued digraph $\mathcal{G}$ of finite out-degree is called *balanced* if and only if for all $z \in \mathcal{V}(\mathcal{G})$, $d^+(z)$ is an even number and $\theta(z) = \frac{d^+(z)}{2}$.

PROPOSITION 4.1.10. *Let* $\mathcal{G}$ *be a valued digraph of finite out-degree, if* $\mathcal{G}$ *is balanced, then* $(IS(\mathcal{G}), \subseteq)$ *is a complete ortho-lattice whose ortho-complement is given by the usual set-complement of* $\mathcal{V}(\mathcal{G})$.

PROOF. This is immediate thanks to Proposition 4.1.8 together with the fact that $\mathcal{G} = \widetilde{\mathcal{G}}$. □

REMARK 4.1.11. Notice that all these previous results do not depend on the fact that $\mathcal{G}$ is simple. Consequently, they naturally generalize to the case where $\mathcal{G}$ is not simple.

4.1.2. The example of *scaffoldings*. In this section, we introduce a family of digraphs called *scaffoldings* (Definition 4.1.13), which will be of central importance in Section 4.3. In the following, an *injective sequence* will be a sequence which is injective once seen as a function from $\mathbb{N}$ to a given set. For any sequence $(a_i)_{i\geq 1}$, we denote by $\mathrm{Im}(a)$ the set $\{a_1, a_2, \ldots\}$.

The definition of scaffoldings requires the notion of *shuffle* of a pair of sequences, which we now define.

DEFINITION 4.1.12. Let $a = (a_i)_{i\geq 1}$ and $b = (b_i)_{i\geq 1}$ be two injective sequences (which can be finite or not) of elements of some disjoint sets X_a and X_b, respectively. A sequence $c = (c_i)_{i\geq 1}$ is called a *shuffle* of a and b if and only if:

(1) c is an injective sequence taking all the values inside $\mathrm{Im}(a) \sqcup \mathrm{Im}(b)$;
(2) both a and b appear in the same relative order in c.

For example, if we set $a = [a_1, a_2, a_3]$ and $b = [b_1, b_2]$, then $[a_1, b_1, a_2, a_3, b_2]$ is a shuffle of a and b, but $[a_1, b_1, a_3, a_2, b_2]$ is not.

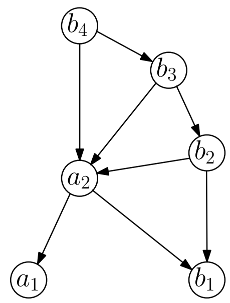
DEFINITION 4.1.13. Let a and b be two injective sequences on two disjoint sets. A digraph G is called a *scaffolding* made of a and b if and only if there exists a sequence c such that:

- c is a shuffle of a and b;
- $\mathcal{V}(\mathcal{G}) = \mathrm{Im}(a) \sqcup \mathrm{Im}(b) = \mathrm{Im}(c)$;
- $\{c_1, c_2\} = \{a_1, b_1\}$;
- the out-degree of c_1 and c_2 is 0;
- for all $j \geq 3$, if we denote by j_a (resp. j_b) the maximal integer such that $j_a < j$ (resp. $j_b < j$) and $c_{j_a} \in \mathrm{Im}(A)$ (resp. $c_{j_b} \in \mathrm{Im}(b)$), then (c_j, c_{j_a}) and (c_j, c_{j_b}) are the only arcs having c_j as starting point in G.

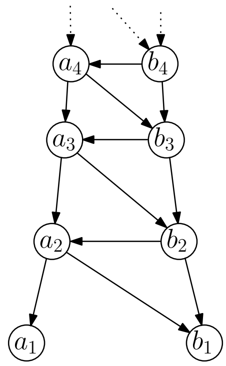
This definition has two immediate consequences, which are given in the next lemma.

LEMMA 4.1.14. *If* G *is a scaffolding made of* a *and* b*, then:*

(1) the out-degree of any vertex z *of* G *is 0 if* $z \in \{a_1, b_1\}$*, and 2 otherwise;*
(2) for all $j \geq 1$*,* (a_{j+1}, a_j) *and* (b_{j+1}, b_j) *are both in* $E(G)$.



$c = [a_1, b_1, a_2, b_2, b_3, b_4]$



$c = [a_1, b_1, a_2, b_2, a_3, b_3, \ldots]$

FIGURE 4.1. Two examples of scaffoldings.

PROOF. Point (1) is clear by definition. Point (2) is an immediate consequence of the fact that G is constructed with a shuffle c of a and b such that $\{c_1, c_2\} = \{a_1, b_1\}$. □

By Point (1) of Lemma 4.1.14, each scaffolding G defines a (unique) balanced valued digraph $\mathcal{G}$. The relatively simple digraph structure of G allows us to determine completely the elements of $IS(\mathcal{G})$, and this is the point of the next proposition. We represent on Figure 4.2 the poset $(IS(\mathcal{G}), \subseteq)$ associated with the scaffolding on the right of Figure 4.1.

PROPOSITION 4.1.15. *Let G be a scaffolding made of a and b, two injective sequences on two disjoint sets, $\mathcal{G}$ be the balanced valued digraph having G as underlying digraph, $\mathcal{A}$ be the set of the initial sections of a (seen as a total order on* $\mathrm{Im}(a)$*) and $\mathcal{B}$ be the set of the initial sections of b. Then, we have*

$$IS(\mathcal{G}) = \{C \subseteq \mathcal{V}(\mathcal{G}) \mid C \text{ or } \mathcal{V}(\mathcal{G}) \setminus C \text{ is in } \mathcal{A} \cup \mathcal{B}\}.$$

PROOF. In this proof, we denote by c the shuffle of a and b associated with G.

We first show that $\mathcal{A} \cup \mathcal{B} \subseteq IS(\mathcal{G})$. Let $A \in \mathcal{A}$ and $z \in \mathcal{V}(\mathcal{G})$, we divide our study into two cases.

- If $z \in A$, then we have two possibilities.
  - If $z = a_1$, then $d^+_A(z) = 0 = \theta(z)$, so that $d^+_A(z) \geq \theta(z)$.
  - If $z = a_k$ with $k > 1$, then a_{k-1} is in A because $A \in \mathcal{A}$. Moreover, we have $(a_k, a_{k-1}) \in E(\mathcal{G})$. Thus, we have $d^+_A(z) \geq 1 = \theta(z)$.
- If $z \in \mathcal{V}(\mathcal{G}) \setminus A$, then we have either $z = a_1$, or $z = b_1$, or there exists an integer j such that $(z, b_j) \in E(\mathcal{G})$. In the first case we have $A = \emptyset$, in the second case we have $d^+_A(z) = 0 = \theta(z)$, and in the third case we have $d^+_A(z) \leq 1 = \theta(z)$ thanks to the fact that $A \cap \mathrm{Im}(b) = \emptyset$ together with the fact that the out-degree of z is smaller than 2 (see Lemma 4.1.14). Thus, we have $d^+_A(z) \leq \theta(z)$.

Consequently, $A \in IS(\mathcal{G})$, hence $\mathcal{A} \subseteq IS(\mathcal{G})$. With similar arguments, one can show that $\mathcal{B} \subseteq IS(\mathcal{G})$. Thus, $\mathcal{A} \cup \mathcal{B} \subseteq IS(\mathcal{G})$. Finally, since $\mathcal{G}$ is balanced and thanks to Proposition 4.1.10, we have

$$\{C \subseteq \mathcal{V}(\mathcal{G}) \mid C \text{ or } \mathcal{V}(\mathcal{G}) \setminus C \text{ is in } \mathcal{A} \cup \mathcal{B}\} \subseteq IS(\mathcal{G}).$$

It remains to show that the converse inclusion holds. In order to do that, let $C \in IS(\mathcal{G})$ be such that $C \neq \emptyset$, we have three cases to consider.

- If $C \subseteq \mathrm{Im}(a)$, then consider $k > 1$ (if such a k exists) such that $a_k \in C$. We have $\theta(a_k) = 1$, and the only arc of $\mathcal{G}$ having a_k as starting point and not having an element of $\mathrm{Im}(b)$ as ending point is (a_k, a_{k-1}). Therefore, $a_{k-1} \in C$, and by induction we can prove that for all $j \leq k$, $a_j \in C$. Thus, $C \in \mathcal{A}$.
- If $C \subseteq \mathrm{Im}(b)$, a similar argument shows that $C \in \mathcal{B}$.
- If C contains some elements from both $\mathrm{Im}(a)$ and $\mathrm{Im}(b)$, then we have three sub-cases.

– If $a_1 \notin C$, then there exists $k > 1$ minimal such that $a_k \in C$. We show that

$$C = \{a_k, a_{k+1}, \dots\} \cup \{b_1, b_2, \dots\} = \mathcal{V}(\mathcal{G}) \setminus \{a_1, a_2, \dots, a_{k-1}\}.$$

By construction, there are only two arcs with a_k as starting point: (a_k, a_{k-1}) and (a_k, b_q) for some $q \geq 1$. By minimality, we have $a_{k-1} \notin C$, hence $b_q \in C$. Note that b_q appears before a_k in c by construction. Therefore, if $q > 1$, then there exists $r < k$ such that (b_q, b_{q-1}) and (b_q, a_r) are the only arcs having b_q as starting point, so that $b_{q-1} \in C$ by minimality of k. Repeating this argument, one can easily show by induction that $\{b_1, \dots, b_q\} \subseteq C$.
In order to conclude the study of this case, let us now denote by $j > 1$ the integer such that $c_j = a_k$. By definition of a scaffolding, both (c_{j+1}, b_q) and (c_{j+1}, a_k) are in $E(\mathcal{G})$, so that $\theta(c_{j+1}) = 1 < d^+_C(c_{j+1})$. Thus, $c_{j+1} \in C$. We then show, using similar arguments, that $c_{j+2} \in C$ (separating the cases $c_{j+2} = a_{k+1}$ and $c_{j+2} = b_{q+1}$), and so on. By induction, we have $\{c_j, c_{j+1}, \dots\} \subseteq C$, but c is a shuffle of a and b, hence we have

$$C = \{a_k, a_{k+1}, \dots\} \cup \{b_1, b_2, \dots\} = \mathcal{V}(\mathcal{G}) \setminus \{a_1, a_2, \dots, a_{k-1}\}.$$

– The case $b_1 \notin C$ is similar, so that there exists $k > 1$ such that

$$C = \mathcal{V}(\mathcal{G}) \setminus \{b_1, b_2, \dots, b_{k-1}\}.$$

– If $\{c_1, c_2\} = \{a_1, b_1\} \subseteq C$, then one can easily check by induction that $C = \{c_1, c_2, c_3, \dots\} = \mathcal{V}(\mathcal{G})$.

In all cases, either C or $\mathcal{V}(\mathcal{G}) \setminus C$ is in $\mathcal{A} \cup \mathcal{B}$, and this ends the proof. □

EXAMPLE 4.1.16. On the following figure, we depict the lattice obtained with the scaffolding on the right of Figure 4.1.

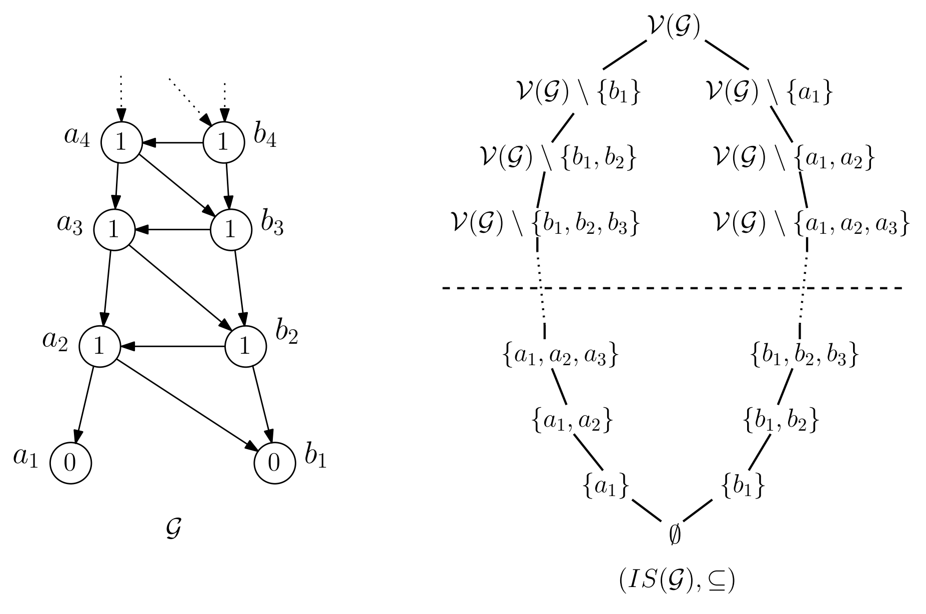


FIGURE 4.2.

4.2. Projective valued digraphs, their peelings, and algebraicity

At this point, an issue arises: when $\mathcal{G}$ is infinite, the study of $(IS(\mathcal{G}), \subseteq)$ seems to be difficult in general. Furthermore, one may ask if it is possible to generalize the notion of peeling sequences (see Definition 3.1.3) to our new context.

These two points are the main motivations for the theoretic construction presented in this section. We will exhibit a family of infinite valued digraphs such that:

(1) the resulting lattice can be seen as a projective limit of finite lattices (Theorem 4.2.17), allowing us to make effective tests despite of the infinite nature of the lattice;
(2) the maximal chains of the resulting lattice are encoded by some specific total orderings of the vertices of the valued digraph (see Theorem 4.2.24).

This construction is in fact a generalization of the theory developed in Chapter 3. Indeed, in the previous chapter we studied the case where the valued digraph $\mathcal{G}$ is acyclic and finite, and we constructed the poset $(IS(\mathcal{G}), \subseteq)$ using all the initial sections of some specific total orderings of $\mathcal{V}(\mathcal{G})$ (which we called peeling sequences of $\mathcal{G}$).

These infinite valued digraphs are what we call *projective valued digraphs* (Definition 4.2.10). That are, infinite valued digraphs which can be seen as a limit (in a certain sense) of a sequence of finite acyclic valued digraphs. We will show that for any projective valued digraph $\mathcal{G}$, there exists a set $PS(\mathcal{G})$ of total orderings of $\mathcal{V}(\mathcal{G})$ (Definition 4.2.4), which completely encodes the maximal chains of $(IS(\mathcal{G}), \subseteq)$ (Theorem 4.2.24). That is, a chain $\mathcal{C}$ of $(IS(\mathcal{G}), \subseteq)$ is maximal if and only if there exists $I \in PS(\mathcal{G})$ such that $\mathcal{C}$ equals set of the initial sections of I. We will also explain how the lattice structure of $(IS(\mathcal{G}), \subseteq)$ can be completely studied trough an associated sequence of finite complete lattices (Theorem 4.2.17), allowing us to make some computational tests. We finish this section showing that $(IS(\mathcal{G}), \subseteq)$ is *algebraic* whenever $\mathcal{G}$ is projective.

4.2.1. Some facts about the finite acyclic case. In this section, we briefly study the case where the valued digraph $\mathcal{G}$ is finite and acyclic, from which all the results of this section come. This case has already been intensively studied in Chapter 3, using combinatorial techniques. Since we want this chapter to be as self contained as possible, we will only use a particular case of Theorem 3.1.5, from which all the following results can be deduced.

THEOREM 4.2.1. *Let $\mathcal{G}$ be a valued digraph, if $\mathcal{G}$ is finite and acyclic, then $(IS(\mathcal{G}), \subseteq)$ is a graded complete lattice with rank function $\rho : A \mapsto |A|$.*

In the finite acyclic case, this graded structure leads to the following construction. Let $\mathcal{G}$ be a finite acyclic valued digraph and $n = |\mathcal{V}(\mathcal{G})|$. Thanks to Theorem 4.2.1, a maximal chain in $(IS(\mathcal{G}), \subseteq)$ is a sequence $(A_i)_{0 \leq i \leq n}$ of elements of $IS(\mathcal{G})$ such that for all $i \geq 1$, there exists $z_i \in \mathcal{V}(\mathcal{G}) \setminus A_{i-1}$ such that $A_i = A_{i-1} \cup \{z_i\}$. Thus, one can associate to each maximal chain of $(IS(\mathcal{G}), \subseteq)$ an injective sequence $[z_1, \dots, z_n]$ of elements of $\mathcal{V}(\mathcal{G})$. Such a sequence is called a *peeling sequence* of $\mathcal{G}$, and we denote by $PS(\mathcal{G})$ the set of all the peeling sequences of $\mathcal{G}$. Let us summarize this construction.

DEFINITION 4.2.2. Let $\mathcal{G}$ be a finite valued digraph, $|\mathcal{V}(\mathcal{G})| = n$ and $L = [z_1, \dots, z_n]$ be an injective sequence of $\mathcal{V}(\mathcal{G})$. We say that L is a peeling sequence of $\mathcal{G}$ if and only if

$$\text{for all } k \in [n],\ \{z_1, z_2, \dots, z_k\} \in IS(\mathcal{G}),$$

and we denote by $PS(\mathcal{G})$ the set of all the peeling sequences of $\mathcal{G}$.

It follows that, in the finite acyclic case, the study of $IS(\mathcal{G})$ is equivalent to the study of $PS(\mathcal{G})$. In Chapter 3, we explained how those peeling sequences can be constructed combinatorially thanks to an algorithmic procedure, called the peeling process. However, in this chapter we will need a more intrinsic characterization, which is given in the following proposition.

PROPOSITION 4.2.3. *Let $\mathcal{G} = (G, \theta)$ be a finite acyclic valued digraph, $|\mathcal{V}(\mathcal{G})| = n$ and $[z_1, \dots, z_n]$ be an injective sequence of elements of $\mathcal{V}(\mathcal{G})$. Then, $[z_1, \dots, z_n] \in PS(\mathcal{G})$ if and only if we have*

$$\textit{for all } i \in [n],\ \theta(z_i) = |\{j \leq i \mid (z_i, z_j) \in E(\mathcal{G})\}|.$$

PROOF. Let us denote by A_k the set $\{z_1, \dots, z_k\}$ for all $k \in [n]$, with the convention that $A_0 = \emptyset$. If $[z_1, \dots, z_n] \in PS(\mathcal{G})$, then for all $i \in [n]$ we have $A_{i-1}, A_i \in IS(\mathcal{G})$. Moreover, $z_i \notin A_{i-1}$, so that by definition

$$\theta(z_i) \geq |\{z \in A_{i-1} \mid (z_i, z) \in E(\mathcal{G})\}| = |\{j < i \mid (z_i, z_j) \in E(\mathcal{G})\}|.$$

We also have $z_i \in A_i$, hence

$$\theta(z_i) \leq |\{z \in A_i \mid (z_i, z) \in E(\mathcal{G})\}| = |\{j \leq i \mid (z_i, z_j) \in E(\mathcal{G})\}|.$$

Finally, since $\mathcal{G}$ is simple, we have $(z_i, z_i) \notin E(\mathcal{G})$, so that

$$\{j < i \mid (z_i, z_j) \in E(\mathcal{G})\} = \{j \leq i \mid (z_i, z_j) \in E(\mathcal{G})\},$$

and this implies that $\theta(z_i) = |\{j \leq i \mid (z_i, z_j) \in E(\mathcal{G})\}|$ as required.

We now prove the converse implication. Note that, by definition, for all integers $i, k \in [n]$, we have

$$d^+_{A_k}(z_i) = |\{j \leq k \mid (z_i, z_j) \in E(\mathcal{G})\}|.$$

Therefore, we have

$$\forall i \in [n],\ \theta(z_i) = d^+_{A_i}(z_i).$$

Fix an integer $i \in [n]$, we prove that $A_i \in IS(\mathcal{G}_i)$. Let $z \in \mathcal{V}(\mathcal{G})$, if $z \in A_i$, then there exists $j \leq i$ such that $z_j = z$. Since $A_j \subseteq A_i$, we have

$$\theta(z) = \theta(z_j) = d^+_{A_j}(z_j) \leq d^+_{A_i}(z_j).$$

If $z \notin A_i$, then there exists $k > i$ such that $z = z_k$. Since $A_i \subseteq A_k$, we have

$$\theta(z) = \theta(z_k) = d^+_{A_k}(z_k) \geq d^+_{A_i}(z_k).$$

This proves that A_i is in $IS(\mathcal{G})$. Thus, by definition, $[z_1, \ldots, z_n] \in PS(\mathcal{G})$, and this ends the proof. □

4.2.2. Peelings of any valued digraph. Proposition 4.2.3 gives us a natural way to extend the definition of peeling sequences to any simple valued digraph. This is the point of this section. We stress the fact that, even if the definition can be extended to any arbitrary valued digraph, the resulting object does not have all the properties of their counter part in the finite acyclic case (see Remark 4.2.8).

DEFINITION 4.2.4. Let $\mathcal{G} = (G, \theta)$ be a valued digraph and $I = (\mathcal{V}(\mathcal{G}), \preceq)$ be a total ordering of $\mathcal{V}(\mathcal{G})$, we say that I is a *peeling* of $\mathcal{G}$ if and only if for all $z \in \mathcal{V}(\mathcal{G})$,

$$\theta(z) = |\{y \preceq z \mid (z, y) \in E(\mathcal{G})\}|.$$

We denote by $PS(\mathcal{G})$ the set of all the peelings of $\mathcal{G}$.

Notice that the set $PS(\mathcal{G})$ may be empty (see Remark 4.2.8). For the sake of clarity, let us introduce the following notation, which we will use throughout this section.

DEFINITION 4.2.5. Let X be a set and $I = (X, \preceq)$ be a total ordering of X, for all $z \in X$ we denote by I_z the set $\{y \in X \mid y \preceq z\}$.

Our aim is now to prove that Definition 4.2.4 indeed generalizes some of the properties of peeling sequences. That is, we will prove that each initial section of a peeling (if a peeling exists) is in $IS(\mathcal{G})$ (see Proposition 4.2.7). We begin with a quite trivial, but useful, lemma.

LEMMA 4.2.6. *Let $\mathcal{G}$ be a valued digraph and I be a total ordering of $\mathcal{V}(\mathcal{G})$. Then, $I \in PS(\mathcal{G})$ if and only if $\theta(z) = d^+_{I_z}(\mathcal{G}, z)$ for all $z \in \mathcal{V}(\mathcal{G})$.*

PROOF. By definition, $d^+_{I_z}(\mathcal{G}, z) = |\{y \in I_z \mid (z, y) \in E(\mathcal{G})\}| = |\{y \preceq z \mid (z, y) \in E(\mathcal{G})\}|$. □

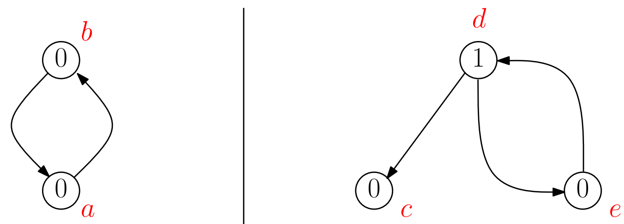
We are now able to prove that each initial section of any peeling is in $IS(\mathcal{G})$. Notice that the relationship between $PS(\mathcal{G})$ and $IS(\mathcal{G})$ is in fact stronger than that, and the following proposition also provides an alternative characterization of the elements of $PS(\mathcal{G})$.

PROPOSITION 4.2.7. *Let $\mathcal{G}$ be a valued digraph and I be a total ordering of $\mathcal{V}(\mathcal{G})$. Then, I is in $PS(\mathcal{G})$ if and only if each initial section of I is in $IS(\mathcal{G})$.*

Proof. Assume that the initial sections of I are in $IS(\mathcal{G})$ and let z be in $\mathcal{V}(\mathcal{G})$. Both I_z and $J := I_z \setminus \{z\}$ are initial sections of I. Since $(z, z) \notin E(\mathcal{G})$ (because $\mathcal{G}$ is simple), we have $d^+_J(z) = d^+_{I_z}(z)$. Furthermore, by definition we have $z \in I_z$ and $z \notin J$, so that $\theta(z) \leq d^+_{I_z}(z)$ and $\theta(z) \geq d^+_J(z) = d^+_{I_z}(z)$, hence $\theta(z) = d^+_{I_z}(z)$. Thus, $I \in PS(\mathcal{G})$.

We now prove the converse. Let A be an initial section of $I \in PS(\mathcal{G})$ and $z \in \mathcal{V}(\mathcal{G})$, if $z \in A$, then $I_z \subseteq A$, so that $\theta(z) = d^+_{I_z}(z) \leq d^+_A(z)$. Similarly, if $z \notin A$, then $A \subseteq I_z$, hence $\theta(z) \geq d^+_A(z)$. Thus, $A \in IS(\mathcal{G})$, and this ends the proof. $\square$

Remark 4.2.8. In the finite acyclic case, there is a one-to-one correspondence between peelings and maximal chains of $(IS(\mathcal{G}), \subseteq)$, but it is not a general situation. For example, $PS(\mathcal{G})$ may be empty. Even in the case where $PS(\mathcal{G}) \neq \emptyset$, there can be some elements of $IS(\mathcal{G})$ which are not initial sections of elements of $PS(\mathcal{G})$ (see Figure 4.3 for two illustrations of these facts). Nevertheless, we can find a family of infinite valued digraphs such that this



$IS(\mathcal{G}) = \{\emptyset, \{a, b\}\}$ $\quad$ $PS(\mathcal{G}) = \emptyset$

$IS(\mathcal{G}) = \{\emptyset, \{c\}, \{e\}, \{e, d\}, \{c, d, e\}\}$ $\quad$ $PS(\mathcal{G}) = \{[e, d, c]\}$

Figure 4.3.

correspondence between maximal chains and generalised peeling sequences is preserved. Those are the projective valued digraphs mentioned in the introduction of this section.

4.2.3. Definition of projective valued digraphs. Let us explain the philosophy behind the notions introduced here. Our aim in this section can be summarized in the following question: *is there a family of infinite valued digraph whose properties generalize those of the finite acyclic case ?* More precisely, can we find an infinite valued digraph $\mathcal{G}$ such that each maximal chain of $(IS(\mathcal{G}), \subseteq)$ is the set of the initial sections of an element of $PS(\mathcal{G})$?

A natural starting point is to consider digraphs (we will consider valued digraphs a little bit after) that can be seen as a limit of a sequence of finite acyclic digraphs. There are many natural way to define a notion of limit of a sequence of digraphs, and the one we will consider here is perhaps one of the simplest: let $(G_i)_{i\geq 1}$ be a sequence of digraphs such that

(1) G_1 is finite, simple and acyclic;
(2) $\mathcal{V}(G_{i+1})$ is obtained from $\mathcal{V}(G_i)$ by adding a new vertex z_{i+1} to it;
(3) $E(G_{i+1})$ is obtained from $E(G_i)$ by adding some new arcs to it, each one having z_{i+1} as starting point (notice that we do not allow to add arcs having z_{i+1} as ending point).

Such a sequence clearly define what corresponds to the intuitive notion of "limit of a sequence of digraphs", that is the digraph G_∞ such that

$$\mathcal{V}(\mathcal{G}_\infty) = \bigcup_{i\geq 0} \mathcal{V}(\mathcal{G}_i) \quad \text{and} \quad E(\mathcal{G}_\infty) = \bigcup_{i\geq 0} E(\mathcal{G}_i).$$

On Figure 4.4 we represent the beginning of such a sequence of digraphs. With these conditions, the sequence of digraph is made of finite acyclic digraphs, so that G_∞ is indeed a limit of simple acyclic digraphs. However, the main advantage of this definition lies on a technical point, related to our theory of valued digraphs. That is, consider A a subset of $\mathcal{V}(G_j)$ and $z \in \mathcal{V}(G_i)$, with $i < j$. Then, we have

$$d^+_A(G_j, z) = d^+_{A\cap\mathcal{V}(G_i)}(\mathcal{G}_j, z) = d^+_{A\cap\mathcal{V}(G_i)}(G_i, z). \tag{4.1}$$

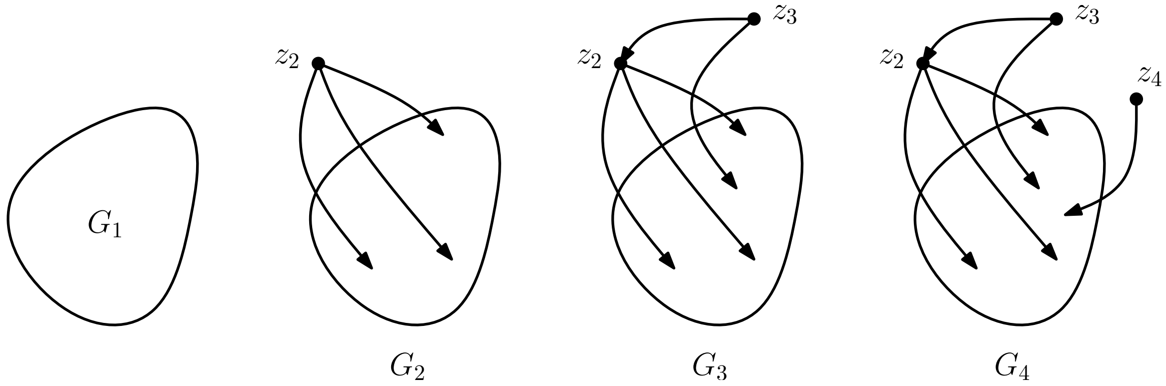


FIGURE 4.4.

Indeed, by definition of the sequence $(G_i)_{i\geq 1}$ there is no arc in G_j having z as starting point and an element of $\mathcal{V}(G_j)\setminus\mathcal{V}(G_i)$ as ending point (a formal and precise proof of (4.1) is given in Lemma 4.2.13). Equation (4.1) has two immediate consequences, which are fundamental in what follows.

First, if we consider a valuation $\theta:\mathcal{V}(\mathcal{G}_j)\to\mathbb{N}$ such that $\mathcal{G}_j=(G_j,\theta)$ is a valued digraph, then for all $z\in\mathcal{V}(G_i)$ we have:

$$0\leq\theta(z)\leq d^+_{\mathcal{V}(G_j)}(G_j,z)\implies 0\leq\theta(z)\leq d^+_{\mathcal{V}(G_i)}(G_i,z).$$

Therefore, the pair $\mathcal{G}_i:=(G_i,\theta)$ (where θ is restricted to the elements of $\mathcal{V}(G_i)$) is also a valued digraph. Even more interesting, we have that for all $A\subseteq\mathcal{V}(\mathcal{G}_j)$, then

$$A\in IS(\mathcal{G}_j)\iff\begin{cases}\theta(z)\leq d^+_A(G_j,z) & \text{for all } z\in A\\ \theta(z)\geq d^+_A(G_j,z) & \text{for all } z\in\mathcal{V}(G_j)\setminus A\end{cases}$$

so that we have

$$A\in IS(\mathcal{G}_j)\implies\begin{cases}\theta(z)\leq d^+_{A\cap\mathcal{V}(G_i)}(G_i,z) & \text{for all } z\in A\cap\mathcal{V}(G_i)\\ \theta(z)\geq d^+_{A\cap\mathcal{V}(G_i)}(G_i,z) & \text{for all } z\in\mathcal{V}(G_i)\setminus A\end{cases}$$

and finally, by definition of $IS(\mathcal{G}_i)$ we have

$$A\in IS(\mathcal{G}_j)\implies\mathcal{V}(G_i)\cap A\in IS(\mathcal{G}_i). \tag{4.2}$$

In other words, we can project $IS(\mathcal{G}_j)$ on $IS(\mathcal{G}_i)$, as depicted in Figure 4.5, and the family

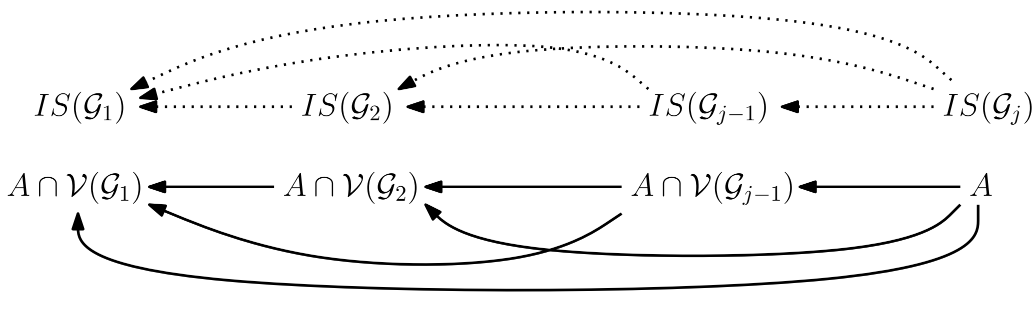


FIGURE 4.5.

$(IS(\mathcal{G}_j))_{i\geq 1}$ is what we call a *projective system* for a well-chosen set of projections (see the definition below).

DEFINITION 4.2.9. A *projective system* is a family $(A_i,p_{j,i})_{1\leq i<j}$, where the indices are taken in $\mathbb{N}\cup\{\infty\}$, A_i is a set and $p_{j,i}$ is a function from A_j to A_i such that

$$p_{k,i}\ o\ p_{j,k}=p_{j,i}\text{ for all } i<j<k.$$

Consequently, for any element A in $IS(\mathcal{G}_\infty)$ (a limit valued digraph, once we fixed a valuation), there exists a sequence of elements $(A_i)_{i\geq 1}$ obtained by projecting A on each $IS(\mathcal{G}_i)$, each one of these A_i being finite. Our aim in this section is twofold: first, we will formalize properly the construction presented above, and then show that the elements of $IS(\mathcal{G}_\infty)$ are completely determined by their sequence of projections. This second point has an important consequence, that is we can study the lattice structure of $(IS(\mathcal{G}_\infty), \subseteq)$ using the sequence of finite lattices $(IS(\mathcal{G}_i), \subseteq)$, allowing us to make effective tests on $\mathcal{G}_\infty$ despite of its infinite nature.

We begin with defining the notion of projective valued digraph.

DEFINITION 4.2.10. A sequence of valued digraphs $(\mathcal{G}_i = (G_i, \theta_i))_{i\geq 1}$ is called *projective* if and only if the four following conditions are satisfied.

(1) The valued digraph $\mathcal{G}_1$ is finite and acyclic.
(2) For all $i \geq 1$, there exists $z_{i+1} \notin \mathcal{V}(\mathcal{G}_i)$ such that $\mathcal{V}(\mathcal{G}_{i+1}) = \mathcal{V}(\mathcal{G}_i) \cup \{z_{i+1}\}$.
(3) For all $i \geq 1$, there exists $X \subseteq \mathcal{V}(\mathcal{G}_i)$ such that $E(\mathcal{G}_{i+1}) = E(\mathcal{G}_i) \cup \{(z_{i+1}, x) \mid x \in X\}$ (note that X may be empty).
(4) For all $i \geq 1$ and $z \neq z_{i+1}$, $\theta_{i+1}(z) = \theta_i(z)$.

To each projective sequence of valued digraphs $(\mathcal{G}_i)_{i\geq 1}$, we associate an infinite valued digraph, denoted by $\mathcal{G}_\infty$, defined by:

$$\mathcal{V}(\mathcal{G}_\infty) = \bigcup_{i\geq 0} \mathcal{V}(\mathcal{G}_i), \quad E(\mathcal{G}_\infty) = \bigcup_{i\geq 0} E(\mathcal{G}_i) \text{ and for all } z \in \mathcal{V}(\mathcal{G}_\infty),\ \theta_\infty(z) = \theta_j(z),$$

where j is an integer such that $z \in \mathcal{V}(\mathcal{G}_j)$. Such a $\mathcal{G}_\infty$ is called projective valued digraph.

Throughout this section, $\mathcal{G}_\infty$ will denote a projective valued digraph, $(\mathcal{G}_i)_{i\geq 1}$ will denote its associated projective sequence and z_k will denote the unique element of $\mathcal{V}(\mathcal{G}_k) \setminus \mathcal{V}(\mathcal{G}_{k-1})$. We begin our study with some general properties of $\mathcal{G}_\infty$.

PROPOSITION 4.2.11. *A projective valued digraph $\mathcal{G}_\infty$ is acyclic and of finite out-degree.*

PROOF. We first prove that $\mathcal{G}_\infty$ is of finite out-degree. Let $z \in \mathcal{V}(\mathcal{G}_\infty)$ and assume by contradiction that $d^+(\mathcal{G}_\infty, z)$ is infinite. By construction, there exists $i \geq 1$ such that $z \in \mathcal{V}(\mathcal{G}_i)$. Since $\mathcal{G}_i$ is finite, $d^+(\mathcal{G}_i, z)$ is finite, hence there exists $k > i$ such that z_k is the unique element of $\mathcal{V}(\mathcal{G}_k)\setminus\mathcal{V}(\mathcal{G}_{k-1})$ and such that $(z, z_k) \in E(\mathcal{G}_k)$, and this contradicts Point (3) of Definition 4.2.10. Thus, $\mathcal{G}_\infty$ is of finite out-degree.

We now prove that $\mathcal{G}_\infty$ is acyclic. Assume by contradiction that there exists $i > 1$ such that $\mathcal{G}_i$ is not acyclic, and let $n > 1$ be minimal such that $\mathcal{G}_n$ contains a cycle ($\mathcal{G}_1$ cannot be acyclic by definition). By minimality, a cycle of $\mathcal{G}_n$ must contain the vertex z_n, hence there exists $z \in \mathcal{V}(\mathcal{G}_n)$ such that $(z, z_n) \in E(\mathcal{G}_n)$, which contradicts Definition 4.2.10-(3). Therefore, $\mathcal{G}_i$ is acyclic for all $i \geq 1$. Finally, assume by contradiction that there exists a cycle $x_1, \ldots, x_q$ in $\mathcal{G}_\infty$. By construction, there exists an integer m such that $x_1, \ldots, x_q$ are all vertices of $\mathcal{G}_m$, thus $x_1, \ldots, x_q$ is also a cycle of $\mathcal{G}_m$, which is absurd. Therefore, $\mathcal{G}_\infty$ is acyclic, and this ends the proof. $\square$

Our aim is now to prove that the family $(IS(\mathcal{G}_i))_{1\leq i\leq\infty}$ is a projective system (Proposition 4.2.14) for a well chosen family of projections $(p_{j,i})_{1\leq i<j\leq\infty}$ (Definition 4.2.12), and then to show that $IS(\mathcal{G}_\infty)$ can be seen as a "kind of" projective limit of $(IS(\mathcal{G}_i), p_{j,i})_{1\leq i<j\leq\infty}$ (a good formulation in terms of categories is missing, this explains the "kind of").

We begin with the definition of the projections.

DEFINITION 4.2.12. For all $1 \leq i < j \leq \infty$, we have $\mathcal{V}(\mathcal{G}_i) \subseteq \mathcal{V}(\mathcal{G}_j)$ by construction. We denote by $p_{j,i}$ the map from the power set of $\mathcal{V}(\mathcal{G}_j)$ to the power set of $\mathcal{V}(\mathcal{G}_i)$, defined by

$$p_{j,i}(A) = A \cap \mathcal{V}(\mathcal{G}_i),$$

for all $A \subseteq \mathcal{V}(\mathcal{G}_j)$. These functions $p_{j,i}$ are called the *projections associated with* $\mathcal{G}_\infty$. Note that we clearly have $p_{j,i}\ o\ p_{k,j} = p_{k,i}$ for all $1 \leq i < j < k \leq \infty$.

We now begin the proof that $(IS(\mathcal{G}_i), p_{j,i})_{1\leq i<j\leq\infty}$ is a projective system. We start with a technical but (very) useful lemma, which is the formal proof of (4.1).

LEMMA 4.2.13. *Let $1 \leq i < j \leq \infty$, $A \subseteq \mathcal{V}(\mathcal{G}_j)$ and $z \in \mathcal{V}(\mathcal{G}_i)$, we have*

$$d^+_A(\mathcal{G}_j, z) = d^+_{p_{j,i}(A)}(\mathcal{G}_i, z).$$

PROOF. We set $C_j = \{y \in A \mid (z, y) \in E(\mathcal{G}_j)\}$ and $C_i = \{y \in p_{j,i}(A) \mid (z, y) \in E(\mathcal{G}_i)\}$. We prove that $C_i = C_j$. Since $p_{j,i}(A) \subseteq A$ and $E(\mathcal{G}_i) \subseteq E(\mathcal{G}_j)$, we have $C_i \subseteq C_j$. Assume by contradiction that there exists $x \in C_j \setminus C_i$, then there exists an integer $k > i$ such that $\mathcal{V}(\mathcal{G}_k) = \mathcal{V}(\mathcal{G}_{k-1}) \cup \{x\}$, *i.e.* $x = z_k$. But $z_k \in C_j$, hence there exists an arc from z to z_k in $\mathcal{G}_k$. But z is in $\mathcal{V}(\mathcal{G}_i)$ so it is in $\mathcal{V}(\mathcal{G}_{k-1})$, and this contradicts Point (3) of Definition 4.2.10. Thus, we have $C_i = C_j$. Finally, we have by definition that $d^+_A(\mathcal{G}_j, z) = |C_j|$ and $d^+_{p_{j,i}(A)}(\mathcal{G}_i, z) = |C_i|$, hence $d^+_A(\mathcal{G}_j, z) = d^+_{p_{j,i}(A)}(\mathcal{G}_i, z)$, and this ends the proof. □

We are now able to prove that $(IS(\mathcal{G}_i), p_{j,i})_{1\leq i<j\leq\infty}$ is a projective system.

PROPOSITION 4.2.14. *For all $1 \leq i < j \leq \infty$, we have*

$$p_{j,i}(IS(\mathcal{G}_j)) = IS(\mathcal{G}_i).$$

PROOF. Let $A \in IS(\mathcal{G}_j)$. We prove that $p_{j,i}(A) \in IS(\mathcal{G}_i)$. For that purpose, fix $z \in \mathcal{V}(\mathcal{G}_i)$, and consider the following two cases.

- If $z \in A$, then we have $\theta_j(z) \leq d^+_A(G_j, z)$. Thanks to Lemma 4.2.13, we have $d^+_A(G_j, z) = d^+_{p_{j,i}(A)}(G_i, z)$. Moreover, $\theta_i(z) = \theta_j(z)$ by construction, hence we have
$$\theta_i(z) = \theta_j(z) \leq d^+_A(\mathcal{G}_j, z) = d^+_{p_{j,i}(A)}(G_i, z).$$
- If $z \in \mathcal{V}(\mathcal{G}_i) \setminus A$, then using the same arguments, we have
$$\theta_i(z) = \theta_j(z) \geq d^+_A(\mathcal{G}_j, z) = d^+_{p_{j,i}(A)}(G_i, z).$$

Thus, $p_{j,i}(A) \in IS(\mathcal{G}_i)$, so that $p_{j,i}(IS(\mathcal{G}_j)) \subseteq IS(\mathcal{G}_i)$.

We now prove that the converse inclusion holds. Equivalently, we prove that the restriction of $p_{j,i}$ to $IS(\mathcal{G}_j)$ is surjective onto $IS(\mathcal{G}_i)$. Let $B \in IS(\mathcal{G}_i)$, we construct explicitly an element B_j in $IS(\mathcal{G}_j)$ such that $p_{j,i}(B_j) = B$. For that purpose, we define recursively a sequence $(B_k)_{i\leq k\leq j}$ as follows:

- $B_i = B$;
- for all $i < k \leq j$, $B_k = B_{k-1} \cup \{z_k\}$ if $\theta_k(z_k) < d^+_{B_{k-1}}(\mathcal{G}_k, z_k)$ (we recall that z_k is the unique element of $\mathcal{V}(\mathcal{G}_k) \setminus \mathcal{V}(\mathcal{G}_{k-1})$), and $B_k = B_{k-1}$ otherwise.

Finally, if $j = +\infty$, then we set $B_\infty = \bigcup_{k\geq i} B_k$. By construction, for all $i \leq k < q \leq j$ the following identities hold:

$$p_{q,k}(B_q) = B_k. \tag{4.3}$$

In particular, $p_{j,i}(B_j) = B_i$. It remains to prove that $B_j \in IS(\mathcal{G}_j)$. Fix $z \in \mathcal{V}(\mathcal{G}_j)$, and consider the following four possible cases.

- If $z \in B_i \subseteq \mathcal{V}(\mathcal{G}_i)$, then $\theta_i(z) \leq d^+_{B_i}(\mathcal{G}_i, z)$. Thanks to Equation (4.3) we have $B_i = p_{j,i}(B_j)$, and by Lemma 4.2.13, we have $d^+_{B_j}(\mathcal{G}_j, z) = d^+_{p_{j,i}(B_j)}(\mathcal{G}_i, z) = d^+_{B_i}(\mathcal{G}_i, z)$. Consequently, we have
$$\theta_j(z) = \theta_i(z) \leq d^+_{B_i}(\mathcal{G}_i, z) = d^+_{B_j}(\mathcal{G}_j, z).$$
- If $z \in B_j \setminus B_i$, then there exists an integer k such that $i < k \leq j$ and $z = z_k$. By construction, we have $\theta_k(z_k) < d^+_{B_{k-1}}(\mathcal{G}_k, z_k)$. Furthermore, since $(z_k, z_k) \notin E(\mathcal{G}_k)$, we have $d^+_{B_{k-1}}(\mathcal{G}_k, z_k) = d^+_{B_k}(\mathcal{G}_k, z_k)$, and by Lemma 4.2.13 we have $d^+_{B_j}(\mathcal{G}_j, z_k) = d^+_{B_k}(\mathcal{G}_k, z_k)$. Thus, we have
$$\theta_j(z_k) = \theta_k(z_k) < d^+_{B_{k-1}}(\mathcal{G}_k, z_k) = d^+_{B_k}(\mathcal{G}_k, z_k) = d^+_{B_j}(\mathcal{G}_j, z_k).$$

- If $z \in \mathcal{V}(\mathcal{G}_i) \setminus B_i$, then using the same arguments of the first case, we have
$$\theta_j(z) = \theta_i(z) \geq d^+_{B_i}(\mathcal{G}_i, z) = d^+_{p_{j,i}(B_j)}(\mathcal{G}_i, z) = d^+_{B_j}(\mathcal{G}_j, z).$$
- If $z \in \mathcal{V}(\mathcal{G}_j) \setminus (B_j \cup \mathcal{V}(\mathcal{G}_i))$, then there exists an integer k such that $i < k \leq j$ and $z = z_k$. By construction, $\theta_k(z_k) \geq d^+_{B_{k-1}}(\mathcal{G}_k, z_k)$, and using the same arguments of the second case, we conclude that
$$\theta_j(z) \geq d^+_{B_j}(\mathcal{G}_j, z).$$

Thus, we have $B_j \in IS(\mathcal{G}_j)$, and this ends the proof. $\square$

Consequently, $(IS(\mathcal{G}_i), p_{j,i})_{1 \leq i < j \leq \infty}$ is a projective system. Let us now explain how this projective structure can be used to study the lattice $(IS(\mathcal{G}_\infty), \subseteq)$. For that purpose, we introduce the following function.

Definition 4.2.15. We denote by $IS_{proj}(\mathcal{G}_1, \mathcal{G}_2, \ldots)$ the set
$$\left\{ (A_1, A_2, \ldots) \in \prod_{i \geq 1} IS(\mathcal{G}_i) \mid p_{j,i}(A_j) = A_i \text{ for all } 1 \leq i < j < \infty \right\},$$
and we define the function $\pi : IS(\mathcal{G}_\infty) \longrightarrow IS_{proj}(\mathcal{G}_1, \mathcal{G}_2, \ldots)$ to be
$$\forall A \in IS(\mathcal{G}_\infty),\ \pi(A) := (p_{\infty,1}(A), p_{\infty,2}(A), \ldots).$$

Since for all $i \geq 1$ and $A \in IS(\mathcal{G}_\infty)$, we have $p_{i+1,i}(p_{\infty,i+1}(A)) = p_{\infty,i}(A)$, thanks to Proposition 4.2.14, the map π is well defined.

Proposition 4.2.16. *The function π is a bijection.*

Proof. This function is clearly injective. To show that it is surjective, consider
$$(A_1, A_2, \ldots) \in IS_{proj}(\mathcal{G}_1, \mathcal{G}_2, \ldots),$$
and set $A := \bigcup_{i \geq 1} A_i$. We show that $A \in IS(\mathcal{G}_\infty)$. In order to do so, fix $z \in \mathcal{V}(\mathcal{G}_\infty)$, and consider the following two cases.

- If $z \in A$, then there exists $n \geq 1$ such that $z \in A_n \subseteq \mathcal{V}(\mathcal{G}_n)$. Since $A_n \in IS(\mathcal{G}_n)$, $\theta_n(z) \leq d^+_{A_n}(\mathcal{G}_n, z)$. Thus, by Lemma 4.2.13 we have
$$\theta_\infty(z) = \theta_n(z) \leq d^+_A(\mathcal{G}_\infty, z).$$
- if $z \in \mathcal{V}(\mathcal{G}_\infty) \setminus A$, then there exists $m \geq 1$ such that $z \in \mathcal{V}(\mathcal{G}_m) \setminus A_m$. Then, by proceeding as in the previous case, we have
$$\theta_\infty(z) \geq d^+_A(\mathcal{G}_\infty, z).$$

Thus, $A \in IS(\mathcal{G}_\infty)$, and we obviously have $\pi(A) = (A_1, A_2, \ldots)$, so that π is surjective and this concludes the proof. $\square$

Proposition 4.2.16 shows that $IS(\mathcal{G}_\infty)$ behaves like a projective limit of the projective system $(IS(\mathcal{G}_i), p_{j,i})_{1 \leq i < j \leq \infty}$. In the following theorem, we show how this projective structure is transferred to the complete lattice $(IS(\mathcal{G}_\infty), \subseteq)$, allowing us to study it through the sequence of finite lattices $(IS(\mathcal{G}_i), \subseteq)_{1 \leq i < \infty}$.

Theorem 4.2.17. *Let $\mathcal{G}_\infty$ be a projective valued digraph, with associated projective sequence $(\mathcal{G}_i)_{i \geq 1}$, $X \subseteq IS(\mathcal{G}_\infty)$ and $A, B \in IS(\mathcal{G}_\infty)$, with $\pi(A) = (A_1, A_2, \ldots)$ and $\pi(B) = (B_1, B_2, \ldots)$. Then, the following four statements are true.*

1. *We have $A \subseteq B$ if and only if $A_i \subseteq B_i$ for all $i \geq 1$.*
2. *The dual valued digraph $\widetilde{\mathcal{G}_\infty}$ (see Definition 4.1.7) is projective with associated sequence $(\widetilde{\mathcal{G}}_i)_{i \geq 1}$.*
3. *For all $i \in \mathbb{N}^* \cup \{\infty\}$, if we denote by $\vee_i$ the join in $(IS(\mathcal{G}_i), \subseteq)$, then*
$$\pi(\vee_\infty X) = (\vee_1\, p_{\infty,1}(X), \vee_2\, p_{\infty,2}(X), \ldots).$$

(4) Similarly, $\pi(\wedge_\infty X) = (\wedge_1 \, p_{\infty,1}(X), \wedge_2 \, p_{\infty,2}(X), \ldots)$.

PROOF. Since $A = \bigcup_{i\geq 1} A_i$ and $B = \bigcup_{i\geq 1} B_i$, Point (1) is obvious. Note that the values of the valuation does not play a role in the definition of a projective valued digraph. Thus, Point (2) is clearly true.

Let us prove Point (3). Let $\pi(\vee_\infty X) = (C_1, C_2, \ldots)$ and denote by X_i the set $p_{\infty,i}(X)$ for all $i \geq 1$. Since $\vee_\infty X$ is an upper bound of X, C_i is an upper bound of X_i for all $i \geq 1$, so that we have $\vee_i \, X_i \subseteq C_i$.

We now prove that the converse inclusion holds. For that purpose, we first show that $\vee_n X_n = p_{n+1,n}(\vee_{n+1} X_{n+1})$ for all $n \geq 1$. We use the notations of the proof of Theorem 4.1.6: let (J_i) be the sequence associated with $\vee_{n+1} X_{n+1}$, and (J'_i) be the sequence associated with $\vee_n X_n$. We prove that $p_{n+1,n}(J_i) = J'_i$ by induction on i.

- Since $J_0 = \bigcup_{A \in X_{n+1}} A$ and $J'_0 = \bigcup_{A\in X_n} A$, we clearly have $p_{n+1,n}(J_0) = J'_0$.
- Let $i \geq 0$ be such that $p_{n+1,n}(J_i) = J'_i$ and $z \in \mathcal{V}(\mathcal{G}_n)$, thanks to Lemma 4.2.13, we have $d^+_{J_i}(\mathcal{G}_{n+1}, z) = d^+_{p_{n+1,n}(J_i)}(\mathcal{G}_n, z) = d^+_{J'_i}(\mathcal{G}_n, z)$, hence, by definition of the sequences J and J', $z \in J_{i+1}$ if and only if $z \in J'_{i+1}$. Thus, we have $p_{n+1,n}(J_{i+1}) = J'_{i+1}$.

This implies that

$$p_{n+1,n}(\vee_{n+1} X_{n+1}) = p_{n+1,n}\left(\bigcup_{i\geq 0} J_i\right) = \bigcup_{i\geq 0} p_{n+1,n}(J_i) = \bigcup_{i\geq 0} J'_i = \vee_n X_n.$$

Therefore, by Proposition 4.2.16 we have $(\bigcup_{i\geq 1} \vee_i X_i) \in IS(\mathcal{G}_\infty)$. Furthermore, it is an upper bound of X by construction, hence $\vee_\infty X \subseteq (\bigcup_{i\geq 1} \vee_i X_i)$. Thus, (1) implies that for all $i \geq 1$, $C_i \subseteq \vee_i X_i$. Consequently, we have $\vee_i X_i = C_i$, and this ends the proof of (3).

Point (4) is an immediate consequence of (3) together with (2) and Proposition 4.1.8. ☐

4.2.4. Link between $PS(\mathcal{G})$ and $IS(\mathcal{G})$ in the projective case. It appears that the elements of $PS(\mathcal{G}_\infty)$ also inherit the projective structure of $\mathcal{G}_\infty$. Using this structure, we will prove that peelings in the projective case encode maximal chains of $(IS(\mathcal{G}_\infty), \subseteq)$ in the same way they do in the finite acyclic case (see Theorem 4.2.24). We begin with showing that $(PS(\mathcal{G}_i))_{i\geq 1}$ is a projective system, introducing first its associated projections.

DEFINITION 4.2.18. Set $1 \leq i < j \leq \infty$ and $J = (\mathcal{V}(\mathcal{G}_j), \prec) \in PS(\mathcal{G}_j)$. We denote by $p_{j,i}(J)$ the total ordering of $\mathcal{V}(\mathcal{G}_i)$ obtained by restricting $\prec$ to $\mathcal{V}(\mathcal{G}_i)$.

Note that for all $1 \leq i < j \leq \infty$, $z \in \mathcal{V}(\mathcal{G}_i)$ and $J \in PS(\mathcal{G}_j)$, we have $p_{j,i}(J_z) = (p_{j,i}(J))_z$. We now prove that $(PS(\mathcal{G}_i), p_{j,i})_{1\leq i<j\leq\infty}$ is a projective system.

PROPOSITION 4.2.19. *For all* $1 \leq i < j \leq \infty$, $p_{j,i}(PS(\mathcal{G}_j)) \subseteq PS(\mathcal{G}_i)$.

PROOF. This is an immediate consequence of Lemma 4.2.6 together with Lemma 4.2.13: let $J \in PS(\mathcal{G}_j)$, $I = p_{j,i}(J)$ and $z \in \mathcal{V}(\mathcal{G}_i)$, we have

$$\theta(z) = d^+_{J_z}(\mathcal{G}_j, z) = d^+_{p_{j,i}(J_z)}(\mathcal{G}_i, z) = d^+_{I_z}(\mathcal{G}_i, z).$$

Thus, by Lemma 4.2.6 $I \in PS(\mathcal{G}_j)$, and this concludes the proof. ☐

Proposition 4.2.19 allows us to define the following function.

DEFINITION 4.2.20. We denote by $PS_{proj}(\mathcal{G}_1, \mathcal{G}_2, \ldots)$ the set

$$\{(I_1, I_2, \ldots) \in \prod_{i\geq 1} PS(\mathcal{G}_i) \mid p_{j,i}(I_j) = I_i \ \textit{for all} \ 1 \leq i < j\},$$

and we define the map $\pi : PS(\mathcal{G}_\infty) \longrightarrow PS_{proj}(\mathcal{G}_1, \mathcal{G}_2, \ldots)$ to be

$$\pi(I) := (p_{\infty,1}(I), p_{\infty,2}(I), \ldots).$$

We now prove the analogous of Proposition 4.2.16 for the peelings in the projective case.

PROPOSITION 4.2.21. *The function π is a bijection.*

PROOF. The function π is clearly injective. It remains to show that it is surjective. Let $(I_i)_{i\geq 1} \in PS_{proj}(\mathcal{G}_1, \ldots)$, we have $p_{j,i}(I_j) = I_i$ for all $1 \leq i < j$. Thus, for all x and y such that x is smaller than y in I_i, then x is also smaller than y in I_j. Therefore, we can define $I = (\mathcal{V}(\mathcal{G}_\infty), \prec)$ the total ordering of $\mathcal{V}(\mathcal{G}_\infty)$ obtained by setting $x \prec y$ whenever there exists $m \geq 1$ such that x is smaller than y in I_m. Clearly, $\pi(I) = (I_1, I_2, \ldots)$ so we just need to show that $I \in PS(\mathcal{G}_\infty)$ to prove the property. Let us now prove that $I \in PS(\mathcal{G}_\infty)$. For that purpose, consider A an initial section of I and denote by A_i the set $p_{\infty,i}(A)$ for all $i \geq 1$. We have $p_{i+1,i}(A_{i+1}) = A_i$, and A_i is an inital section of I_i for all $i \geq 1$. But $I_i \in PS(\mathcal{G}_i)$ with $\mathcal{G}_i$ finite and acyclic, hence A_i is in $IS(\mathcal{G}_i)$. Thus, by Proposition 4.2.16 we have $A \in IS(\mathcal{G}_\infty)$. Finally, thanks to Proposition 4.2.7 I is in $PS(\mathcal{G}_\infty)$, and this ends the proof. □

We finish this section studying the maximal chains in $(IS(\mathcal{G}_\infty), \subseteq)$: we will show that, in the projective case, they are encoded by peelings (Theorem 4.2.24). We start with two technical lemmas.

LEMMA 4.2.22. *Let $k \in \mathbb{N}^*$, $\mathcal{C}$ be a chain in $(IS(\mathcal{G}_{k+1}), \subseteq)$ and $I \in PS(\mathcal{G}_k)$ be such that $p_{k+1,k}(\mathcal{C})$ is included in the set of the initial sections of I (since $\mathcal{G}_{k+1}$ is finite and acyclic, such a I always exists). Then, there exists $J \in PS(\mathcal{G}_{k+1})$ such that:*

(1) $p_{k+1,k}(J) = I$;
(2) $\mathcal{C}$ is included in the set of the initial sections of J.

PROOF. For the sake of clarity, in this proof we denote by θ the valuation θ_{k+1}.

We construct explicitly the peeling J. Without loss of generality, we can assume that both $\emptyset$ and $\mathcal{V}(\mathcal{G}_{k+1})$ are in $\mathcal{C}$. Since $\mathcal{G}_k$ is finite, we can write I as a sequence $[x_1, \ldots, x_n]$ with $n := |\mathcal{V}(\mathcal{G}_k)|$. By hypothesis, there exists $0 = i_1 < i_2 < \ldots < i_q = n$ such that

$$p_{k+1,k}(\mathcal{C}) = \{A_{i_j} \mid 1 \leq j \leq q\},$$

where $A_{i_j} = \{x_1, \ldots, x_{i_j}\}$ and with the convention that $A_0 = A_{i_1} = \emptyset$. Let us denote by B_j the smallest element in $\mathcal{C}$ such that $p_{k+1,k}(B_j) = A_{i_j}$ and by z the unique element of $\mathcal{V}(\mathcal{G}_{k+1}) \setminus \mathcal{V}(\mathcal{G}_k)$. Since $\emptyset$ and $\mathcal{V}(\mathcal{G}_{k+1})$ are both in $\mathcal{C}$, there exists $1 \leq m < q$ minimal such that $z \notin B_m$ and $z \in B_{m+1}$. We now split our study into two cases.

- If $|\mathcal{C}| > |p_{k+1,k}(\mathcal{C})|$, then $\mathcal{C} = \{B_1, \ldots, B_m, B, B_{m+1}, \ldots, B_q\}$ with $B = B_m \cup \{z\}$. We set J the sequence defined by

$$J := [x_1, \ldots, x_{i_m}, z, x_{i_m+1}, \ldots, x_n].$$

Let us now prove that $J \in PS(\mathcal{G}_{k+1})$. Since for all $j \in [n]$ we have $(x_j, z) \notin E(\mathcal{G}_{k+1})$, we also have that $d^+_{I_{x_j}}(x_j) = d^+_{J_{x_j}}(x_j)$. Thus, we have

$$\theta(x_j) = d^+_{I_{x_j}}(x_j) = d^+_{J_{x_j}}(x_j) \text{ for all } j \in [n].$$

Furthermore, since $z \notin B_m$ and $z \in B$, we have $\theta(z) \geq d^+_{B_m}(\mathcal{G}_{k+1}, z)$ and $\theta(z) \leq d^+_B(\mathcal{G}_{k+1}, z)$. But $(z, z) \notin E(\mathcal{G}_{k+1})$, so that $d^+_{B_m}(\mathcal{G}_{k+1}, z) = d^+_B(\mathcal{G}_{k+1}, z)$. Consequently, we have

$$\theta(z) = d^+_{B_m}(\mathcal{G}_{k+1}, z).$$

Thus, thanks to Lemma 4.2.6 we have $J \in PS(\mathcal{G}_{k+1})$, and J clearly satisfies (1) and (2).

- If $|\mathcal{C}| = |p_{k+1,k}(\mathcal{C})|$, then $\mathcal{C} = \{B_1, \ldots, B_q\}$. Consider the finite sequence $(d_i)_{0\leq i\leq n}$ defined by $d_i = |\{j \leq i \mid (z, x_j) \in E(\mathcal{G}_{k+1})\}| - \theta(z)$, this sequence is weakly increasing and we have $d_{i+1} - d_i \leq 1$ for all $0 \leq i < n$. By definition, $z \in B_{m+1} \setminus B_m$ with B_m and B_{m+1} both in $IS(\mathcal{G}_{k+1})$, hence $d_{i_m} \leq 0$ and $d_{i_{m+1}} \geq 0$, so that there exists r minimal such that $i_m \leq r \leq i_{m+1}$ and $d_r = 0$. Consequently, if we set

$$J := [x_1, \ldots, x_r, z, x_{r+1}, \ldots, x_n],$$

then thanks to a similar argument as in the previous case we have

$$\theta(x_k) = d^+_{I_{x_k}}(x_k) = d^+_{J_{x_k}}(x_k) \text{ for all } k \in [n],$$

and $\theta(z) = d^+_{J_z}(z)$ by definition of d_r. Thus, $J \in PS(\mathcal{G}_{k+1})$ and satisfy Point (1) and (2). □

We now give the second technical lemma.

Lemma 4.2.23. *Let $I = (X, \preceq_I)$ and $J = (X, \preceq_J)$ be two total orderings of a set X. If the set of the initial sections of I is included in the set of the initial sections of J, then $I = J$.*

Proof. We show that $x \preceq_I y$ if and only if $x \preceq_J y$. First, note that for all $x, y \in X$, $x \preceq_I y$ if and only if there exists an initial section A of I such that $x \in A$ and $y \notin A$, and similarly for $\preceq_J$. Therefore, if $x \preceq_I y$, then $x \preceq_J y$. Conversely, if $x \preceq_J y$ then we have that either $x \preceq_I y$, or $y \preceq_I x$ (because I is a total order). Assume by contradiction that $y \preceq_I x$, then there exists an initial section B of I such that $y \in B$ and $x \notin B$. However, by hypothesis, B is also an initial section of J, so that $y \preceq_J x$ and this is absurd. Thus, we have $x \preceq_I y$.

Eventually, we proved that $x \preceq_I y$ if and only if $x \preceq_J y$, hence $I = J$, and this ends the proof. □

We are now able to state and prove the main theorem of this section.

Theorem 4.2.24. *Let $\mathcal{G}_\infty$ be a projective valued digraph, $(\mathcal{G}_i)_{i \geq 1}$ be its associated sequence of valued digraphs and $\mathcal{C}$ be a chain of $(IS(\mathcal{G}_\infty), \subseteq)$. Then, there exists $I \in PS(\mathcal{G}_\infty)$ such that $\mathcal{C}$ is included in the set of the initial sections of I and $\mathcal{C}$ is maximal if and only if we have equality.*

Proof. **Step 1**: first, we construct a peeling I such that $\mathcal{C}$ is included in the set of the initial sections of I. We denote by $\mathcal{C}_i$ the set $p_{\infty,i}(\mathcal{C})$ (notice that $p_{j,i}(\mathcal{C}_j) = \mathcal{C}_i$). Since $\mathcal{C}_1$ is a chain in $(IS(\mathcal{G}_1), \subseteq)$ and since $\mathcal{G}_1$ is finite and acyclic, by Theorem 4.2.1 there exists $I_1 \in PS(\mathcal{G}_1)$ such that $\mathcal{C}_1$ is included in the set of the initial sections of I_1. Thus, using recursively Lemma 4.2.22, for all $i > 1$ there exists $I_i \in PS(\mathcal{G}_i)$ such that:

- $\mathcal{C}_i$ is included in the set of the initial sections of I_i;
- $p_{i+1,i}(I_{i+1}) = I_i$.

Eventually, thanks to Proposition 4.2.21 there exists $I = (\mathcal{V}(\mathcal{G}_\infty), \preceq) \in PS(\mathcal{G}_\infty)$ such that $\pi(I) = (I_1, I_2, \ldots)$.

We now prove that each element of $\mathcal{C}$ is an initial section of I. Let $A \in \mathcal{C}$, by construction each $A_i = p_{\infty,i}(A)$ is an initial section of I_i. Let $u \preceq v$ be such that $v \in A$, there exists $k \geq 1$ such that both u and v are in $\mathcal{V}(\mathcal{G}_k)$. Then, we have $v \in A_k$, but A_k is an initial section of I_k, so that $u \in A_k$. Consequently, $u \in A$ so A is an initial section of I. Thus, $\mathcal{C}$ is included in the set of the initial sections of I, and this ends the first part of our proof.

Step 2: We now prove that $\mathcal{C}$ is maximal if and only if $\mathcal{C}$ is equal to the set of the initial sections of I. Since the set of the initial section of I is a chain that contains $\mathcal{C}$, if $\mathcal{C}$ is maximal then we have equality. Conversely, assume that there exists $I \in PS(\mathcal{G}_\infty)$ such that $\mathcal{C}$ is equal to the set of the initial sections of I. There exists $\mathcal{C}'$ a maximal chain such that $\mathcal{C} \subseteq \mathcal{C}'$ (note that this is a consequence of Zorn's Lemma, and its use can be avoided but the proof is somewhat longer). By Step 1 there exists $J \in PS(\mathcal{G}_\infty)$ such that $\mathcal{C}'$ is the set of the initial sections of J. As a consequence, the initial sections of I are initial sections of J. Thus, thanks to Lemma 4.2.23 we have $I = J$, so that $\mathcal{C} = \mathcal{C}'$, which obviously implies that $\mathcal{C}$ is maximal. □

Theorem 4.2.24 has the following corollary, which concludes this section.

Corollary 4.2.25. *Let $A \in IS(\mathcal{G}_\infty)$, there exists $I \in PS(\mathcal{G}_\infty)$ such that A is an initial section of I.*

PROOF. Since A is included in a maximal chain, this corollary is clear. If one wants to avoid the use of Zorn's Lemma hidden in this little argument, note that we can in fact construct the peeling I using recursively Lemma 4.2.22. $\square$

4.2.5. Projective implies algebraic. Let us first recall the definitions of compact elements of a complete lattice and of algebraic complete lattices.

DEFINITION 4.2.26. Let $\mathcal{P} = (P, \leq)$ be a complete lattice and $c \in P$, we say that c is *compact* if and only if for all subset X of $\mathcal{P}$ such that $c \leq \vee X$, there exists $Y \subseteq X$ finite such that $c \leq \vee Y$.

DEFINITION 4.2.27. A complete lattice $\mathcal{P} = (P, \leq)$ is called *algebraic* if and only if:

$$\text{for all } x \in P, \text{ we have } x = \vee\{c \leq x \mid c \text{ compact}\}.$$

It is obvious that a finite complete lattice is algebraic, since in this case all the elements are compact. Therefore, $(IS(\mathcal{G}), \subseteq)$ is algebraic whenever $\mathcal{G}$ is finite, so that the lattice coming from a projective valued digraph is a limit of algebraic lattices, and it is natural to ask if the property "to be algebraic" also passes to the limit.

We will show that it is indeed the case (Theorem 4.2.30). In the following of this section, $\mathcal{G}_\infty$ will denote a projective valued digraph with associated sequence $(\mathcal{G}_i)_{i\geq 1}$, and, as usual, for all $k > 1$ we will denote by z_k the unique element of $\mathcal{V}(\mathcal{G}_k) \setminus \mathcal{V}(\mathcal{G}_{k-1})$. We begin with introducing a family of subsets of $\mathcal{V}(\mathcal{G}_\infty)$, which turn out to be compact elements of $(IS(\mathcal{G}_\infty), \subseteq)$ (Proposition 4.2.29).

DEFINITION 4.2.28. Let $k \in \mathbb{N}^*$ and $A \in IS(\mathcal{G}_k)$, we recursively define a sequence (A_i) as follows:

- for all $1 \leq i < k$, $A_i = p_{k,i}(A)$;
- $A_k = A$;
- for all $j \geq k$, $A_{j+1} = A_j \cup \{z_{j+1}\}$ if $\theta(z_{j+1}) < d^+_{A_j}(\mathcal{G}_{j+1}, z_{j+1})$, and $A_{j+1} = A_j$ otherwise.

Finally, we denote by $\overleftarrow{A}$ the set $\bigcup_{i\geq 1} A_i$.

PROPOSITION 4.2.29. *For all $k \in \mathbb{N}^*$ and $A \in IS(\mathcal{G}_k)$, we have that $\overleftarrow{A}$ is in $IS(\mathcal{G}_\infty)$ and is compact in $(IS(\mathcal{G}_\infty), \subseteq)$.*

PROOF. We first show that $\overleftarrow{A} \in IS(\mathcal{G}_\infty)$. Let (A_i) be the sequence associated with $\overleftarrow{A}$, since $A \in IS(\mathcal{G}_k)$, we have that for all $i \leq k$, $A_i \in IS(\mathcal{G}_i)$.

We now prove by induction on j that A_j is in $IS(\mathcal{G}_j)$ for all $j \geq 1$. First, notice that $A_k \in IS(\mathcal{G}_k)$ by definition, and for all $i \leq k$ we have $A_i = p_{k,i}(A_k) \in IS(\mathcal{G}_i)$. Let $j \geq k$ be such that $A_j \in IS(\mathcal{G}_j)$, and let $z \in \mathcal{V}(\mathcal{G}_{j+1})$. There are two cases.

- If $z \neq z_{j+1}$, then $z \in \mathcal{V}(\mathcal{G}_j)$ and $d^+_{A_{j+1}}(\mathcal{G}_{j+1}, z) = d^+_{A_j}(\mathcal{G}_j, z)$ by Lemma 4.2.13. Therefore, if $z \in A_{j+1}$, then $z \in A_j$ and $\theta(z) \leq d^+_{A_j}(\mathcal{G}_j, z) = d^+_{A_{j+1}}(\mathcal{G}_{j+1}, z)$, otherwise we have $\theta(z) \geq d^+_{A_{j+1}}(\mathcal{G}_{j+1}, z)$.
- If $z = z_{j+1}$, then the fact that $(z_{j+1}, z_{j+1}) \notin E(\mathcal{G}_{j+1})$ implies that
$$d^+_{A_j}(\mathcal{G}_{j+1}, z_{j+1}) = d^+_{A_{j+1}}(\mathcal{G}_{j+1}, z_{j+1}).$$
Thus, by definition of A_{j+1}, if $z_{j+1} \in A_{j+1}$, then $\theta(z_{j+1}) < d^+_{A_{j+1}}(\mathcal{G}_{j+1}, z_{j+1})$, and $\theta(z_{j+1}) \geq d^+_{A_{j+1}}(\mathcal{G}_{j+1}, z_{j+1})$ otherwise.

In all cases, $A_{j+1} \in IS(\mathcal{G}_{j+1})$ for all j. Moreover for all $j \geq 1$, $p_{j+1,j}(A_{j+1}) = A_j$ and $p_{\infty,j}(\overleftarrow{A}) = A_j$, thus, by Proposition 4.2.16, $\overleftarrow{A}$ is in $IS(\mathcal{G}_\infty)$.

We now show that $\overleftarrow{A}$ is compact. Let $X \subseteq IS(\mathcal{G}_\infty)$ be such that $\overleftarrow{A} \subseteq \vee_\infty X$, and denote by Y_k the set $p_{\infty,k}(X)$, which is a finite subset of $IS(\mathcal{G}_k)$. There exists Y a finite subset of X such that $p_{\infty,k}(Y) = Y_k$. Let us denote by Y_i the set $p_{\infty,i}(Y)$ for all $i \geq 1$, and note that thanks to Theorem 4.2.17 (3) we have $\pi(\vee Y) = (\vee_1 Y_1, \vee_2 Y_2, \ldots)$. Consequently, if we prove that $A_i \subseteq \vee_i Y_i$ for all $i \geq 1$, then by Theorem 4.2.17 (1) we have that $\overleftarrow{A} \subseteq \vee Y$, which is the expected result. It remains to show that $A_i \subseteq \vee_i Y_i$, and we proceed by induction on i.

- By construction, for all $1 \leq i \leq k$ we have $A_i \subseteq \vee_i Y_i$.
- Let $j \geq k$ be such that $A_j \subseteq \vee_j Y_j$. If $z_{j+1} \in A_{j+1}$, then

$$\theta(z_{j+1}) < d^+_{A_j}(\mathcal{G}_{j+1}, z_{j+1}) \leq d^+_{\vee_j Y_j}(\mathcal{G}_{j+1}, z_{j+1}) \leq d^+_{\vee_{j+1} Y_{j+1}}(\mathcal{G}_{j+1}, z_{j+1}).$$

  Thus, $z_{j+1} \in \vee_{j+1} Y_{j+1}$, so that $A_{j+1} \subseteq \vee_{j+1} Y_{j+1}$. If $z_{j+1} \notin A_{j+1}$, then $A_j = A_{j+1} \subseteq \vee_j Y_j \subseteq \vee_{j+1} Y_{j+1}$.

By induction, we have the desired property, and this ends the proof. □

We are now able to prove that "projective implies algebraic".

THEOREM 4.2.30. *Let $\mathcal{G}$ be a valued digraph, if $\mathcal{G}$ is projective, then $(IS(\mathcal{G}), \subseteq)$ is algebraic.*

PROOF. Let $A \in IS(\mathcal{G})$, $(\mathcal{G}_i)_{i \geq 1}$ be its associated sequence of valued digraphs, z_i be the unique element of $\mathcal{V}(\mathcal{G}_i) \setminus \mathcal{V}(\mathcal{G}_{i-1})$ and $k \in \mathbb{N}^*$, we denote by A_i the set $p_{\infty,i}(A)$ for all $i \geq 1$. We prove that $\overleftarrow{A_k} \subseteq A$. In order to do so, let us denote by B_i the set $p_{\infty,i}\left(\overleftarrow{A_k}\right)$ for all $i \geq 1$. We prove by induction on i that $B_i \subseteq A_i$.

By definition, it is clear that for all $i \leq k$, $B_i = A_i$. Let $i \geq k$ be such that $B_i \subseteq A_i$ and $z \in B_{i+1}$. We have two cases.

- If $z = z_{i+1}$, then we have $\theta(z) < d^+_{B_i}(\mathcal{G}_i, z)$ by definition. Moreover, we have that $(z_{i+1}, z_{i+1}) \notin E(\mathcal{G}_{i+1})$, which implies that

$$\theta(z) < d^+_{B_i}(\mathcal{G}_i, z) \leq d^+_{A_i}(G_i, z) = d^+_{A_i}(\mathcal{G}_{i+1}, z) \leq d^+_{A_{i+1}}(\mathcal{G}_{i+1}, z),$$

  hence $z \in A_{i+1}$.
- If $z \in B_i$, then $z \in B_i \subseteq A_i \subseteq A_{i+1}$.

In all cases, we have $B_{i+1} \subseteq A_{i+1}$, and this ends the induction. Consequently, thanks to Theorem 4.2.17, we have that $\overleftarrow{A_k} \subseteq A$.

Eventually, we now are able to prove that $(IS(\mathcal{G}), \subseteq)$ is algebraic. For all $z \in A$, there exists $k \in \mathbb{N}^*$ such that $z \in \mathcal{V}(\mathcal{G}_k)$, thus $z \in p_{\infty,k}(A)$, so that $z \in \overleftarrow{A_k}$. Therefore, we have

$$A \subseteq \bigcup_{k \geq 1} \overleftarrow{A_k} \subseteq \bigcup_{\substack{C \subseteq A \\ C \text{ compact}}} C \subseteq A.$$

This implies that $(IS(\mathcal{G}), \subseteq)$ is algebraic, and this ends the proof. □

4.3. Application to the study of Dyer's conjectures

In this section, we follow the terminology used in [**DH**].

4.3.1. Dyer's conjectures and statement of the results of this section. Let us begin with recalling some facts about Coxeter groups and their root systems. In this section we consider a Coxeter group W with finite generating set S, and Coxeter matrix $M = (m_{st})_{s,t \in S}$. That is, M is a symmetric matrix with $m_{ss} = 1$ and, for $s \neq t$, $m_{st} = m_{ts} \in \{2, 3, \ldots\} \cup \{\infty\}$. The relations among the generators are of the form $(st)^{m_{st}} = 1$ if $m_{st} < \infty$. For all $w \in W$, we denote by $\ell(w)$ the minimum length of any decomposition $w = s_1 \cdots s_l$ with $s_i \in S$. This statistic ℓ on W is the rank function of a poset structure $(W, \leq_R)$, called the *(right) weak order* on W, and which is defined as follows: we say that $w \leq_R \tau$ if and only if there exists $s_1, \ldots, s_k$ in S such that $\tau = \omega s_1 \cdots s_k$ and $\ell(\tau) = \ell(w) + k$. In the remaining of this chapter, we will exclusively work with the right weak order so we use from now on the term "weak order" instead of "right weak order".

Let us consider a quadratic space (V, B), where V is a vector space of finite dimension $|S|$ with basis $\Delta = \{\alpha_s \mid s \in S\}$, and where B is a bilinear form which satisfies the following conditions: for all s and t in S,

$$B(\alpha_s, \alpha_t) = -\cos(\frac{\pi}{m_{st}}) \text{ if } m_{st} < +\infty,$$
$$B(\alpha_s, \alpha_t) \in]-\infty; -1] \text{ otherwise.}$$

We denote by $\mathcal{O}_B(V)$ the group of linear maps that preserve B. A vector $v \in V$ is called isotropic if and only if $B(v, v) = 0$, and to each non-isotropic vector α we associate the B-reflection $s_\alpha(u) = u - 2\frac{B(u,\alpha)}{B(\alpha,\alpha)}\alpha$, $s_\alpha \in \mathcal{O}_B(V)$. Then we have a *geometric representation* of W, that is a faithful representation of W as a subgroup of $\mathcal{O}_B(V)$ such that S is mapped into the set of B-reflections associated to Δ. We denote by $\Phi := W(\Delta)$ the corresponding *root system* with basis Δ, which is partitioned into *positive roots* Φ^+ (the elements of Φ that are linear combination of elements of Δ with non-negative coefficients) and *negative roots* $\Phi^- = -\Phi^+$. We call the couple (Φ, Δ) a geometric system of W, and we call Δ a *simple system* of this geometric system. To each $\omega \in W$ one can associate its *inversion set*, namely $\mathrm{Inv}(\omega) = \Phi^+ \cap \omega(\Phi^-)$. The inversion set is a finite set of positive roots, which has the two following classical properties (see [**BB**]).

(1) For all $\omega \in W$, $\ell(\omega) = |\mathrm{Inv}(\omega)|$.
(2) For all ω and τ in W, we have that $\omega \leq_R \tau$ if and only if $\mathrm{Inv}(\omega) \subseteq \mathrm{Inv}(\tau)$.

For more details about geometric representations and general properties of Coxeter groups, the reader may consult, for instance, [**H**] and [**BB**].

In what follows, we will represent root systems using *projective representations* of Φ, as in [**DH**], [**DHR**], and [**HLR**]. Since $\Phi = \Phi^+ \sqcup \Phi^-$ is encoded by the set of positive roots, we represent Φ by an affine cut $\widehat{\Phi}$. That is, there exists an affine hyperplane V_1 in V such that for all $\gamma \in \Phi^+$, the ray $\mathbb{R}^+\gamma$ intersects V_1 in a unique nonzero point $\widehat{\gamma}$ (in the classical geometric representation, one can choose V_1 as the hyperplane of equation $x_1 + x_2 + \cdots + x_n = 1$). Finally, $\widehat{\Phi}$ is the set obtained considering all those intersection points $\widehat{\gamma}$, and $\widehat{\Phi}$ is what is called a projective representation of Φ. We depict in Figure 4.6 a projective representation of the root system $\Phi = \{e_j - e_i \mid i, j \in [n-1],\ i \neq j\}$ of A_3, where e_i denotes the i-th vector of the canonical base of $\mathbb{R}^n$. Clearly, three points are aligned in a projective representation if and only if their corresponding roots are coplanar. Since this property is fundamental for what follows, we represent collinearity with a portion of line.

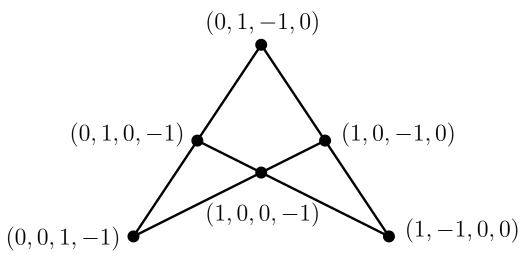


FIGURE 4.6. A projective representation of a root system of A_3, where we omited the symbole "$\widehat{\cdot}$".

We will particularly focus on some subsets of Φ^+, that we define below.

DEFINITION 4.3.1. Let $A \subseteq \Phi^+$, we say that A is *closed* if and only if

$$\text{for all } \alpha, \beta, \gamma \in \Phi^+ \text{ such that } \gamma = a\alpha + b\beta,\ a > 0,\ b > 0, \text{ if } \alpha, \beta \in A, \text{ then } \gamma \in A.$$

We say that A is *bi-closed* if and only if both A and $\Phi^+ \setminus A$ are closed. We denote by $\mathcal{B}(\Phi^+)$ the set of all the bi-closed sets of Φ^+.

Using this notion, there is a geometrical characterization of the inversion sets of the elements of W, as shown in Proposition 4.3.2. For a proof of this proposition, the interested read may consult, for instance, [**P**].

PROPOSITION 4.3.2. *The finite bi-closed sets of* Φ^+ *are exactly the inversion sets of the elements of* W.

Therefore, the poset $(\mathcal{B}(\Phi^+), \subseteq)$ provides a natural extension of the weak order.

CONJECTURE 4.3.3 ([**D1**], Remark 2.14). The poset $(\mathcal{B}(\Phi^+), \subseteq)$ is a complete ortho-lattice with ortho-complement given by the set complement of Φ^+.

Let us now focus on a second conjecture of Dyer, which gives a generalisation of the reduced decompositions of the maximal element in the finite case, in terms of *reflection orderings*.

DEFINITION 4.3.4. A reflection ordering is a total order $I = (\Phi^+, \preceq)$ of Φ^+, such that

$$\text{for all } \alpha, \beta, \gamma \in \Phi^+, \text{if } \gamma = a\alpha + b\beta,\ a > 0,\ b > 0,\ \text{then } \alpha \preceq \gamma \preceq \beta,\ \text{or } \beta \preceq \gamma \preceq \alpha.$$

It is known that, in the finite case, we have a one-to-one correspondence between reduced decomposition of the maximal element and reflection orderings. More precisely, if W is finite, then a chain $\mathcal{C}$ of $(\mathcal{B}(\Phi^+), \subseteq)$ is maximal if and only if there exists a reflection ordering I such that $\mathcal{C}$ is constituted exactly of the *initial sections* (*i.e.* lower sets) of I. The second conjecture says that this situation is preserved in the infinite case.

CONJECTURE 4.3.5 ([**D4**], Section 2.2). A chain $\mathcal{C}$ of $(\mathcal{B}(\Phi^+), \subseteq)$ is maximal if and only if there exists a reflection ordering I such that $\mathcal{C}$ is constituted exactly of the initial sections of I.

Our aim in the following sections is to apply the notion of valued digraph to the study of Dyer's conjectures. First, we introduce in Section 4.3.2 a family of valued digraph, called well-assembled on Φ^+ that we relate to Conjectures 4.3.3 and 4.3.5. In Section 4.3.3, we prove that there exists a projective well-assembled on Φ^+ valued digraph, constructing explicitly such a valued digraph.

In Section 4.3.4, we explore some of the consequences of the existence of a projective well-assembled on Φ^+ valued digraph $\mathcal{G}$. More precisely, we prove the following theorem.

THEOREM 4.3.6. *Let* W *be a Coxeter group of finite rank,* $\Phi = \Phi^+ \sqcup \Phi^-$ *be the associated root system,* $\mathcal{B}(\Phi^+)$ *be its set of bi-closed sets and* $\mathcal{G}$ *be a projective well-assembled on* Φ^+ *valued digraph, we have the two following properties.*

(1) *The poset* $(IS(\mathcal{G}), \subseteq)$ *is a complete algebraic ortho-lattice such that* $\mathcal{B}(\Phi^+) \subseteq IS(\mathcal{G})$, *and each reflection ordering of* Φ^+ *is in* $PS(\mathcal{G})$.
(2) *We have that* $IS(\mathcal{G}) = \mathcal{B}(\Phi^+)$ *if and only if* $PS(\mathcal{G})$ *is equal to the set of the reflection orderings of* Φ^+.

Then we explain how the projective structure of $\mathcal{G}$ can be used to test if $IS(\mathcal{G})$ has a chance to be equal to $\mathcal{B}(\Phi^+)$ or not (Proposition 4.3.25). Using Proposition 4.3.25, we provide an example of a projective valued digraph $\mathcal{G}'$ being well-assembled on a root system of the Coxeter group $\widetilde{C_3}$ such that $IS(\mathcal{G}') \neq \mathcal{B}(\Phi^+)$. Nevertheless, we make the following conjecture.

CONJECTURE 4.3.7. Let W be a Coxeter group of finite rank and $\Phi = \Phi^+ \sqcup \Phi^-$ be a root system of W, there exists at least one projective valued digraph $\mathcal{G}$ well-assembled on Φ^+ such that $IS(\mathcal{G}) = \mathcal{B}(\Phi^+)$.

Finally, in Section 4.3.5 we use results from Sections 4.3.3 and 4.3.4 to prove that our conjecture holds in the case of finite *simply laced* Coxeter groups (Corollary 4.3.30), that are Coxeter groups A_n, D_n, E_6, E_7 and E_8.

4.3.2. Definition of well-assembled valued digraphs. A natural way to approach Conjecture 4.3.3 with the theory developed in the previous sections is to look for a balanced valued digraph $\mathcal{G}$ such that $\mathcal{V}(\mathcal{G}) = \Phi^+$ and $IS(\mathcal{G}) = \mathcal{B}(\Phi^+)$. Let us first see if this can be done on the simplest infinite example: the infinite dihedral Coxeter group.

EXAMPLE 4.3.8. We represent on Figure 4.7 the poset obtained by ordering by inclusion the bi-closed sets of a root system of the infinite dihedral group, *i.e.* the Coxeter group with two generators s and t such that $m_{s,t} = +\infty$. As one can see, the lattice obtained is isomorphic to the one depicted in Example 4.1.16, identifying α_i with a_i and β_i with b_i.

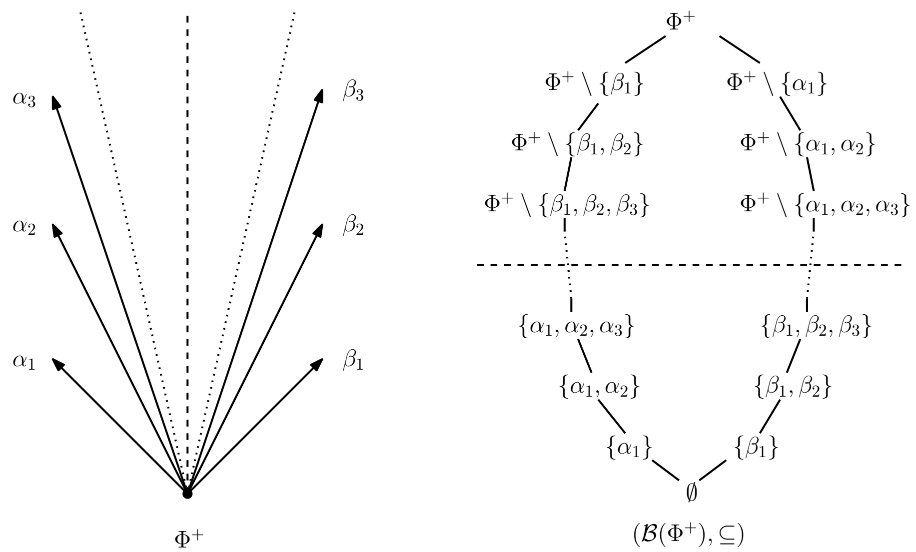


FIGURE 4.7.

As it is shown in the previous example, one can describe the lattice $(\mathcal{B}(\Phi^+), \subseteq)$ using scaffolding whenever W is a dihedral group (of course, we have to use a finite scaffolding whenever W is a finite dihedral group). In what follows, we propose a family of candidates for a valued digraph describing bi-closed sets (see Definition 4.3.11), using scaffoldings as fundamental compounds. Let us first introduce some definitions and properties about the geometry of Φ^+.

A maximal dihedral sub-system is a set $\Phi' = \Phi \cap P$ where P is a plan of V, such that $|\Phi'^+| \geq 2$ (where $\Phi'^+ = \Phi^+ \cap \Phi'$). Then, there exists $\Delta' = \{\alpha, \beta\} \subseteq \Phi'^+$ and a dihedral Coxeter group W' such that:

- each element of Φ'^+ is a linear combination of α and β with non-negative coefficients;
- (Φ', Δ') is a geometric system of W', where the bilinear form is given by B restricted to P.

For a proof of these facts, we refer the reader to [**H**], Section 8.2.

DEFINITION 4.3.9. We denote by $\mathcal{M}$ the set of all the maximal dihedral sub-systems of Φ. For each positive root γ, we define the set $\mathcal{M}_\gamma \subseteq \mathcal{M}$ to be

$$\mathcal{M}_\gamma = \{\Phi' \in \mathcal{M} \mid \gamma \in \Phi' \text{ and } \gamma \notin \Delta'(\Phi')\},$$

where $\Delta'(\Phi')$ denotes the simple system of Φ'.

We now explain what we will do in this section. The key point is the following immediate property:

$$\mathcal{B}(\Phi^+) = \{A \subseteq \Phi^+ \mid A \cap \Phi' \in \mathcal{B}(\Phi'^+) \text{ for all } \Phi' \in \mathcal{M}\}, \tag{4.4}$$

i.e. the property "to be bi-closed" is fundamentally 2-dimensional. However, in the case of dihedral groups, we already have a family of valued digraph describing the lattice of bi-closed sets (see Figures 4.2 and 4.7). Therefore, considering this fact together with (4.4), we have a natural way to construct our candidate of valued digraph $\mathcal{G}$ following a two step method:

(1) associate to each $\Phi' \in \mathcal{M}$ a scaffolding $\mathcal{G}_{\Phi'}$ such that $\mathcal{V}(\mathcal{G}_{\Phi'}) = \Phi'^{+}$ and $IS(\mathcal{G}_{\Phi'}) = \mathcal{B}(\Phi'^{+})$;
(2) "gluing" together these scaffoldings in order to obtain a valued digraph $\mathcal{G}$ such that $\mathcal{V}(\mathcal{G}) = \Phi^{+}$ (more formally, we identify the vertices corresponding to the same positive root, and the valuation is obtained by adding the valuations coming from each scaffolding).

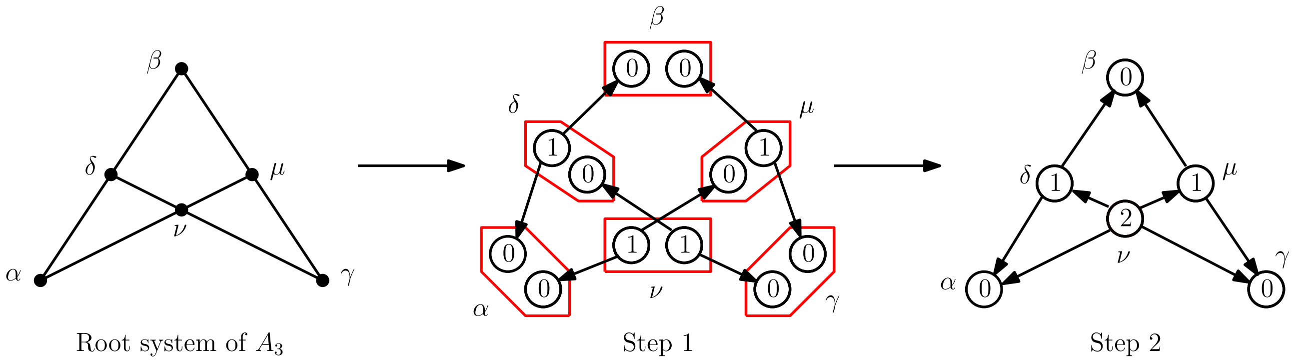


FIGURE 4.8. Representation of our method in the case of a root system of A_3. Note that the resulting valued digraph is the same one introduced in Chapter 3 Section 3.3.1.

We give the formal definition of this family of valued digraphs after a preliminary lemma.

LEMMA 4.3.10. *For all $\gamma \in \Phi^{+}$, the set $\mathcal{M}_{\gamma}$ is finite.*

PROOF. There exists ω in W such that $\gamma \in \mathrm{Inv}(\omega)$, and $\mathrm{Inv}(\omega)$ is finite and bi-closed. Moreover, if we consider $\Phi' \in \mathcal{M}_{\gamma}$ with associated simple system $\Delta' = \{\alpha, \beta\}$, then γ is a linear combination with positive coefficients of α and β. Thus, we have that $\alpha \in \mathrm{Inv}(\omega)$ or $\beta \in \mathrm{Inv}(\omega)$. Clearly, if we consider $\Phi'' \in \mathcal{M}_{\gamma} \setminus \{\Phi'\}$ with associated simple system Δ'', then $\Delta'' \cap \Phi' = \emptyset$. Furthermore, we also have $\Delta'' \cap \mathrm{Inv}(\omega) \neq \emptyset$. Thus, we have $|\mathcal{M}_{\gamma}| \leq |\mathrm{Inv}(\omega)|$, but $\mathrm{Inv}(\omega)$ is finite, so that we have the expected property. □

In the following definition, we define the claimed family of balanced valued digraphs, each one of them giving rise to a complete ortho-lattice containing $(\mathcal{B}(\Phi^{+}), \subseteq)$ as a sub-poset (Theorem 4.3.13). We will explain just after how this definition fits in our two step procedure.

DEFINITION 4.3.11. Let $\mathcal{G} = (G, \theta)$ be a valued digraph such that $\mathcal{V}(\mathcal{G}) = \Phi^{+}$, we say that $\mathcal{G}$ is *well-assembled* on Φ^{+} if and only if:

(1) for all $\Phi' \in \mathcal{M}$ with simple system Δ', there exists two disjoint injective sequences $(a_i)_{i\geq 1}$ and $(b_i)_{i\geq 1}$ taking values in $\Phi'^{+} := \Phi' \cap \Phi^{+}$ such that:
   (a) $\{a_1, b_1\} = \Delta'$;
   (b) $\Phi'^{+} = \mathrm{Im}(a) \sqcup \mathrm{Im}(b)$;
   (c) the initial sections of a and b are bi-closed in Φ' (but not necessarily in Φ);
   (d) the digraph G' obtained by restraining G to Φ'^{+} is a scaffolding made of a and b.
(2) for all $\gamma \in \Phi^{+}$, $\theta(\gamma) = |\mathcal{M}_{\gamma}|$;

Points 1-(a) to (d) correspond to the first step of our program: these technical points are here to ensure that the restriction of a well-assembled on Φ^{+} valued digraph to any $\Phi' \in \mathcal{M}$ is a scaffolding (Point 1-(d)), and such that the balanced valued digraph $\mathcal{G}_{\Phi'}$ defined by this scaffolding satisfies $IS(\mathcal{G}_{\Phi'}) = \mathcal{B}(\Phi'^{+})$ (Points 1-(a) to (c)). Point (2) corresponds to the second step: on Figure 4.8, we see that the value of the valuation obtained after we glued the scaffolding precisely equals $|\mathcal{M}_{\gamma}|$.

Before we move to the study of well-assembled valued digraphs, we have to be sure that such a valued digraph exists. This is the point of the following lemma.

Lemma 4.3.12. *For all Coxeter group W with associated root system $\Phi = \Phi^+ \sqcup \Phi^-$, there exists a well-assembled on Φ^+ valued digraph.*

Proof. Let $\Phi' \in \mathcal{M}$ and consider its projective representation, we fix an arbitrary orientation of this representation and we define two sequences $a_{\Phi'}$ and $b_{\Phi'}$ as depicted in Figure 4.9 ($a_{\Phi'}$ corresponds to the left side of the representation, and $b_{\Phi'}$ to the right side). It is clear

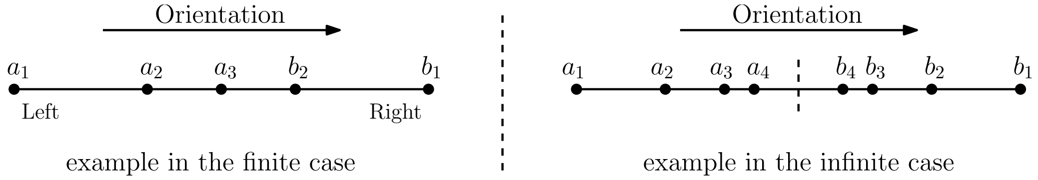


Figure 4.9. For the sake of clarity, we omit the index Φ' on this figure

(see Figure 4.9) that the sequences $a_{\Phi'}$ and $b_{\Phi'}$ satisfy points (1.a), (1.b) and (1.c) of Definition 4.3.11.

We now construct a well-assembled on Φ^+ valued digraph. For this purpose, we first construct the underlying digraph. For all $\Phi' \in \mathcal{M}$, fix $G_{\Phi'}$ an arbitrary scaffolding made of $a_{\Phi'}$ and $b_{\Phi'}$ and then let G be the digraph such that $\mathcal{V}(G) = \Phi^+$ and

$$E(G) = \bigsqcup_{\Phi' \in \mathcal{M}} E(G_{\Phi'}).$$

By construction, G satisfies points (1.a), (1.b), (1.c) and (1.d) of Definition 4.3.11.

We now show that the valuation $\theta : \mathcal{V}(G) \to \mathbb{N}$, defined by $\theta(\gamma) = |\mathcal{M}_\gamma|$, satisfy

$$\forall \gamma \in \Phi^+,\ 0 \leq \theta(\gamma) \leq d^+(G, \gamma). \tag{4.5}$$

In order to prove (4.5), we explicitly compute the value of $d^+(G,\gamma)$. Let $\Phi' \in \mathcal{M}$, with simple system Δ', and $\gamma \in \Phi'$. By construction, if $\gamma \in \Delta'$, then $d^+(G_{\Phi'}, \gamma) = 0$, and $d^+(G_{\Phi'}, \gamma) = 2$ otherwise. Furthermore, we clearly have

$$d^+(G, \gamma) = \sum_{\substack{\Phi' \in \mathcal{M} \\ \gamma \in \Phi'}} d^+(G_{\Phi'}, \gamma). \tag{4.6}$$

Thus, by definition of $\mathcal{M}_\gamma$, we have $d^+(G,\gamma) = 2|\mathcal{M}_\gamma|$. Consequently, θ satisfy (4.5) and (G, θ) is a valued digraph, which is well-assembled on Φ^+ by construction. This complete the proof. $\square$

Theorem 4.3.13. *Let W be a Coxeter group, $\Phi = \Phi^+ \cup \Phi^-$ be a root system of W, $\mathcal{B}(\Phi^+)$ be the set of the bi-closed sets of Φ^+ and $\mathcal{G} = (G, \theta)$ be a well-assembled on Φ^+ valued digraph. Then, $\mathcal{G}$ is balanced and $\mathcal{B}(\Phi^+) \subseteq IS(\mathcal{G})$.*

Proof. Since $\mathcal{G}$ clearly satisfy Equation (4.6), $\mathcal{G}$ is balanced.

We now prove that $\mathcal{B}(\Phi^+) \subseteq IS(\mathcal{G})$. Let $B \in \mathcal{B}(\Phi^+)$, $\gamma \in \Phi^+$ and $\Phi' \in \mathcal{M}_\gamma$, we denote by $(a_i)_i$ and $(b_i)_i$ the sequences associated with Φ' (see Definition 4.3.11 (1)) and by $\mathcal{A}$ and $\mathcal{B}$ their respective sets of initial sections. We divide our study into two cases.

- Case $\gamma \in B$: since $B \in \mathcal{B}(\Phi^+)$, the set $B' := B \cap \Phi'$ is bi-closed in $\Phi'^+ = \Phi' \cap \Phi^+$. By definition of a well-assembled valued digraph, $\mathcal{A}$ and $\mathcal{B}$ are both constituted of bi-closed sets of Φ' and $\{a_1, b_1\}$ is the simple system Φ'. Therefore, the projective representation of Φ' is, once again, as depicted in Figure 4.9. Thanks to this graphical representation and to the fact that B' is bi-closed in Φ'^+, it is clear that we have either $B' \in \mathcal{A} \cup \mathcal{B}$, or $\Phi'^+ \setminus B' \in \mathcal{A} \cup \mathcal{B}$. Thus, thanks to the fact that the restriction of G to Φ' is a scaffolding made of a and b and to Proposition 4.1.15, there exists $\mu \in \Phi'^+ \cap B'$ such that $(\gamma, \mu) \in E(G)$. Since this is true for any choice of Φ' in $\mathcal{M}_\gamma$, we have $d^+_B(G,\gamma) \geq |\mathcal{M}_\gamma| = \theta(z)$.

- Case $\gamma \notin B$: using similar arguments as in the previous case, one can show that $\theta(z) \geq d^+(G,\gamma)$.

The conclusion of this study is that $B \in IS(\mathcal{G})$. Thus, $\mathcal{B}(\Phi^+) \subseteq IS(\mathcal{G})$ and this ends the proof. $\square$

COROLLARY 4.3.14. *For any well-assembled on Φ^+ valued digraph $\mathcal{G}$, there exists an injective poset morphism from $(\mathcal{B}(\Phi^+), \subseteq)$ to the complete ortho-lattice $(IS(\mathcal{G}), \subseteq)$.*

REMARK 4.3.15. Notice that the valued digraphs introduced in Chapter 3 in order to describe the weak order on A_{n-1} and B_n can both be seen as well-assembled on a root system corresponding to A_{n-1} and B_n, respectively. However, note that it is not the case for the valued digraph describing the weak order on $\widetilde{A_n}$, since in this case some vertices have an odd out-degree.

4.3.3. Depth-increasing sequence and associated projective valued digraph. Our aim in this section is to construct a projective well-assembled on Φ^+ valued digraph. First, notice that any scaffolding is obviously projective, as depicted in Figure 4.10. Therefore, a

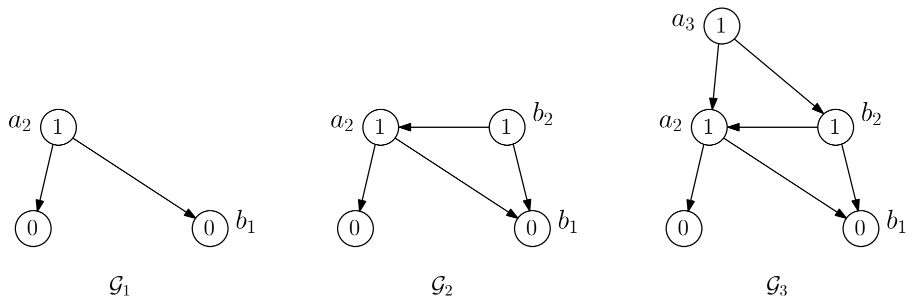


FIGURE 4.10. Scaffoldings are projective.

natural idea to construct a projective well assembled on Φ^+ valued digraph $\mathcal{G}$ would be to find a way to construct simultaneously the scaffoldings of $\mathcal{G}$, according to the pattern described on Figure 4.10. For that purpose, we consider a classical statistic on the positive root system (see [**BB**]). Let $\beta \in \Phi^+$, then the *depth* of β is

$$\mathrm{dp}(\beta) := \min\{\ell(\omega) \mid \omega \in W,\ \omega(\beta) \in \Phi^-\}.$$

We will prove in what follows that for any $\Phi' \in \mathcal{M}$ with simple system Δ', there exists two injective sequences (a_i) and (b_i) of elements of Φ'^+ such that:

- for all $i < j$, we have $\mathrm{dp}(a_i) < \mathrm{dp}(a_j)$ and $\mathrm{dp}(b_i) < \mathrm{dp}(b_j)$;
- $\Phi'^+ = \mathrm{Im}(a) \sqcup \mathrm{Im}(b)$;
- $\Delta' = \{a_1, b_1\}$;
- each initial section of $(a_i)_i$ or $(b_i)_i$ is a bi-closed set of Φ'^+.

Therefore, if one considers an injective sequence (c_i) such that $\Phi^+ = \mathrm{Im}(c)$ and for all $i, j \in \mathbb{N}$, if $\mathrm{dp}(c_i) < \mathrm{dp}(c_j)$ then $i < j$, then the restriction $(c_i')_i$ of $(c_i)_i$ to the elements of Φ' is a shuffle of a and b, and the corresponding scaffolding satisfies points (1)-(a) to (d) of Definition 4.3.11. Thus, one can construct a projective well-assembled on Φ^+ valued digraph using the sequence $(c_i)_i$.

Let us begin the formal construction with a well-known property of the depth (see for instance [[**BB**], Lemma 4.6.2]).

PROPOSITION 4.3.16. *Let $s \in S$ and $\beta \in \Phi^+ \setminus \{\alpha_s\}$, we have* $\mathrm{dp}(\alpha_s) = 1$ *and*

$$\mathrm{dp}(s(\beta)) = \begin{cases} \mathrm{dp}(\beta) - 1 & \text{if} \quad B(\alpha_s, \beta) > 0 \\ \mathrm{dp}(\beta) & \text{if} \quad B(\alpha_s, \beta) = 0 \\ \mathrm{dp}(\beta) + 1 & \text{if} \quad B(\alpha_s, \beta) < 0 \end{cases}$$

We also introduce a useful notation.

DEFINITION 4.3.17. Let γ be a positive root, and $\Phi' \in \mathcal{M}$ be such that $\gamma \in \Phi'$, we denote by $\mathrm{dp}_{\Phi'}(\gamma)$ the depth of γ, seen as an element of Φ'.

The following proposition will allow us to prove Proposition 4.3.19, which is the key property allowing us to construct a projective well-assembled on Φ^+ valued digraph.

PROPOSITION 4.3.18. *Let $\gamma \in \Phi^+$, we have the three following properties.*

(1) We have $\mathrm{dp}(\gamma) = 1 + \sum_{\Phi' \in \mathcal{M}_\gamma} (\mathrm{dp}_{\Phi'}(\gamma) - 1)$.

(2) For all $\Phi' \in \mathcal{M}_\gamma$ with associated simple system Δ', there exists $\alpha \in \Delta'$ such that: $B(\gamma, \alpha) > 0$, and for all $\mu \in \Phi'^+$, if $B(\mu, \alpha) > 0$ and if $\mathrm{dp}_{\Phi'}(\mu) < \mathrm{dp}_{\Phi'}(\gamma)$, then $\mathrm{dp}(\mu) < \mathrm{dp}(\gamma)$.

(3) For all $\Phi' \in \mathcal{M}_\gamma$ and $\alpha \in \Delta'$, we have $\mathrm{dp}(\alpha) < \mathrm{dp}(\gamma)$.

PROOF. **Point (1)**: our proof relies on the construction of a well-chosen subset $A \subseteq \Phi^+$, such that A is the inversion set of some $\omega \in W$, $\gamma \in A$ and $|A| = \mathrm{dp}(\gamma)$. For that purpose, let $\Phi' \in \mathcal{M}_\gamma$ with simple system Δ' and α be an element of Δ' such that $B(\gamma, \alpha) > 0$ (such a α always exists), we denote by $\mathcal{U}_\alpha(\gamma)$ the set of all the roots $\mu \in \Phi'^+$ such that $\mathrm{dp}_{\Phi'}(\mu) < \mathrm{dp}_{\Phi'}(\gamma)$ and $B(\mu, \alpha) > 0$. Graphically, $\mathcal{U}_\alpha(\gamma)$ is of the form depicted in Figure 4.11.

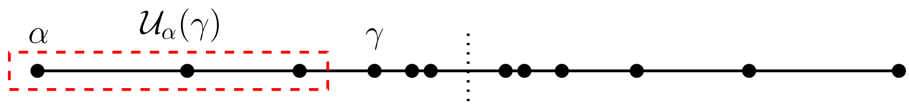


FIGURE 4.11. Illustration in the case of an infinite dihedral sub-system.

Note that both $\mathcal{U}_\alpha(\gamma) \cup \{\gamma\}$ and $\Phi'^+ \setminus \mathcal{U}_\alpha(\gamma)$ are bi-closed in Φ'^+. We also have

$$|\mathcal{U}_\alpha(\gamma)| = \mathrm{dp}_{\Phi'}(\gamma) - 1, \text{ and } |\Phi'^+ \setminus \mathcal{U}_\alpha(\gamma)| > |\mathcal{U}_\alpha(\gamma)|. \tag{4.7}$$

Choose an arbitrary labelling of the elements of $\mathcal{M}_\gamma$, that is

$$\mathcal{M}_\gamma = \{\Phi_1(\gamma), \Phi_2(\gamma), \dots, \Phi_{\mathcal{M}_\gamma}(\gamma)\},$$

and let α_i be an element of the simple system of $\Phi_i(\gamma)$ such that $B(\gamma, \alpha_i) > 0$. We now consider $\omega \in W$ such that $\gamma \in \mathrm{Inv}(\omega)$, since $\mathrm{Inv}(\omega)$ is bi-closed, we have that $\mathcal{U}_{\alpha_i}(\gamma) \cup \{\gamma\}$ or $\Phi'^+ \setminus \mathcal{U}_{\alpha_i}(\gamma)$ is included in $\mathrm{Inv}(\omega)$. Since the sets $\mathcal{U}_{\alpha_1}(\gamma), \mathcal{U}_{\alpha_2}(\gamma), \dots$ are pairwise disjoint, thanks to Equation (4.7) we have the following inequality:

$$|\mathrm{Inv}(\omega)| \geq |\{\gamma\} \cup \left(\bigcup_{1 \leq i \leq |\mathcal{M}_\gamma|} \mathcal{U}_{\alpha_i}(\gamma) \right)| = 1 + \sum_{1 \leq i \leq |\mathcal{M}_\gamma|} \mathrm{dp}_{\Phi_i(\gamma)}(\gamma) - 1.$$

We now prove by induction on the depth of γ, that there exists a finite sequence of positive roots $(\alpha_i)_{1 \leq i \leq |\mathcal{M}_\gamma|}$ such that:

- for all $1 \leq i \leq |\mathcal{M}_\gamma|$, α_i is in the simple system of $\Phi_i(\gamma)$ and $B(\gamma, \alpha_i) > 0$;
- $\{\gamma\} \cup \left(\bigcup_{1 \leq i \leq |\mathcal{M}_\gamma|} \mathcal{U}_{\alpha_i}(\gamma) \right)$ is the inversion set of a given ω in W.

If $\mathrm{dp}(\gamma) = 1$, then γ is a simple root, so that $\mathcal{M}_\gamma = \emptyset$ and the property is obviously true.

Let $n+1$ be the depth of γ, and assume that the property is true for all roots of depth at most n. Then, there exists μ of depth n and α a simple root of Φ^+ such that $s_\alpha(\mu) = \gamma$. By induction, there exists $(\alpha_i)_{1 \leq i \leq |\mathcal{M}_\mu|}$ such that $\{\mu\} \cup \left(\bigcup_{1 \leq i \leq |\mathcal{M}_\mu|} \mathcal{U}_{\alpha_i}(\mu) \right)$ is the inversion set of a permutation ω. We divide our study into two cases.

- If for all $1 \leq i \leq |\mathcal{M}_\mu|$ we have $\alpha \notin \Phi_i(\mu)$, then $\alpha \notin \mathrm{Inv}(\omega)$, hence $\mathrm{Inv}(s_\alpha \omega) = \{\alpha\} \cup s_\alpha(\mathrm{Inv}(\omega))$. We have the following facts:
  - $s_\alpha(\Phi_i(\mu)) \in \mathcal{M}_\gamma$ for all $1 \leq i \leq \mathcal{M}_\mu$;

  – $s_\alpha(\alpha_i)$ is in the simple system of $s_\alpha(\Phi_i(\mu))$;
  – $B(\gamma, s_\alpha(\alpha_i)) = B(s_\alpha(\mu), s_\alpha(\alpha_i)) = B(\mu, \alpha_i) > 0$;
  – $\mathrm{dp}_{s_\alpha(\Phi_i(\mu))}(s_\alpha(\beta)) = \mathrm{dp}_{\Phi_i(\mu)}(\beta)$ for all $\beta \in \Phi_i(\mu)$.

  Therefore, if we set $\beta_i = s_\alpha(\alpha_i)$ we have $s_\alpha(\mathcal{U}_{\alpha_i}(\mu)) = \mathcal{U}_{\beta_i}(\gamma)$, so that $\mathrm{Inv}(s_\alpha\omega) = \{\alpha, \gamma\} \cup \left(\bigcup_{1\leq i\leq |\mathcal{M}_\mu|} \mathcal{U}_{\beta_i}(\gamma)\right)$. Note that $\{\alpha, \mu\}$ is the simple system of an element of $\mathcal{M}_\gamma$. Moreover, $\mathrm{dp}(\gamma) > \mathrm{dp}(s_\alpha(\gamma))$ thus $B(\alpha, \gamma) > 0$ by Proposition 4.3.16, so that $\mathcal{U}_\alpha(\gamma) = \{\alpha\}$ and if we denote $\beta_{|\mathcal{M}(\mu)|+1} = \alpha$ we have $\mathrm{Inv}(s_\alpha\omega) = \{\gamma\} \cup \left(\bigcup_{1\leq i\leq |\mathcal{M}_\mu|+1} \mathcal{U}_{\beta_i}(\gamma)\right)$. In order to finish the study of this case, we just have to show that $|\mathcal{M}_\gamma| = |\mathcal{M}_\mu| + 1$, and this is immediate since the intersection of $\mathrm{Inv}(s_\alpha\omega)$ with any element of $\mathcal{M}_\gamma$ is non-empty.
- If there exists $1 \leq j \leq |\mathcal{M}_\mu|$ such that $\alpha \in \Phi_j$, then α is in the simple system of Φ_j (because it is a simple root). Moreover, by Proposition 4.3.16 we have $B(\alpha, \mu) < 0$, so that we have $\alpha \notin \mathcal{U}_{\alpha_j}(\mu)$. Hence, $\mathrm{Inv}(s_\alpha\omega) = \{\alpha\} \cup s_\alpha(\mathrm{Inv}(\omega))$. From that, we conclude with a similar method as in the previous case.

In all cases, the property is proved. As a consequence, point (1) is true.

Point (2): consider a positive root γ, $\Phi' \in \mathcal{M}_\gamma$, and α in the simple system of Φ' such that $\mathcal{U}_\alpha(\gamma)$ is in the inversion set of an element $\omega \in W$ such that $|\mathrm{Inv}(\omega)| = \mathrm{dp}(\gamma)$ (such a ω always exists, thanks to the proof of Point (1)). Let $\mu \in \mathcal{U}_\alpha(\gamma)$, by minimality of $|\mathrm{Inv}(\omega)|$ there exists $\omega' \in W$ such that $\mu \in \mathrm{Inv}(\omega') \subsetneq \mathrm{Inv}(\omega)$, then $\mathrm{dp}(\mu) \leq |\mathrm{Inv}(\omega')| < |\mathrm{Inv}(\omega)| = \mathrm{dp}(\gamma)$, and this ends the proof of the point (2).

Point (3): this can easily be shown by induction, with a similar method as in the proof of Point (1).

This ends the proof of this Proposition. □

PROPOSITION 4.3.19. *For any $\Phi' \in \mathcal{M}$ with simple system Δ', there exists two injective sequences (a_i) and (b_i) of elements of Φ'^+ such that:*

- *for all $i < j$, we have $\mathrm{dp}(a_i) < \mathrm{dp}(a_j)$ and $\mathrm{dp}(b_i) < \mathrm{dp}(b_j)$;*
- *$\Phi'^+ = \mathrm{Im}(a) \sqcup \mathrm{Im}(b)$;*
- *$\Delta' = \{a_1, b_1\}$;*
- *each initial section of $(a_i)_i$ or $(b_i)_i$ is a bi-closed set of Φ'^+.*

PROOF. We construct recursively the two sequences. First, note that we have the situation depicted in Figure 4.12. We arbitrary set a_1 and b_1 such that $\{a_1, b_1\} = \Delta'$. Then, we consider

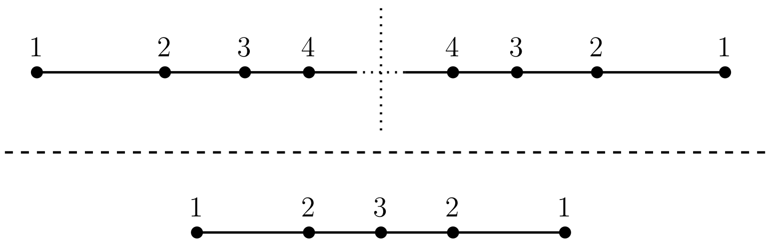


FIGURE 4.12. Values of $\mathrm{dp}_{\phi'}$ for an infinite (on top) and a finite (on bottom) maximal dihedral sub-systems.

$\gamma \in \Phi'^+$ such that $\mathrm{dp}_{\phi'}(\gamma) = 2$. Thanks to Proposition 4.3.18-(2), there exists $\alpha \in \Delta'$ such that $B(\alpha, \gamma) > 0$ and $\mathrm{dp}(\alpha) < \mathrm{dp}(\gamma)$. Therefore, if $\alpha = a_1$ then we set $\gamma = a_2$, and we set $\gamma = b_2$ otherwise. If there exists $\mu \in \Phi'^+$ such that $\mu \neq \gamma$ and $\mathrm{dp}_{\phi'}(\mu) = 2$, then we have $B(\alpha, \mu) \geq 0$ and we set $\mu = b_2$ if $\alpha = a_1$, $\mu = a_2$ otherwise. We repeat this procedure for the roots of depth 3, and so on. At the end, we have the configuration represented on Figure 4.13, and the sequences $(a_i)_i$ and $(b_i)_i$ satisfy the conditions of the property. This ends the proof. □

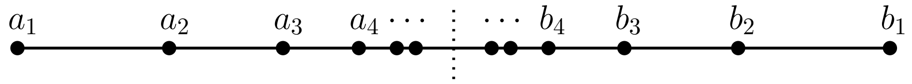


FIGURE 4.13.

Thanks to the previous proposition, we are now able to construct a projective well-assembled on Φ^+ valued digraph (Definition 4.3.22). This construction relies on the concept of *depth-increasing sequences* of Φ^+, so that we begin with defining those sequences.

DEFINITION 4.3.20. Let $c = (c_i)_{i\geq 1}$ be an injective sequence of elements of Φ^+, we say that c is a *depth-increasing sequence of positive roots* if and only if $\Phi^+ = \mathrm{Im}(c)$ and $i < j$ whenever $\mathrm{dp}(c_i) < \mathrm{dp}(c_j)$.

The fact that a depth-increasing sequence of positive roots exists is clear, since there are only a finite number of roots which have a given depth. Before moving to the construction of the valued digraph, we introduce a useful notation, which considerably simplifies its construction.

DEFINITION 4.3.21. Let $(c_i)_i$ be a depth-increasing sequence of root, $k \in \mathbb{N}^*$ and $\Phi' \in \mathcal{M}_{c_k}$, we denote by $\mathcal{F}_{\Phi'}(c_k) = \{c_i, c_j\}$ the set of the two unique roots in Φ'^+ such that $c_k = ac_i + bc_j$ with $a > 0$, $b > 0$, and with i, j both maximal and strictly smaller than k.

The fact that $\mathcal{F}_{\Phi'}(c_k)$ is non empty whenever $\Phi' \in \mathcal{M}_{c_k}$ is clear by Point (3) of Proposition 4.3.19. We now have everything required to define the claimed valued digraphs.

DEFINITION 4.3.22. Let $c = (c_i)_{i\geq 1}$ be a depth-increasing sequence of root, we define recursively the sequence of digraph $(G_i(c))_{i\geq 1}$ by:
- $\mathcal{V}(G_1(c)) = \{c_1\}$ and $E(G_1(c)) = \emptyset$;
- $\mathcal{V}(G_{i+1}(c)) = \mathcal{V}(G_i(c)) \cup \{c_{i+1}\}$;
- $E(G_{i+1}(c)) = E(G_i(c)) \cup \{(c_{i+1}, \alpha) \mid \alpha \in \cup_{\Phi' \in \mathcal{M}_{c_{i+1}}} \mathcal{F}_{\Phi'}(c_{i+1})\}$.

We denote by $G_W(c)$ the simple digraph such that

$$\mathcal{V}(G_W(c)) = \cup_{i\geq 1}\mathcal{V}(G_i(c)) \text{ and } E(G_W(c)) = \cup_{i\geq 1}E(G_i(c)).$$

By construction and thanks to Lemma 4.3.10, the out-degree of any root c_k is an even number equal to $2|\mathcal{M}_{c_k}|$, and we set $\mathcal{G}_W(c)$ the balanced valued digraph associated with $G_W(c)$.

PROPOSITION 4.3.23. *Let W be a Coxeter group and $\Phi = \Phi^+ \sqcup \Phi^-$ its associated root system, for each depth-increasing sequence c, we have that $\mathcal{G}_W(c)$ is projective and well-assembled on Φ^+.*

PROOF. First, notice that $\mathcal{G}_W(c)$ is obviously projective by construction.

We now prove that $\mathcal{G}_W(c)$ is well-assembled on Φ^+. Let ϕ' be a maximal dihedral sub-system of Φ and $(a_i)_i, (b_i)_i$ be two sequences coming from Proposition 4.3.19. Then, there exists $(i_k)_k$ and $(j_k)_k$ two sequences of increasing indices such that

$$\text{for all } k \geq 1,\ c_{i_k} = a_k \text{ and } c_{j_k} = b_k.$$

Let us denote by c' the sequence obtained restraining c to the elements of Φ'. By definition, c' is a shuffle of a and b, and thanks to Proposition 4.3.19 together with the fact the c is depth increasing, we have that the digraph obtained by restraining $\mathcal{G}_W(c)$ to Φ' is a scaffolding made of a and b, the initial sections of a and b are bi-closed in Φ', and the simple system of Φ' equals $\{\alpha_1, \beta_1\}$. Thus, $\mathcal{G}_W(c)$ is well-assembled on Φ^+, and this concludes the proof. $\square$

4.3.4. Consequences of this construction. In this section, we give some consequences of the existence of a projective well-assembled on Φ^+ valued digraph. We begin with proving Theorem 4.3.6.

Proof of Theorem 4.3.6. **Point (1)**: Thanks to Theorem 4.3.13 and Theorem 4.2.30, $(IS(\mathcal{G}), \subseteq)$ is a complete algebraic ortho-lattice such that $\mathcal{B}(\Phi^+) \subseteq IS(\mathcal{G})$. Moreover, any initial section of a reflection ordering is a bi-closed set, then it is in $IS(\mathcal{G})$. Consequently, thanks to Proposition 4.2.7, each reflection ordering is in $PS(\mathcal{G})$ and this ends the proof of Point (1).

Point (2): Assume that $IS(\mathcal{G}) = \mathcal{B}(\Phi^+)$ and let $I = (\Phi^+, \preceq) \in PS(\mathcal{G})$ and $\alpha, \beta, \gamma \in \Phi^+$ such that $\gamma = a\alpha + b\beta$, with $a > 0$ and $b > 0$. Without loss of generality, we can suppose that $\alpha \prec \beta$. Since $I_\beta \in \mathcal{B}(\Phi^+)$, we have $\gamma \in I_\beta$ so that $\gamma \prec \beta$. Thus, we have that $\beta \notin I_\gamma$, but $I_\gamma \in \mathcal{B}$ so that $\alpha \in I_\gamma$ and $\alpha \prec \gamma$. Finally, we have $\alpha \prec \gamma \prec \beta$, thus I is a reflection ordering. The converse is clear, and this concludes the proof of Point (2). □

In what follows, we explain how the projective structure of a projective well-assembled on Φ^+ valued digraph $\mathcal{G}$ can be used to test if $IS(\mathcal{G})$ has a chance to be equal to $\mathcal{B}(\Phi^+)$.

Definition 4.3.24. Let $A \subseteq \Phi^+$ and $B \subseteq A$, we say that B is A-closed if and only if

$$\forall \alpha, \beta, \gamma \in \mathcal{A} \text{ such that } \gamma = a\alpha + b\beta,\ a > 0,\ b > 0,\ \text{if } \alpha, \beta \in B,\ \text{then } \gamma \in B.$$

We say that B is A-bi-closed if and only if B and $A \setminus B$ are both A-closed. We denote by $\mathcal{B}(A)$ the set of the A-bi-closed sets.

Proposition 4.3.25. *Let $\mathcal{G}$ be a projective well-assembled on Φ^+ valued digraph, with associated sequence of valued digraphs $(\mathcal{G}_i)_{i\geq 1}$, we have that $IS(\mathcal{G}) = \mathcal{B}(\Phi^+)$ if and only if $IS(\mathcal{G}_i) \subseteq \mathcal{B}(\mathcal{V}(\mathcal{G}_i))$ for all $i \geq 1$.*

Proof. Let $A \in \mathcal{B}(\Phi^+)$, we have that $A \cap \mathcal{V}(\mathcal{G}_i)$ is in $\mathcal{B}(\mathcal{V}(\mathcal{G}_i))$, thus if $IS(\mathcal{G}) = \mathcal{B}(\Phi^+)$, then $IS(\mathcal{G}_i) \subseteq \mathcal{B}(\mathcal{V}(\mathcal{G}_i))$.

We now prove the converse. Let $\alpha, \beta, \gamma \in \Phi^+$ be such that $\gamma = a\alpha + b\beta$ with $a > 0$ and $b > 0$, $n \geq 1$ be such that $\alpha, \beta, \gamma \in \mathcal{V}(\mathcal{G}_n)$ and $A \in IS(\mathcal{G})$. If α and β are in A, then α and β are in $p_{\infty,n}(A)$ which is $\mathcal{V}(\mathcal{G}_n)$-bi-closed, so that $\gamma \in p_{\infty,n}(A)$ and, as a consequence, $\gamma \in A$. We conclude with a similar argument that if α and β are both in $\mathcal{V}(\mathcal{G})) \setminus A$, then $\gamma \in \mathcal{V}(\mathcal{G}) \setminus A$. Thus, A is bi-closed and this concludes the proof. □

Using Proposition 4.3.25, we are able to provide an example of a projective well-assembled on Φ^+ valued digraph $\mathcal{G}$ such that $IS(\mathcal{G}) \neq \mathcal{B}(\Phi^+)$ in the case of $W = \widetilde{C_3}$ (the author wants to thank Matthew Dyer, who found this simple example). We recall that $\widetilde{C_3}$ is the Coxeter group of rank 3 with generators a, b, and c such that $m_{a,b} = m_{b,c} = 4$ and $m_{a,c} = 2$. Consider a root system $\Phi = \Phi^+ \cup \Phi^-$ of $\widetilde{\mathcal{C}}_3$, with simple system $\{\alpha, \beta, \gamma\}$ corresponding respectively to a, b, and c, and let us define a depth increasing sequence d of Φ^+ whose ten first terms are ordered as follows:

$$d = [\alpha,\ \beta,\ \gamma,\ a(\beta),\ b(\alpha),\ b(\gamma),\ c(\beta),\ ab(\gamma),\ cb(\alpha),\ ac(\beta), \ldots].$$

On the projective representation of Φ^+, the configuration is as depicted in Figure 4.14.

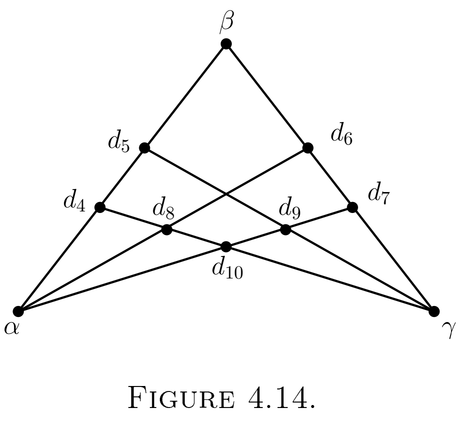


Figure 4.14.

In that case, we have that $\{\alpha, d_4, d_5, d_9\}$ is in $IS(\mathcal{G}_{10})$, but is not $\{d_1, \ldots, d_{10}\}$-bi-closed since d_{10} is a linear combination of α and d_9. Hence, thanks to Proposition 4.3.25, we have that

$IS(\mathcal{G}_W(d)) \neq \mathcal{B}(\Phi^+)$. However, note that this situation does not occur if we swap the positions of d_9 and d_{10}. Despite of this counter-example, we finish this section with a (quite long) remark to motivate Conjecture 4.3.7.

REMARK 4.3.26. At least in rank 3, there is a family of depth-increasing sequences which never led to a counter example, despite of multiple tests. Those are the sequences defined as follows. Consider an ordering $L_1 = [\alpha_1, \dots, \alpha_n]$ of the simple system of the positive root of a Coxeter group of rank n, and denote by s_i the reflection corresponding to α_i. Then we define the following sequence:

$$L' := [s_1(\alpha_1), s_1(\alpha_2), \dots, s_1(\alpha_n), s_2(\alpha_1), \dots, s_2(\alpha_n), s_3(\alpha_1), \dots, s_n(\alpha_n)],$$

and we define $L_2 = [\beta_1, \dots, \beta_k]$, the sequence obtained from L' by, first, keeping only the first occurrence of each root in L', and then deleting the roots which are not positive of depth 2. Similarly, we define:

$$L'' := [s_1(\beta_1), \dots, s_1(\beta_k), s_2(\beta_1), \dots, s_n(\beta_k)],$$

and we define L_3 from L'' by deleting the roots which are not of depth 3, and deleting the repetitions, and so on. It follows that we defined a family of sequences L_i, and by construction L_i is a total ordering of the positive roots which are of depth i. Thus, the sequence c obtained concatenating the sequences L_i is a depth increasing sequence.

The reason why these sequences seem to satisfy Conjecture 4.3.7 (at least in rank 3) remains quite mysterious. Indeed, this is an empiric observation based on the study of rank 3 Coxeter groups, and the author does not know why the situation seems to work that way. However, it seems that a proof or a refutation of our conjecture (using these sequences or not) would be based on a better understanding of the geometry of Φ^+, as it is the case in the Section 4.3.5.

4.3.5. Finite simply-laced case. In this section, we prove that Conjecture 4.3.7 holds when W is a finite simply-laced Coxeter group (Corollary 4.3.30). Our proof relies on the fact that in the finite simply-laced case, there is only one projective valued digraph well-assembled on Φ^+, and on some geometric properties of Φ^+ (see Definition 4.3.27 and Proposition 4.3.28).

We begin with recalling the definition of a simply laced Coxeter group. We say that a Coxeter system (W, S) of finite rank is *simply-laced* if and only if for all s and t in S, we have either $m_{st} = 2$, or $m_{st} = 3$. The finite simply-laced Coxeter groups are exactly the Coxeter groups A_n, D_n, E_6, E_7, and E_8.

In the remaining of this section, we assume that W is a finite simply-laced Coxeter group. By ([**K**], Lemma 3.2.3 (a)), we have that for all $\Phi' \in \mathcal{M}$, $|\Phi' \cap \Phi^+| = 2$ or 3. Since there is only one scaffolding with respectively 2 and 3 vertices (see Figure 4.15), there is only one valued digraph well-assembled on Φ^+, so that we simply denote by $\mathcal{G}_W$ this (projective) well-assembled valued digraph.

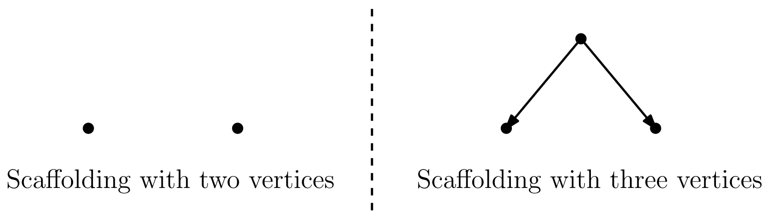


FIGURE 4.15.

Our aim is now to prove that $IS(\mathcal{G}_W) = \mathcal{B}(\Phi^+)$. In order to do so, the first key remark is that the construction and the properties of projective valued digraphs remain true when the sequence of valued digraph is stationary after a given rank. That is, consider a finite sequence $(\mathcal{G}_i)_{1 \le i \le N}$ of finite simple acyclic valued digraphs such that for all $1 \le i \le N-1$, $\mathcal{G}_{i+1}$ is obtained from $\mathcal{G}_i$ adding to it a vertex and some arcs with respect to the conditions of Definition 4.2.10, then the sets $IS(\mathcal{G}_i)$ form a projective system, and $\mathcal{G}_N$ has all the properties that $\mathcal{G}_\infty$ has in the infinite case. Hence, if we consider a depth-increasing sequence $c = (c_i)_{1\leq i\leq |\Phi^+|}$ of Φ^+ and $(\mathcal{G}_i)$ the associated sequence of valued digraph, then we have a projective system (with all the good properties) and $\mathcal{G}_{|\Phi^+|} = \mathcal{G}_W$. In particular, Proposition 4.3.25 holds and we can use it to prove that $IS(\mathcal{G}_W) = \mathcal{B}(\Phi^+)$ by induction.

In what follows, we denote by c a depth-increasing sequence of Φ^+, by $(\mathcal{G}_i)$ the finite sequence of valued digraphs associated with $\mathcal{G}_W$, and by $\mathcal{B}_k$ the set of $\{c_1,\dots,c_k\}$-bi-closed sets. Clearly, for all k smaller than the rank of W, we have $IS(\mathcal{G}_i) = \mathcal{B}_i$ (this is true for any Coxeter group W).

We show by induction on i that $IS(\mathcal{G}_i) = \mathcal{B}_i$. For that purpose, we need to study the geometry of Φ^+. This is the point of the following definition and properties.

DEFINITION 4.3.27. Let $\gamma \in \Phi^+$ such that $|\mathcal{M}_\gamma| \geq 2$, and Φ', Φ'' be two distinct elements of $\mathcal{M}_\gamma$ with respective simple system Δ' and Δ''. We say that Φ' and Φ'' are in a *kite-configuration* with respect to γ if and only if there exists $\mu \in \Phi^+$ and a labelling $\Delta' = \{\alpha_1, \beta_1\}$ and $\Delta'' = \{\alpha_2, \beta_2\}$ such that α_1 (resp. β_2) is a linear combination with positive coefficients of μ and α_2 (resp. μ and β_1). This kind of configuration has a simple projective representation, depicted in Figure 4.16.

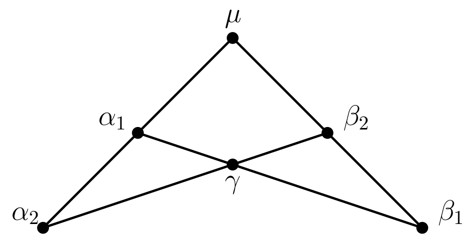


FIGURE 4.16. Projective representation of a kite-configuration

PROPOSITION 4.3.28. *Let $\gamma \in \Phi^+$ be such that $|\mathcal{M}_\gamma| \geq 2$ and Φ', Φ'' be two distinct elements of $\mathcal{M}_\gamma$, we have that Φ' and Φ'' are in a kite configuration with respect to γ.*

PROOF. We prove this proposition by induction on the depth n of γ. First, note that if $\mathrm{dp}(\gamma) = 1$ or 2, then $|\mathcal{M}_\gamma| = 0$ or 1 respectively, hence each positive root of depth 1 or 2 satisfy the property.

Let $n \geq 2$ be such that the property is true for all positive root of depth at most n, and assume that γ is of depth $n+1$. Notice that there exists a simple root μ and a positive root ρ of depth n such that $\gamma = s_\mu(\rho)$. Let $\Phi', \Phi'' \in \mathcal{M}_\gamma$, we have two cases.

- There exists $\Phi_1 \in \mathcal{M}_\rho$ such that $\Phi' = s_\mu(\Phi_1)$ and $\Phi'' \cap \Phi^+ = \{\rho, \gamma, \mu\}$. Let $\{\alpha_1, \beta_1\}$ be the simple system of Φ_1, and consider the maximal dihedral subsystems which contain respectively $\{\mu, \alpha_1\}$ and $\{\mu, \beta_1\}$. Since W is finite, we have that those two maximal dihedral subsystems cannot both contain 3 positive roots: otherwise W would have a reflection sub-group isomorphic to $\widetilde{A}_2$. Hence, we have the three possible cases depicted in Figure 4.17 (up to relabelling). In case 1, we have that $s_\mu(\rho) = \rho$, and this is impossible since $\mathrm{dp}(\rho) < \mathrm{dp}(\gamma) = \mathrm{dp}(s_\mu(\rho))$. In case 3, by Proposition 4.3.19 we have $\mathrm{dp}(\rho) > \mathrm{dp}(\gamma)$, so that this case cannot occur. At the end, the only remaining possibility is case 2, and this proves that Φ' and Φ'' are in a kite configuration with respect to γ.
- There exists Φ_1 and Φ_2 in $\mathcal{M}_\rho$ with respective simple system Δ_1 and Δ_2, such that $\Phi' = s_\mu(\Phi_1)$ and $\Phi'' = s_\mu(\Phi_2)$. By induction, Φ_1 and Φ_2 are in a kite configuration with respect to ρ, hence there exists $\nu \in \Phi^+$ and a labelling $\Delta_1 = \{\alpha_1, \beta_1\}$ and $\Delta_2 = \{\alpha_2, \beta_2\}$ such that we have the situation depicted in Figure 4.18. Since $\mathrm{dp}(s_\mu(\rho)) > \mathrm{dp}(\rho)$, we have that $\mu \notin \{\nu, \alpha_2, \beta_1\}$, but s_μ is a linear map such that $s_\mu(\Phi^+ \setminus \{\mu\}) = \Phi^+ \setminus \{\mu\}$, so that $s_\mu(\{\alpha_1, \alpha_2, \beta_1, \beta_2, \nu\}) \subseteq \Phi^+$, $s_\mu(\alpha_1)$ is a linear combination with

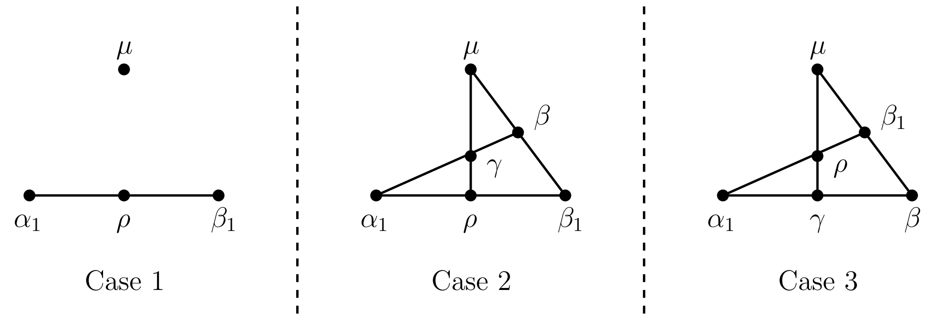


FIGURE 4.17.

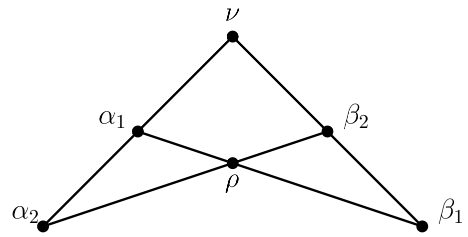


FIGURE 4.18.

positive coefficients of $s_\mu(\nu)$ and $s_\mu(\alpha_2)$, and $s_\mu(\beta_2)$ is a linear combination with positive coefficients of $s_\mu(\nu)$ and $s_\mu(\beta_1)$. That is, Φ' and Φ'' are in a kite configuration with respect to γ.

By induction, the proof is done. □

With this geometric property, we are now able to prove the main result of this section.

PROPOSITION 4.3.29. *For all $1 \leq i \leq |\Phi^+|$, we have $IS(\mathcal{G}_i) \subseteq \mathcal{B}_i$.*

PROOF. We prove that $IS(\mathcal{G}_i) \subseteq \mathcal{B}_i$ by induction on i. As stated earlier in this section, this property is obviously true when i is smaller than the rank of W.

Let i be such that $IS(\mathcal{G}_i) \subseteq \mathcal{B}_i$ and $A \in IS(\mathcal{G}_{i+1})$. First, note that, thanks to Proposition 4.2.14, if we denote by A' the set $A \cap \mathcal{V}(\mathcal{G}_i)$, then $A' \in IS(\mathcal{G}_i)$. Our aim is now to prove that both A and $\mathcal{V}(\mathcal{G}_{i+1}) \setminus A$ are $\mathcal{V}(\mathcal{G}_{i+1})$-closed. In order to do so, let $\alpha, \beta, \gamma \in \{c_1, \dots, c_{i+1}\}$ such that $\gamma = a\alpha + b\beta$ with $a > 0$ and $b > 0$. That is, $\{\alpha, \beta\}$ is the simple system of an element Φ' in $\mathcal{M}_\gamma$.

Step 1: We first prove that A is $\mathcal{V}(\mathcal{G}_{i+1})$-closed. Assume that $\alpha, \beta \in A$, then we have two cases.

- Case $|\mathcal{M}_\gamma| = 1$: we have that $\{y \in A | (\gamma, y) \in E(\mathcal{G}_{i+1})\} = \{\alpha, \beta\}$, thus
$$d_A^+(\mathcal{G}_{i+1}, \gamma) = |\{\alpha, \beta\}| = 2 > 1 = \theta(\gamma),$$
so that $\gamma \in A$.
- Case $|\mathcal{M}_\gamma| \geq 2$: let $\Phi'' \in \mathcal{M}_\gamma \setminus \{\Phi'\}$, thanks to Proposition 4.3.28 Φ' and Φ'' are in a kite configuration with respect to γ. Thus, there exists $\mu \in \Phi^+$ and a relabelling $\{\alpha, \beta\} = \{\alpha_1, \beta_1\}$ and a labelling $\{\alpha_2, \beta_2\}$ of the simple system of Φ'' such that α_1 (resp. β_2) is a linear combination with positive coefficients of μ and α_2 (resp. μ and β_1). By Proposition 4.3.19, this implies that the depth of γ is strictly bigger than the depth of α_1, α_2, β_1, β_2, and μ. Hence $\{\alpha_1, \alpha_2, \beta_1, \beta_2, \mu\} \subseteq \mathcal{V}(\mathcal{G}_i)$.

  By induction hypothesis, we have $A' \in B_i$. Moreover, both α_1 and β_1 are in A', but α_1 is a linear combination with positive coefficient of μ and α_2. Hence, we have that $\alpha_2 \in A'$ or $\mu \in A'$. If $\mu \in A'$, then $\beta_2 \in A'$ for the same reason. In fact, we just proved that for any $\Phi'' \in \mathcal{M}_\gamma$, the intersection of A with the simple system of Φ'' is non empty, however, by construction of $\mathcal{G}_{i+1}$, there is an arc from γ to each element of the simple system of Φ'', so that we have

$$d_A^+(\mathcal{G}_{i+1}, \gamma) > |\mathcal{M}_\gamma| = \theta_{i+1}(\gamma),$$

which implies that $\gamma \in A$.

Consequently, A is $\mathcal{V}(\mathcal{G}_{i+1})$-closed.

Step 2: since $\mathcal{G}_{i+1}$ is balanced, we have that $\mathcal{V}(\mathcal{G}_{i+1}) \setminus A$ is in $IS(\mathcal{G}_{i+1})$. Thus, by Step 1, we have that $\mathcal{V}(\mathcal{G}_{i+1}) \setminus A$ is $\mathcal{V}(\mathcal{G}_{i+1})$-closed.

Therefore, $A \in \mathcal{B}_{i+1}$, and this ends the proof. $\square$

This proposition has the following immediate corollary, which concludes this section.

COROLLARY 4.3.30. *In the finite simply-laced case,* $(W, \leq_R)$ *and* $(IS(\mathcal{G}_W), \subseteq)$ *are isomorphic.*

4.4. Link with convex geometry and closure operators

REMARK 4.4.1. The author want to thank Matthew Dyer for his useful suggestions, which led to the content of this section.

In this section, we briefly highlight some connections existing between our construction and the notion of *abstract convex geometry* and their associated *closure operators*. Before stating the results of this section, we recall the definition of these two notions (for more informations about abstract convex geometries, the interested reader may consult, for instance, [**EJ**]).

Let X be a set (finite or not), a closure operator on X is a map Γ from the power set of X to itself such that:

(1) for all $A \subseteq X$, $A \subseteq \Gamma(A)$;
(2) for all A and B subsets of X, if $A \subseteq B$, then $\Gamma(A) \subseteq \Gamma(B)$;
(3) for all $A \subseteq X$, $\Gamma(\Gamma(A)) = \Gamma(A)$.

We say that Γ is of *finite type* if and only if for all $A \subseteq X$,

$$\Gamma(A) = \bigcup_{\substack{B \subseteq A \\ B \text{ finite}}} \Gamma(B).$$

Let us denote by $\mathcal{C}$ the set $\{\Gamma(A) | A \subseteq X\}$, we say that the couple $(\mathcal{C}, \Gamma)$ (or simply $\mathcal{C}$, when there is no ambiguity) is an abstract convex geometry if and only if Γ satisfies the *anti-exchange property*, that is, for all $K \in \mathcal{C}$ and $p, q \notin K$ such that $p \neq q$, if $q \in \Gamma(K \cup \{p\})$, then $p \notin \Gamma(K \cup \{q\})$.

Let us now state the results of this section. In Section 4.4.1, we define, for any valued digraph $\mathcal{G}$, a subset $\mathcal{C}(\mathcal{G})$ of the power set of $\mathcal{V}(\mathcal{G})$ and a closure operator Γ on $\mathcal{V}(\mathcal{G})$ (Definition 4.4.2). We then show that $\mathcal{C}(\mathcal{G})$ is closed under arbitrary intersection and that $\mathcal{C}(\mathcal{G}) = \{\Gamma(A) | A \subseteq \mathcal{V}(\mathcal{G})\}$ (Proposition 4.4.3). Notice that the link between $(\mathcal{C}(\mathcal{G}), \Gamma)$ and the theory developed in previous sections is guaranteed by the facts that $IS(\mathcal{G}) \subseteq \mathcal{C}(\mathcal{G})$ and for all $X \subseteq IS(\mathcal{G})$, $\Gamma\left(\bigcup_{a \in X} A\right) = \vee X$. Finally, we prove that if $\mathcal{G}$ is acyclic, then Γ has the anti-exchange property, so that $(\mathcal{C}(\mathcal{G}), \Gamma)$ is a convex geometry (Proposition 4.4.4).

In section 4.4.2, we assume that $\mathcal{G}$ is projective, and we prove that the set of the intersections of elements of $IS(\mathcal{G})$ is also a convex geometry (Proposition 4.4.7).

Finally, in Section 4.4.3 we study closure operators of the form

$$\forall X \subseteq \mathcal{V}(\mathcal{G}),\ \Gamma_{\mathcal{S}}(X) := \bigcap_{\substack{A \in \mathcal{S} \\ X \subseteq A}} A,$$

for some well-chosen subset $\mathcal{S}$ of $IS(\mathcal{G})$. In particular, we show that if $\mathcal{G}$ is projective and $\mathcal{S}$ is *projective complete* (Definition 4.4.8), then $\Gamma_{\mathcal{S}}$ is of finite type (Theorem 4.4.12). A consequence of this result is that both $\Gamma_{IS(\mathcal{G})}$ and $\Gamma_{\mathcal{B}(\Phi^+)}$ are of finite type (Corollary 4.4.13).

4.4.1. The acyclic case. In this section, we explain how we can associate to each acyclic valued digraph $\mathcal{G} = (G, \theta)$ a convex geometry $(\mathcal{C}(\mathcal{G}), \Gamma)$ such that $IS(\mathcal{G}) \subseteq \mathcal{C}(\mathcal{G})$, and such that for all $X \subseteq IS(\mathcal{G})$, we have $\vee X = \Gamma\left(\bigcup_{A \in X} A\right)$. Note that the closure operator introduced in this section is of fundamental importance for our study of Cambrian lattices in Section 4.5.

Definition 4.4.2. We denote by $\mathcal{C}(\mathcal{G})$ the subset of the power set of $\mathcal{V}(\mathcal{G})$ defined by

$$\mathcal{C}(\mathcal{G}) := \{A \subseteq \mathcal{V}(\mathcal{G}) \mid \theta(z) \geq d_A^+(\mathcal{G}, z) \text{ for all } z \in \mathcal{V}(\mathcal{G}) \setminus A\},$$

and by Γ the operator defined by

$$\forall X \subseteq \mathcal{V}(\mathcal{G}),\ \Gamma(X) = \bigcup_{i \geq 0} J_i,$$

where (J_i) is a sequence of subset of $\mathcal{V}(\mathcal{G})$ recursively defined by:

- $J_0 = X$;
- for all $i \geq 0$, $J_{i+1} = J_i \cup \{z \in \mathcal{V}(\mathcal{G}) \setminus J_i \mid \theta(z) < d_{J_i}^+(\mathcal{G}, z)\}$.

We begin with some general properties of $\mathcal{C}(\mathcal{G})$ and Γ.

Proposition 4.4.3. *Let $\mathcal{G}$ be a valued digraph, the set $\mathcal{C}(\mathcal{G})$ and the operator Γ have the following properties:*

(1) $IS(\mathcal{G}) \subseteq \mathcal{C}(\mathcal{G})$ and for all $X \subseteq IS(\mathcal{G})$, $\vee X = \Gamma\left(\bigcup_{A \in X} A\right)$;
(2) the intersection of any family of elements of $\mathcal{C}(\mathcal{G})$ is also in $\mathcal{C}(\mathcal{G})$;
(3) $\mathcal{C}(\mathcal{G}) = \{\Gamma(A) | A \subseteq \mathcal{V}(\mathcal{G})\}$;
(4) Γ is a closure operator of finite type.

Proof.

Point (1): the fact that $IS(\mathcal{G}) \subseteq \mathcal{C}(\mathcal{G})$ is clear. The second part of this point can be proved following the exact same method as in the proof of Theorem 4.1.6.

Point (2): let $X \subseteq \mathcal{C}(\mathcal{G})$, Y be the intersection of all the elements of X and $z \in \mathcal{V}(\mathcal{G}) \setminus Y$. There exists $A \in X$ such that $z \notin A$, hence $\theta(z) \geq d_A^+(\mathcal{G}, z)$, but $Y \subseteq A$ so that $d_A^+(\mathcal{G}, z) \geq d_Y^+(\mathcal{G}, z)$. Consequently, $\theta(z) \geq d_Y^+(\mathcal{G}, z)$ so that $Y \in \mathcal{C}(\mathcal{G})$.

Point (3): following the exact same method as in the third point of the proof of Theorem 4.1.6, we have that $\Gamma(A) \in \mathcal{C}(\mathcal{G})$ for all A. We now prove the converse, let $B \in \mathcal{C}(\mathcal{G})$ and denote by (J_i) the non-decreasing sequence of sets associated with $\Gamma(B)$. We have that for all $z \in \mathcal{V}(\mathcal{G}) \setminus J_0$, $\theta(z) \geq d_{J_0}^+(\mathcal{G}, z)$ by definition of $\mathcal{C}(\mathcal{G})$, hence $J_1 = J_0 = B$, and by induction we have $J_i = B$ for all $i \geq 0$. It proves that $\Gamma(B) = B$, and this ends the proof of Point (3).

Point (4): we first show that Γ is a closure operator. Thanks to the proof Point (3), we have that $\Gamma(\Gamma(A)) = \Gamma(A)$ for all $A \subseteq \mathcal{V}(\mathcal{G})$. Moreover, it is obvious by definition of Γ that $A \subseteq \Gamma(A)$. It remains to show that for all $A \subseteq B \subseteq \mathcal{V}(\mathcal{G})$, $\Gamma(A) \subseteq \Gamma(B)$. In order to do so, let $A \subseteq B \subseteq \mathcal{V}(\mathcal{G})$ and denote by $(J_i)_i$ and $(J'_i)_i$ the non-decreasing sequences of sets respectively associated with $\Gamma(A)$ and $\Gamma(B)$. We prove by induction that $J_i \subseteq J'_i$ for all $i \geq 0$. First, note that $J_0 = A \subseteq B = J'_0$ by definition. Let $i \geq 0$ be such that $J_i \subseteq J'_i$ and set $z \in J_{i+1} \setminus J_i$. We have that $\theta(z) < d_{J_i}^+(\mathcal{G}, z) \leq d_{J'_i}^+(\mathcal{G}, z)$ by induction hypothesis, so that $z \in J'_{i+1}$, thus we have $J_{i+1} \subseteq J'_{i+1}$ and this ends the induction. Consequently, we have that $\Gamma(A) \subseteq \Gamma(B)$, so that Γ is a closure operator.

We now prove that Γ is of finite type. Let $A \subseteq \mathcal{V}(\mathcal{G})$, $(J_i)_i$ the non-decreasing sequence of sets associated with $\Gamma(A)$ and $z \in \Gamma(A)$. By construction, there exists $k \geq 0$ such that $z \in J_k$, and we prove by induction on k that there exists $X \subseteq A$ finite such that $z \in \Gamma(X)$. If $z \in J_0$, then the set $\{z\}$ obviously works. Let $k \geq 0$ be such that the property is true, and assume that $z \in J_{k+1} \setminus J_k$. Then, we have $\theta(z) < d_{J_k}^+(\mathcal{G}, z)$, thus there exists $z_1, \ldots, z_{\theta(z)+1}$ pairwise distinct in J_k such that $(z, z_j) \in E(\mathcal{G})$ for all $1 \leq j \leq \theta(z) + 1$. By induction hypothesis, for all

$1 \leq j \leq \theta(z)+1$ there exists a finite subset X_j of A such that $z_j \in \Gamma(X_j)$. Let us denote by X the union of all these X_j, which is finite and a subset of A. Since Γ is a closure operator, we have $z_j \in \Gamma(X)$ for all j, so that $\theta(z) < d^+_{\Gamma(X)}(\mathcal{G}, z)$, but $\Gamma(X) \in C(\mathcal{G})$, hence $z \in \Gamma(X)$, and this prove that Γ is of finite type. $\square$

The above properties lead us to a natural question: is the couple $(\mathcal{C}(\mathcal{G}), \Gamma)$ an abstract convex geometry ? In the general case, we do not know precisely when the closure operator Γ has the anti-exchange property (see the example on the right of Figure 4.3 for a valued digraph such that the associated closure operator does not have the anti-exchange property). However, the couple $(\mathcal{C}(\mathcal{G}), \Gamma)$ is an abstract convex geometry when $\mathcal{G}$ is acyclic, as stated in the following proposition.

PROPOSITION 4.4.4. *Let $\mathcal{G}$ be a valued digraph and Γ be its associated closure operator, if $\mathcal{G}$ is acyclic, then Γ has the anti-exchange property.*

PROOF. We divide the proof into two distinct steps. In the first step, we show a technical property of Γ, then, in the second step, we use this property to prove that Γ is of finite type.

Step 1: let $A \in \mathcal{C}(\mathcal{G})$, $p \in \mathcal{V}(\mathcal{G}) \setminus A$ and $z \in (\Gamma(A \cup \{p\})) \setminus (A \cup \{p\})$, we show that there exists a sequence $z_1, \ldots, z_k$ such that $(z, z_1), (z_1, z_2), \ldots, (z_{k-1}, z_k), (z_k, p)$ are all in $E(\mathcal{G})$. For that purpose, let $(J_i)_i$ be the sequence of sets associated with $\Gamma(A \cup \{p\})$. Note that there exists $j \geq 0$ such that $z \in J_{j+1} \setminus J_j$. We prove the property by induction on j.

If $j = 0$, then we have that $\theta(z) < d^+_{A\cup\{p\}}(\mathcal{G}, z)$, but $A \in \mathcal{C}(\mathcal{G})$, so that $\theta(z) \geq d^+_A(\mathcal{G}, z)$. consequently, there is an arc from z to p in $\mathcal{G}$. Let $j \geq 0$ be such that the property is true, and assume that $z \in J_{j+2} \setminus J_{j+1}$. We have that $\theta(z) < d^+_{J_{i+1}}(\mathcal{G}, z)$ and $\theta(z) \geq d^+_{J_i}(\mathcal{G}, z)$ (otherwise, z would have been in J_{i+1}), thus there exists z_1 in $J_{i+1} \setminus J_i$ such that $(z, z_1) \in E(\mathcal{G})$, and by induction hypothesis there exists $z_2, \ldots, z_k$ such that $(z_1, z_2), \ldots, (z_{k-1}, z_k), (z_k, p)$ are all in $E(\mathcal{G})$. This ends the induction.

step 2: we now prove that Γ has the anti-exchange property. Let $q, p \in \mathcal{V}(\mathcal{G}) \setminus A$ such that $q \neq p$ and $q \in \Gamma(A \cup \{p\})$. Thanks to Step 1, there exists a finite sequence of arcs starting at q and ending at p. Assume by contradiction that $p \in \Gamma(A \cup \{q\})$, then we have a finite sequence of arcs starting at p and ending at q, and this is a contradiction since $\mathcal{G}$ is acyclic. As a consequence, $p \notin \Gamma(A \cup \{q\})$ so that Γ has the anti-exchange property. $\square$

4.4.2. A "smaller" convex geometry in the projective case. Let us begin this section with a remark. Let W be a dihedral group with root system $\Phi = \Phi^+ \cup \Phi^-$ and $\mathcal{B}(\Phi^+)$ be the set of the bi-closed sets of Φ^+. It is clear that the set $\mathcal{A}$ of all the intersections of elements of $\mathcal{B}(\Phi^+)$ is the set of the closed sets of Φ^+, and that $\mathcal{A}$ is an abstract convex geometry. However, in general for a valued digraph $\mathcal{G}$ well-assembled on Φ^+, we have that $\mathcal{C}(\mathcal{G})$ is strictly bigger than $\mathcal{A}$. This leads us to the following natural question: is there (in the case where $\mathcal{G}$ is acyclic) a convex geometry smaller than $\mathcal{C}(\mathcal{G})$, which contains $IS(\mathcal{G})$, and such that the closure operator associated with this geometry gives the join when applied to the union of a family of elements of $IS(\mathcal{G})$?

In general, this question remains open. However, in the projective case we can show that the natural choice, *i.e.* the set of the intersections of elements of $IS(\mathcal{G})$, is a convex geometry. In what follows, $\mathcal{G}_\infty$ will denote a projective valued digraph, with associated sequence of valued digraph $(\mathcal{G}_i)_{i\geq 1}$, and $\overline{IS(\mathcal{G}_\infty)}$ will denote the set of the intersections of elements of $IS(\mathcal{G}_\infty)$. Let us begin with defining a family of closure operator $\mathcal{V}(\mathcal{G}_\infty)$

DEFINITION 4.4.5. Let $\mathcal{S}$ be a subset of $IS(\mathcal{G}_\infty)$, we define the closure operator $\Gamma_\mathcal{S}$ by

$$\forall X \subseteq \mathcal{V}(\mathcal{G}_\infty),\ \Gamma_\mathcal{S}(X) = \bigcap_{\substack{A \in \mathcal{S} \\ X \subseteq A}} A.$$

Note that, by definition, $\overline{IS(\mathcal{G}_\infty)}$ is closed under intersection and we have

$$\overline{IS(\mathcal{G}_\infty)} = \{\Gamma_{IS(\mathcal{G}_\infty)}(X) \mid X \subseteq \mathcal{V}(\mathcal{G}_\infty)\}.$$

We now prove that $(\overline{IS(\mathcal{G}_\infty)}, \Gamma_{IS(\mathcal{G}_\infty)})$ is an abstract convex geometry. For that purpose, we first give a characterisation of $\Gamma_{IS(\mathcal{G}_\infty)}$.

PROPOSITION 4.4.6. *Let $L \in PS(\mathcal{G}_\infty)$, $IS(L)$ be the set of the initial sections of L and denote by Γ_L the closure operator defined by*

$$\forall X \subseteq \mathcal{V}(\mathcal{G}_\infty),\ \Gamma_L(X) = \bigcap_{\substack{A \in IS(L)\\ X \subseteq A}} A.$$

We have that

$$\forall X \subseteq \mathcal{V}(\mathcal{G}_\infty),\ \Gamma_{IS(\mathcal{G}_\infty)}(X) = \bigcap_{L \in PS(\mathcal{G}_\infty)} \Gamma_L(X).$$

PROOF. We first show that for all $L = (\mathcal{V}(\mathcal{G}_\infty), \preceq) \in PS(\mathcal{G}_\infty)$ and $X \subseteq \mathcal{V}(\mathcal{G}_\infty)$, $\Gamma_L(X) \in IS(L)$. Notice that we clearly have

$$\Gamma_L(X) = \{z \in \mathcal{V}(\mathcal{G}_\infty) \mid \exists x \in X,\ z \preceq x\}.$$

Let $y, z \in \mathcal{V}(\mathcal{G}_\infty)$ be such that $y \preceq z$ and assume that $z \in \Gamma_L(X)$. Then, there exists $x \in X$ such that $z \preceq x$. Thus, $y \preceq x$ so $y \in \Gamma_L(X)$. Therefore, we have $\Gamma_L(X) \in IS(L)$, but $IS(L) \subseteq IS(\mathcal{G}_\infty)$, hence

$$\Gamma_{IS(\mathcal{G}_\infty)}(X) \subseteq \bigcap_{L \in PS(\mathcal{G}_\infty)} \Gamma_L(X).$$

We now prove that the reverse inclusion holds. Let $A \in IS(\mathcal{G}_\infty)$ such that $X \subseteq A$, there exists L' in $PS(\mathcal{G})$ such that A is an initial section of L'. Then, we have $\Gamma_{L'}(X) \subseteq A$. As a consequence, we have

$$\bigcap_{L \in PS(\mathcal{G}_\infty)} \Gamma_L(X) \subseteq \bigcap_{\substack{A \in IS(\mathcal{G}_\infty)\\ X \subseteq A}} A = \Gamma_{IS(\mathcal{G}_\infty)}(X).$$

This ends the proof. □

We are now able to prove that $(\overline{IS(\mathcal{G}_\infty)}, \Gamma_{IS(\mathcal{G}_\infty)})$ is an abstract convex geometry.

PROPOSITION 4.4.7. *Let $\mathcal{G}_\infty$ be a projective valued digraph, the closure operator $\Gamma_{IS(\mathcal{G}_\infty)}$ has the anti-exchange property.*

PROOF. Let $A \in \overline{IS(\mathcal{G}_\infty)}$ and $p, q \in \mathcal{V}(\mathcal{G}_\infty) \setminus A$ be such that $p \neq q$, we have that $p \notin A = \Gamma_{IS(\mathcal{G}_\infty)}(A)$, thus, thanks to Proposition 4.4.6, there exists $L' = (\mathcal{V}(\mathcal{G}_\infty), \preceq) \in PS(\mathcal{G}_\infty)$ such that $p \notin \Gamma_{L'}(A)$. So for all $z \in A$, $z \prec p$ (*i.e.* p is an upper bound of A in L'). Assume that $q \in \Gamma_{IS(\mathcal{G}_\infty)}(A \cup \{p\})$, then we have $q \in \Gamma_{L'}(A \cup \{p\})$, so that $q \prec p$. Thus, for all $z \in A \cup \{q\}$ we have $z \prec p$, hence $p \notin \Gamma_{L'}(A \cup \{q\})$, and this implies that $p \notin \Gamma_{IS(\mathcal{G}_\infty)}(A \cup \{q\})$. Consequently, $\Gamma_{IS(\mathcal{G}_\infty)}$ has the anti-exchange property. □

4.4.3. Some closure operators of finite type in the projective case. It follows from Proposition 4.4.7 that $(\overline{IS(\mathcal{G})}, \Gamma_{IS(\mathcal{G})})$ is an abstract convex geometry. One may ask if $\Gamma_{IS(\mathcal{G})}$ is of finite type. As it will turn out, it is indeed the case, and we prove it in this section (Corollary 4.4.13). Furthermore, our proof can be extended to show that $\Gamma_{\mathcal{B}(\Phi^+)}$ is also of finite type.

As usual, we denote by $\mathcal{G}_\infty$ a projective valued digraph with associated sequence of valued digraphs $(\mathcal{G}_i)_{i \geq 1}$.

DEFINITION 4.4.8. Let $\mathcal{S}_\infty \subseteq IS(\mathcal{G}_\infty)$ and $\mathcal{S}_i \subseteq IS(\mathcal{G}_i)$ for all $i \geq 1$, we say that the family $(\mathcal{S}_i)_{1 \leq i \leq \infty}$ is *projective complete* if and only if the following properties are true:

(1) for all $1 \leq i < j \leq \infty$, $p_{j,i}(\mathcal{S}_j) \subseteq \mathcal{S}_i$;
(2) conversely, for all sequences $A_i \in \mathcal{S}_i$, if $p_{i+1,i}(A_{i+1}) = A_i$ for all $i \geq 1$, then $\cup_i A_i \in \mathcal{S}_\infty$.

There are two main examples of projective complete families. The first one is simply the family $(IS(\mathcal{G}_i))_{1\leq i\leq\infty}$. The second one comes from the bi-closed sets of a Coxeter group. That is, let $\mathcal{G}$ be a projective well-assembled on Φ^+ valued digraph, $(\mathcal{G}_i)_i$ be its associated sequence and $\mathcal{B}(\mathcal{V}(\mathcal{G}_n))$ be the set of $\mathcal{V}(\mathcal{G}_n)$-bi-closed sets of Φ^+. Then, we have $\mathcal{B}_n(\mathcal{V}(\mathcal{G}_n)) \subseteq IS(\mathcal{G}_n)$, and following the same method as in the proof of Proposition 4.3.25, one can prove that $(\mathcal{B}(\mathcal{V}(\mathcal{G}_i)))_{1\leq i\leq\infty}$ is projective complete.

In what follows, $(\mathcal{S}_i)_{1\leq i\leq\infty}$ will denote a projective complete family such that $\mathcal{V}(\mathcal{G}_i) \in \mathcal{S}_i$ for all $1 \leq i \leq \infty$. In general, $\Gamma_{\mathcal{S}_\infty}$ does not have the anti-exchange property, but we will show that it is of finite type. The proof is a bit tricky, and we need to introduce several notations. In what follows, X will denote a subset of $\mathcal{V}(\mathcal{G}_\infty)$. We define the following sets:

$$\forall i,j \text{ such that } 1 \leq i < j \leq \infty, \quad \begin{cases} X_i := p_{\infty,i}(X), \\ \mathcal{S}_i(X) := \{A \in IS(\mathcal{G}_i) \mid X_i \subseteq A\}, \\ \mathcal{S}_{j,i}(X) := p_{j,i}(\mathcal{S}_j(X)). \end{cases}$$

In order to prove that $\Gamma_{\mathcal{S}_\infty}$ is of finite type, we first prove three technical lemmas.

LEMMA 4.4.9. *For all $i \geq 1$, we have*

$$\mathcal{S}_{i+1,i}(X) \supseteq \mathcal{S}_{i+2,i}(X) \supseteq \mathcal{S}_{i+3,i}(X) \supseteq \cdots .$$

PROOF. Let $j > i$ and $A \in \mathcal{S}_j(X)$. Then, $X_j \subseteq A$ so $X_i = p_{j,i}(X_j) \subseteq p_{j,i}(A) \in \mathcal{S}_i$. Thus, we have $\mathcal{S}_{j,i}(X) \subseteq \mathcal{S}_i(X)$. Let $k > j$, we have

$$\mathcal{S}_{k,i}(X) = p_{k,i}(\mathcal{S}_k(X)) = p_{j,i}(p_{k,j}(\mathcal{S}_k(X))) = p_{j,i}(\mathcal{S}_{k,j}(X)) \subseteq p_{j,i}(\mathcal{S}_j(X)) = \mathcal{S}_{j,i}(X). \qquad \square$$

Thanks to Lemma 4.4.9 and to the fact that $\mathcal{V}(\mathcal{G}_i) \in \mathcal{S}_{j,i}(X)$ for all $j > i$, we have that $(\mathcal{S}_{j,i}(X))_{j>i}$ is a non-increasing sequence of finite non-empty sets. Thus, there exists $n > i$ such that for all $m > n$, $\mathcal{S}_{m,i}(X) = \mathcal{S}_{n,i}(X)$. We denote by $\mathcal{S}_{\infty,i}(X)$ the set $\mathcal{S}_{n,i}(X)$ (which is non-empty).

LEMMA 4.4.10. *For all $j > i$, $p_{j,i}(\mathcal{S}_{\infty,j}(X)) = \mathcal{S}_{\infty,i}(X)$.*

PROOF. There exists $n > j$ such that $p_{n,j}(\mathcal{S}_n(X)) = \mathcal{S}_{\infty,j}(X)$ and $p_{n,i}(\mathcal{S}_n(X)) = \mathcal{S}_{\infty,i}(X)$. Thus, we have

$$p_{j,i}(\mathcal{S}_{\infty,j}(X)) = p_{j,i}(p_{n,j}(\mathcal{S}_n(X))) = p_{n,i}(\mathcal{S}_n(X)) = \mathcal{S}_{\infty,i}(X). \qquad \square$$

LEMMA 4.4.11. *Let $(A_i)_{i\geq 1}$ be such that $A_i \in \mathcal{S}_{\infty,i}(X)$ and $A = \bigcup_{i\geq 1} A_i$. If for all $i \geq 1$, $p_{i+1,i}(A_{i+1}) = A_i$, then $A \in \mathcal{S}_\infty(X)$.*

PROOF. Since $(\mathcal{S}_i)_{1\leq i\leq\infty}$ is projective complete and $\mathcal{S}_{\infty,i}(X) \subseteq \mathcal{S}_i$, we have $A \in \mathcal{S}_\infty$. Consider $x \in X$, there exists $i \geq 1$ such that $x \in \mathcal{V}(\mathcal{G}_i)$. Moreover, $x \in X_i$ and $A_i \in \mathcal{S}_i(X)$, so that $x \in A_i$. Therefore, $x \in A$, thus $X \subseteq A$. That is, $A \in \mathcal{S}_\infty(X)$. $\square$

Thanks to these lemmas, we are now able to prove the main result of this section.

THEOREM 4.4.12. *Let $(\mathcal{S}_i)_{1\leq i\leq\infty}$ be a projective complete family such that $\mathcal{V}(\mathcal{G}_i) \in \mathcal{S}_i$ for all $i \geq 1$. Then, $\Gamma_{\mathcal{S}_\infty}$ is of finite type.*

PROOF. Let $X \subseteq \mathcal{V}(\mathcal{G}_\infty)$ and $z \in \Gamma_{\mathcal{S}_\infty}(X)$. Assume by contradiction that for all $n \geq 1$, there exists $m > n$ such that there exists $B \in \mathcal{S}_m(X)$ such that $z \notin B$. In particular, consider $i \geq 1$ such that $z \in \mathcal{V}(\mathcal{G}_i)$ and let $n > i$ be such that $\mathcal{S}_{\infty,i}(X) = \mathcal{S}_{n,i}(X)$. Then, there exists $m > n$ and $B \in \mathcal{S}_m(X)$ such that $z \notin B$, but $p_{m,i}(\mathcal{S}_m(X)) = \mathcal{S}_{\infty,i}(X)$, so that $p_{m,i}(B)$ is in $\mathcal{S}_{\infty,i}(X)$. Notice that $z \notin p_{m,i}(B)$ by construction. Thanks to Lemma 4.4.10, there exists a sequence $(A_j)_{j\geq 1}$ such that $A_j \in \mathcal{S}_{\infty,j}(X)$ for all $j \geq 1$, $A_i = p_{m,i}(B)$ and $p_{j+1,j}(A_{j+1}) = A_j$ for all $j \geq 1$. Thanks to Lemma 4.4.11, $A := \cup_j A_j$ is in $\mathcal{S}_\infty(X)$. Moreover, we have that $z \notin A$ by construction, so that $z \notin \Gamma_{\mathcal{S}_\infty}(X)$ and this is absurd.

Therefore, there exists $n \geq 1$ such that for all $m > n$ and for all $B \in \mathcal{S}_m(X)$, $z \in B$. Let $A \in \mathcal{S}_\infty$ be such that $X_{n+1} \subseteq A$, where X_{n+1} is seen as a subset of $\mathcal{V}(\mathcal{G}_\infty)$. Then,

$X_{n+1} \subseteq p_{\infty,n+1}(A) \in \mathcal{S}_{n+1}$, hence $p_{\infty,n+1}(A) \in \mathcal{S}_{n+1}(X)$, so that $z \in p_{\infty,n+1}(A)$ and this finally implies that $z \in A$. As a consequence, we have $z \in \Gamma_{\mathcal{S}_\infty}(X_{n+1})$, but X_{n+1} is a finite subset of X, thus $\Gamma_{\mathcal{S}_\infty}$ is of finite type. □

This theorem has the following corollary.

COROLLARY 4.4.13.

(1) Let $\mathcal{G}$ be a projective valued digraph, the operator $\Gamma_{IS(\mathcal{G})}$ is of finite type.
(2) Let W be a Coxeter group of finite rank, $\Phi = \Phi^+ \cup \Phi^-$ be a root system of W and $\mathcal{B}(\Phi^+)$ be the set of bi-closed sets of Φ^+, the operator $\Gamma_{\mathcal{B}(\Phi^+)}$ is of finite type.

We conclude this section with a remark about these various closure operators.

REMARK 4.4.14. Note that in general, we do not know if the operator $\Gamma_{\mathcal{B}(\Phi^+)}$ has the anti-exchange property. Still in relation with Dyer's conjecture, one can define another closure operator using reflection orderings. That is, consider $\mathcal{A}$ the set of the initial sections of all the reflection orderings of Φ^+, then the closure operator

$$\forall X \subseteq \Phi^+,\ \Gamma_{\mathcal{A}}(X) := \bigcap_{\substack{A \in \mathcal{A} \\ X \subseteq A}} A$$

has the anti-exchange property. However, we do not know if $\mathcal{A}$ is projective complete, hence we do not know if it is of finite type. Finally, note that if there exists a projective valued digraph $\mathcal{G}$ well-assembled on Φ^+ such that $IS(\mathcal{G}) = \mathcal{B}(\Phi^+)$, then the associated closure operator $\Gamma_{IS(\mathcal{G})}$ is equal to the closure operators $\Gamma_{\mathcal{B}(\Phi^+)}$ and $\Gamma_{\mathcal{A}}$. Hence, the study of those two closure operators $\Gamma_{\mathcal{B}(\Phi^+)}$ and $\Gamma_{\mathcal{A}}$ (and proving or disproving that they are of finite type with the anti-exchange property) should be a good starting point to the study of Conjecture 4.3.7.

4.5. Toward Cambrian lattices?

REMARK 4.5.1. The author wants to thank Christophe Hohlweg for many useful discussions, which led to the content of this section.

In this section we begin with applying our theory to the problem of extending Cambrian semi-lattices. Note that this is just an approach of this problem, and if we indeed provide an extension of Cambrian semi-lattices into complete lattice, we are not even able to prove that our extension coincide with Cambrian lattices in type A. However, this problem motivates the introduction of a new development of our theory, which seems to be interesting in its own (see Section 4.5.3).

4.5.1. Definition of Cambrian semi-lattices. We begin with recalling the definition of Cambrian semi-lattices, following **[RS3]**. In this section W denotes a Coxeter group of finite rank n with generating set S, and $\mathbf{c}$ denotes a Coxeter element of W, that is an element which has a reduced word of the form $s_1 s_2 \cdots s_n$, where $S = \{s_1, \dots, s_n\}$.

DEFINITION 4.5.2. For all subset J of $[n]$, denote by R_J the word $s_{j_1} s_{j_2} \cdots s_{j_k}$, where $J = \{j_1, \dots, j_k\}$ and $j_1 < j_2 < \cdots < j_k$. An element w in W is called $\mathbf{c}$-sortable if and only if there exists a finite sequence of set $[n] \supseteq J_1 \supseteq J_2 \supseteq \cdots \supseteq J_r$ such that the concatenation $R_{J_1} R_{J_2} \cdots R_{J_r}$ is a reduced word for w (note that $\mathbf{c}$-sortable elements do not depend on the choice of a particular reduced decomposition of $\mathbf{c}$). The *Cambrian semi-lattice* associated with $\mathbf{c}$ is the set of all the $\mathbf{c}$-sortable elements of W ordered by the weak order on W.

The fact that Cambrian semi-lattices are effectively semi-lattices is not trivial, and a complete proof can be found in **[RS3]** (or in **[R1]** in the case where W is finite). It appears that $\mathbf{c}$-sortable elements of W can be recognized by the geometry of their inversion sets. Before, we need to introduce the general notion of *orientation* of a root system.

DEFINITION 4.5.3. Consider $\Phi = \Phi^+ \cup \Phi^-$ a root system of W, and denote by $\mathcal{M}_{>2}$ the set of the maximal dihedral sub-systems of Φ^+ containing strictly more than two positive roots. An orientation $\overrightarrow{\Phi}$ of Φ consists in a subset X of $\mathcal{M}_{>2}$ (the elements of X are called the *oriented elements* of $\mathcal{M}_{>2}$) together with a function $\psi : X \to \Phi^+ \times \Phi^+$ associating to each $\Phi' \in X$ a couple (α, β), such that $\{\alpha, \beta\}$ is the simple system of Φ'.

Now consider $A \subseteq \Phi^+$ and $\overrightarrow{\Phi}$ an orientation of Φ, we say that A is $\overrightarrow{\Phi}$-aligned if and only if the two following conditions are satisfied:

(1) A is bi-closed;
(2) for all $\Phi' \in X$ with $\psi(\Phi') = (\alpha, \beta)$, if $A \cap \Phi'$ contains both β and $s_\beta(\alpha)$, then $\Phi' \subseteq A$.

An orientation of a Coxeter group can easily be represented on the projective representation of its root system, as depicted in Figure 4.19 on an example of rank two Coxeter group. In this case, the aligned elements are exactly the subsets

$$\emptyset,\ \{\alpha\},\ \{\alpha, \gamma_1\},\ \{\alpha, \gamma_1, \gamma_2\},\ \{\alpha, \gamma_1, \gamma_2, \gamma_3\},\ \{\alpha, \gamma_1, \gamma_2, \gamma_3, \beta\},\ \{\beta\}.$$

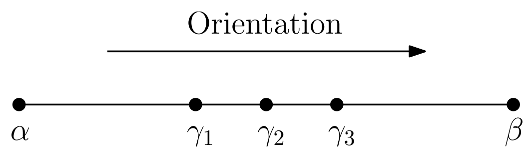


FIGURE 4.19.

Note that these bi-closed sets are exactly the inversion sets of the $s_\alpha s_\beta$-sortable elements. Hence, once we ordered these sets by inclusion, we obtain the Cambrian lattice associated with $s_\alpha s_\beta$ (see Figure 4.20).

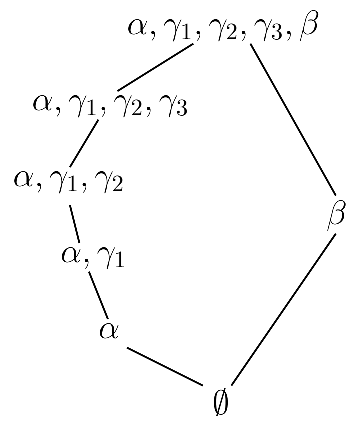


FIGURE 4.20.

This fact is not a coincidence, and the following general result was showed by Reading and Speyer in [**RS3**] (and first by Reading in [**R1**] in the case W is finite).

THEOREM 4.5.4. *For each Coxeter element* $\mathbf{c}$*, there exists an orientation* $\overrightarrow{\Phi}$ *such that the inversion sets of* $\mathbf{c}$*-sortable elements of* W *are* $\overrightarrow{\Phi}$*-aligned. Such an orientation is called a* $\mathbf{c}$*-*orientation *of* Φ*. If* W *is finite, we have that the inversion sets of* $\mathbf{c}$*-sortable elements of* W *are exactly the* $\overrightarrow{\Phi}$*-aligned elements.*

In what follows, we will work on general orientations of roots system, and not only on $\mathbf{c}$-orientations. Thus, the fact that such a $\mathbf{c}$-orientation exists will be sufficient for the rest of this section, and we refer the interested reader to [**RS3**] or [**R1**] for a precise construction of a $\mathbf{c}$-orientation.

4.5.2. An interesting phenomenon. In this section, we will follow the same method as in Section 4.3.2, by first giving a complete description of Cambrian semi-lattices and lattices in dihedral Coxeter group, and then applying this construction to provide a candidate of valued digraph in the general case.

Consider a dihedral Coxeter group W with generating set $S = \{s_1, s_2\}$, and let $\Phi = \Phi^+ \cup \Phi^-$ be a root system of W. Denote $\{\alpha_1, \alpha_2\}$ the simple system of Φ^+, where α_i is the simple root corresponding to s_i. Finally, consider the orientation of $\overrightarrow{\Phi}$ given by $\psi(\Phi) = (\alpha_1, \alpha_2)$. As

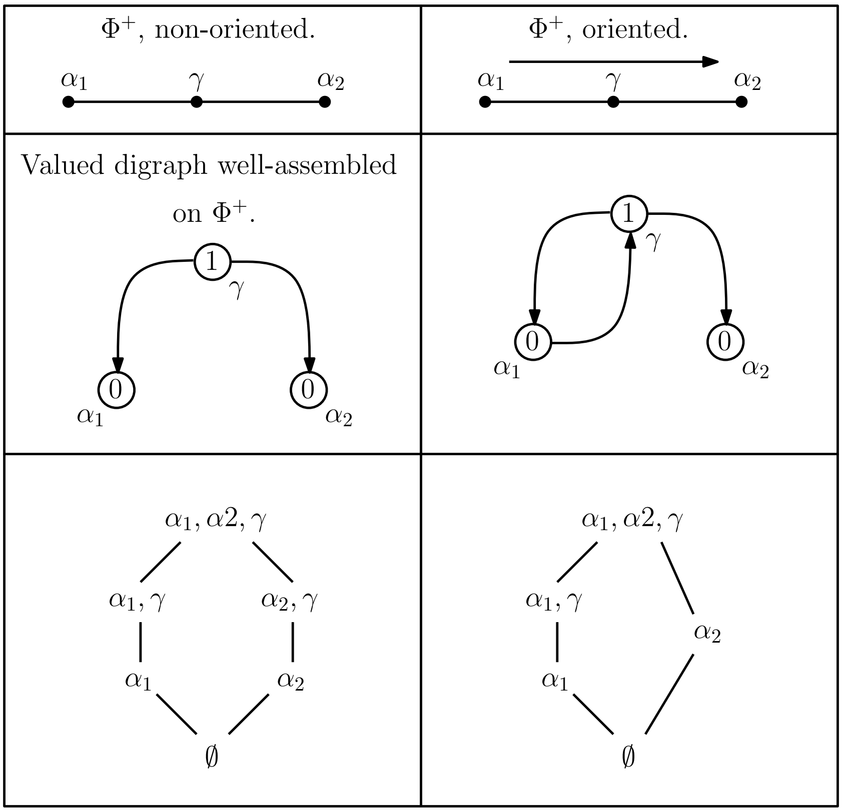


FIGURE 4.21. An interesting valued digraph.

depicted in Figure 4.21, in the case where $(s_1 s_2)^3 = Id$ we have a valued digraph $\overrightarrow{\mathcal{G}}$ such that $IS(\overrightarrow{\mathcal{G}})$ is constituted exactly of the $\overrightarrow{\Phi}$-aligned elements of Φ^+. In fact, this situation is not limited to this specific case, and we can find such a valued digraph for each dihedral group.

PROPOSITION 4.5.5. *Let $\mathcal{G}$ be a valued digraph well assembled on Φ^+, and denote by $\overrightarrow{\mathcal{G}}$ the valued digraph obtained from $\mathcal{G}$ by adding an arc from α_1 to $s_2(\alpha_1)$. Then the elements of $IS(\overrightarrow{\mathcal{G}})$ are exactly the $\overrightarrow{\Phi}$-aligned elements of Φ^+.*

PROOF. Let $A \in IS(\overrightarrow{\mathcal{G}})$ and $z \in \mathcal{V}(\overrightarrow{\mathcal{G}}) \backslash \{\alpha_1\}$, by construction we have $d_A^+(\mathcal{G}, z) = d_A^+(\overrightarrow{\mathcal{G}}, z)$. Therefore, if $z \in A$, then $\theta(z) \leq d_A^+(\mathcal{G}, z)$, and $\theta(z) \geq d_A^+(\mathcal{G}, z)$ otherwise. Note that $\theta(\alpha_1) = 0$ and there is no arc in $\mathcal{G}$ having α_1 as starting point, hence in all cases we have $\theta(\alpha_1) = d_A^+(\mathcal{G}, \alpha_1) = 0$, so that $A \in IS(\mathcal{G})$, which implies that A is bi-closed since W is dihedral. With similar arguments we show that $\{\alpha_2\} \in IS(\overrightarrow{\mathcal{G}})$ and each bi-closed set which contains α_1 is in $IS(\overrightarrow{\mathcal{G}})$,

It remains only to show that these bi-closed sets are the only one in $IS(\overrightarrow{\mathcal{G}})$. Note that if A contains both α_2 and $s_2(\alpha_1)$, then $d_A^+(\overrightarrow{\mathcal{G}}, \alpha_1) = 1 > \theta(\alpha_1)$, so that $\alpha_1 \in A$, but A is bi-closed, thus $A = \mathcal{V}(\overrightarrow{\mathcal{G}})$ and this ends the proof. $\square$

This construction naturally extends to any Coxeter group W.

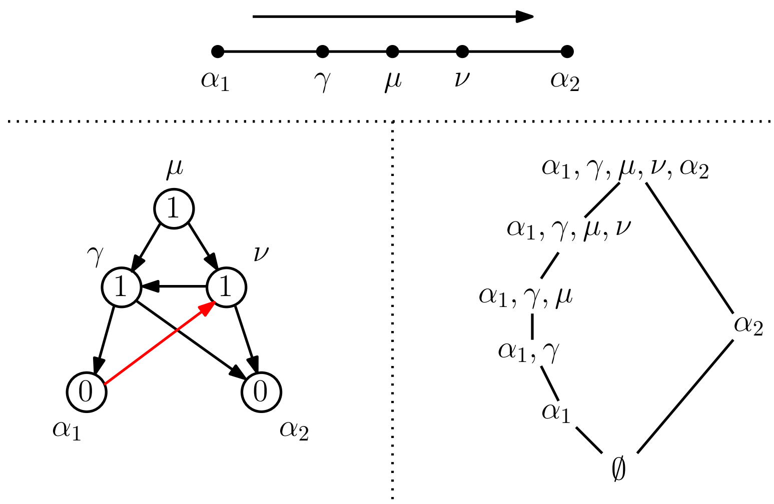


FIGURE 4.22. An example on a dihedral group of cardinality 10.

DEFINITION 4.5.6. Let W be a Coxeter group of finite rank, $\Phi = \Phi^+ \cup \Phi^-$ be a root system of W, $\mathcal{G}$ be a valued digraph well assembled on Φ^+ and $\overrightarrow{\Phi}$ be an orientation of Φ. We denote by $G_{\overrightarrow{\Phi}}$ the valued digraph associated with $\overrightarrow{\Phi}$, obtained from $\mathcal{G}$ by adding to it the arcs $(\alpha, s_\beta(\alpha))$, where $\{\alpha, \beta\}$ is the simple system of an oriented element Φ' of $\mathcal{M}_{>2}$ such that $\psi(\Phi') = (\alpha, \beta)$.

Let us see what happens in the case of various orientations of A_3. On Figure 4.23, we depict the valued digraph obtained considering the $s_1s_2s_3$-orientation and the $s_1s_3s_2$-orientation. In both cases, one can check that the elements of $IS(\mathcal{G}_{\overrightarrow{A_3}})$ are exactly the inversion sets of the respective aligned elements. Thus, the resulting lattice is the corresponding Cambrian lattice in both cases. Finally, let us consider an orientation which does not come from a Coxeter element of A_3 (see Figure 4.24). In this case, the corresponding aligned elements are all in the obtained lattice. However, note that there are elements in $IS(\mathcal{G}_{\overrightarrow{\Phi}})$ which are not aligned (see the elements in the red boxes of Figure 4.24).

Even if the general situation seems to be quite complicated, notice that in the case of any $\mathbf{c}$-orientation of A_3, the resulting lattice is always the corresponding Cambrian lattice. Furthermore, this construction always gives rise to a complete lattice, for finite and infinite Coxeter groups. Consequently, this construction may lead to an extension of Cambrian lattices in infinite Coxeter groups. In order to investigate this possibility, we need to develop new tools. Indeed, a fundamental property of Cambrian lattices is that they are sub-posets of the weak order on the corresponding Coxeter group. However, if we consider a valued digraph $\mathcal{G}$ and a second valued digraph $\mathcal{G}'$ obtained from $\mathcal{G}$ by adding some arcs to $\mathcal{G}$, then $IS(\mathcal{G}')$ is generally not included in $IS(\mathcal{G})$ (see Figure 4.25). In fact, the relationship between $IS(\mathcal{G}')$ and $IS(\mathcal{G})$ seems to heavily depend on the considered valued digraphs, and $IS(\mathcal{G}')$ can be really different from $IS(\mathcal{G})$. To overcome this difficulty, we develop in the next section a different (and more subtle) approach, by explaining how $\mathcal{G}'$ can be used to define a sub-poset of $(IS(\mathcal{G}), \subseteq)$, which turns out to be a lattice (but generally not a sub-lattice of $(IS(\mathcal{G}), \subseteq)$).

4.5.3. Adding arcs in a valued digraph and induced lattice. In this section, $\mathcal{G}$ and $\mathcal{G}'$ will denote two valued digraphs such that $\mathcal{V}(\mathcal{G}) = \mathcal{V}(\mathcal{G}')$, $E(\mathcal{G}) \subseteq E(\mathcal{G}')$, and for all $z \in \mathcal{V}(\mathcal{G})$, $\theta(z) = \theta'(z)$. Note that we do not restrict our study to simple digraphs here, hence $E(\mathcal{G})$ and $E(\mathcal{G}')$ can be two multi-sets. Therefore, the formula $E(\mathcal{G}) \subseteq E(\mathcal{G}')$ means that for each $a \in E(\mathcal{G})$, the multiplicity of a in $E(\mathcal{G})$ is weakly smaller than the multiplicity of a in $E(\mathcal{G}')$. Before we explain what we will do exactly in this section, let us first introduce a few notations and concepts.

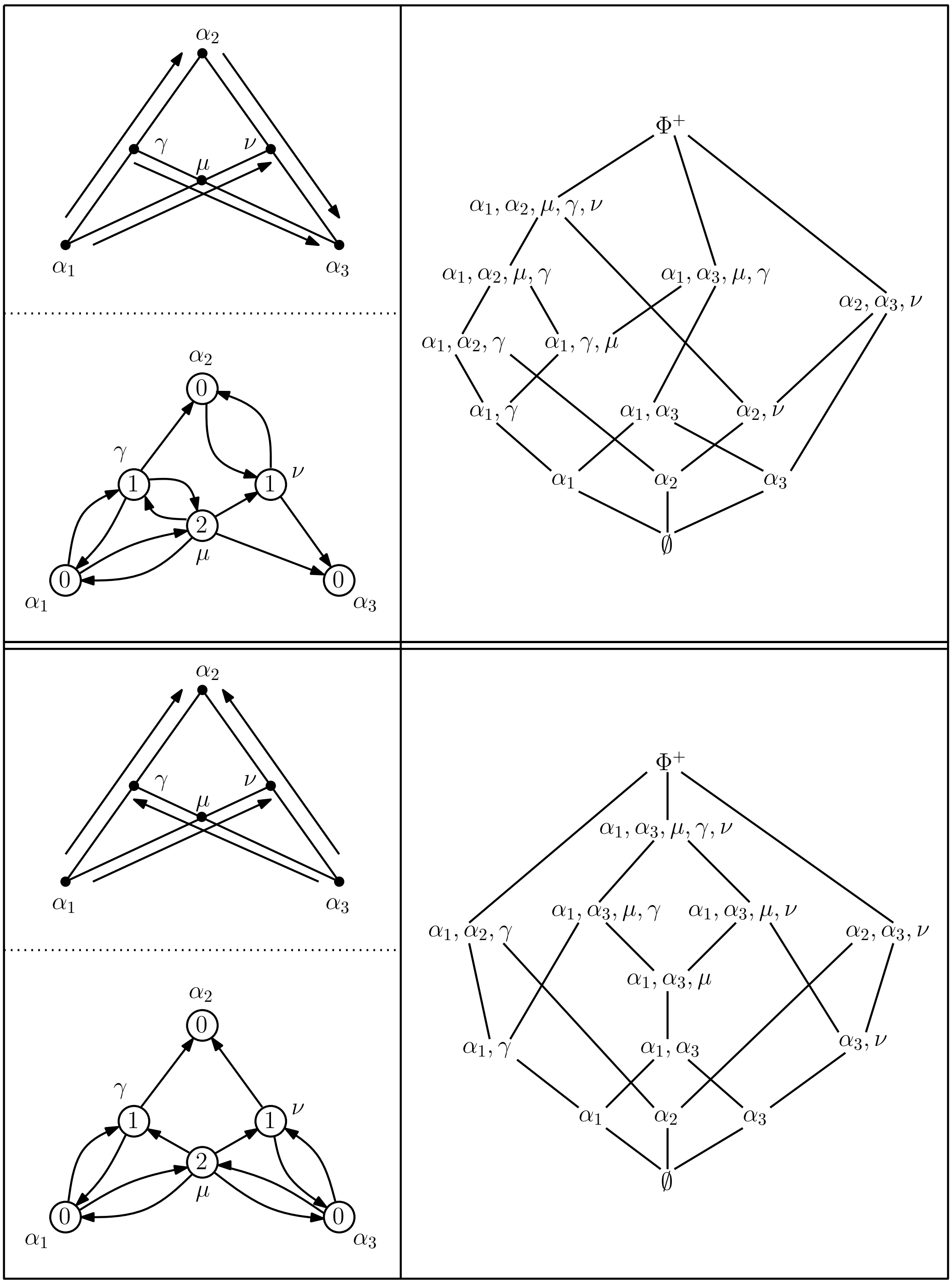


FIGURE 4.23. On the top: valued digraph and lattice obtained with the $s_1s_2s_3$-orientation of A_3. On the bottom: the same thing considering the $s_1s_3s_2$-orientation.

We denote by Γ the closure operator associated with $\mathcal{G}$ defined in Definition 4.4.2. That is, for all $X \subseteq \mathcal{V}(\mathcal{G})$, we have

$$\Gamma(X) = \bigcup_{i \geq 0} J_i,$$

where $(J_i)_{i \geq 0}$ is recursively defined by $J_0 = X$, and

$$J_{i+1} = J_i \cup \{z \in \mathcal{V}(\mathcal{G}) \setminus J_i \mid \theta(z) < d^+_{J_i}(\mathcal{G}, z)\}.$$

We denote by Γ' the closure operator similarly associated with $\mathcal{G}'$. Let us start our study by giving some fundamental properties of Γ and Γ'. We begin with a useful definition.

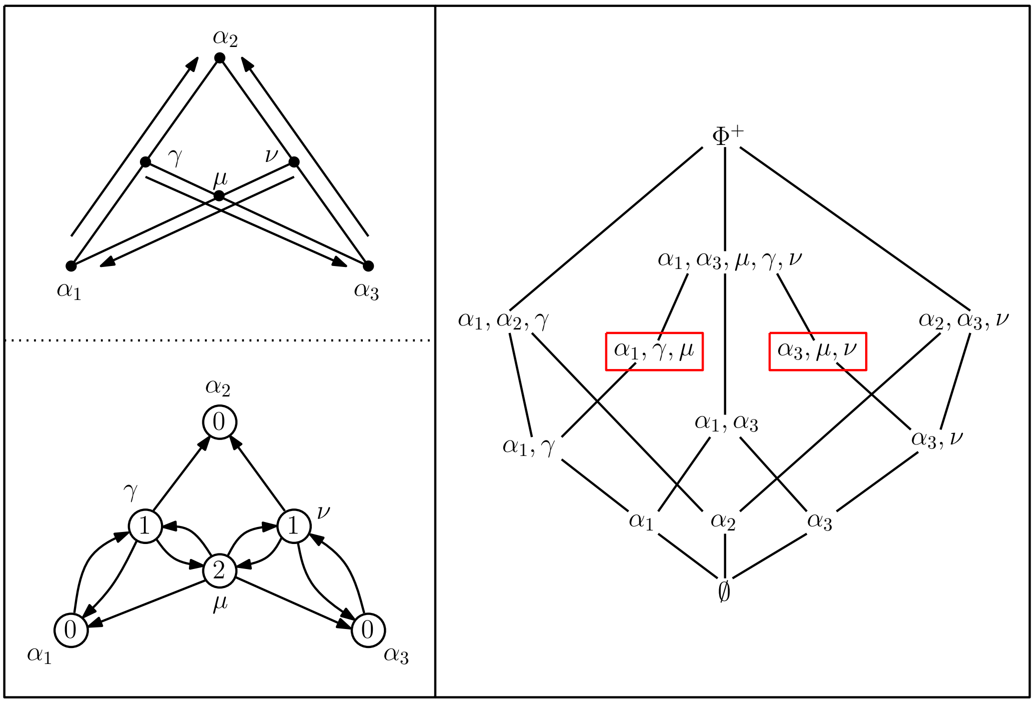


FIGURE 4.24. Valued digraph and lattice obtained with an orientation which is not a **c**-orientation. The elements in the red boxes are not aligned.

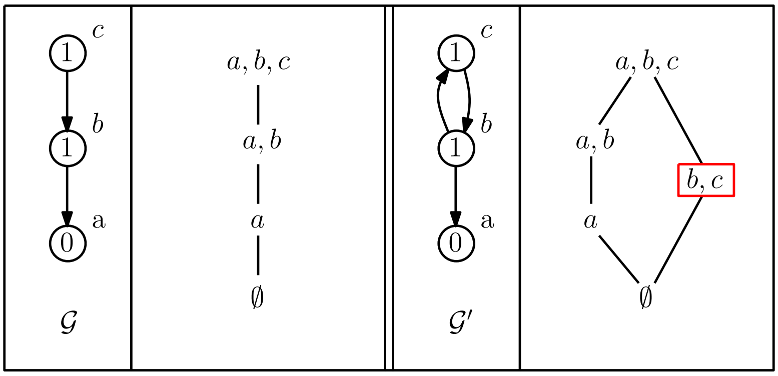


FIGURE 4.25. The element in the red box is in $IS(\mathcal{G}')$, but not in $IS(\mathcal{G})$.

DEFINITION 4.5.7. Set $X \subseteq \mathcal{V}(\mathcal{G})$, we say that X is a *germ* in $\mathcal{G}$ if and only if we have

$$\text{for all } x \in X,\ \theta(x) \leq d^+_X(\mathcal{G}, x).$$

LEMMA 4.5.8. *If $X \subseteq \mathcal{V}(\mathcal{G})$ is a germ in $\mathcal{G}$, then $\Gamma(X)$ is in $IS(\mathcal{G})$.*

PROOF. The proof is similar as the one of Theorem 4.1.6. $\square$

LEMMA 4.5.9. *For all $z \in \mathcal{V}(\mathcal{G}) = \mathcal{V}(\mathcal{G}')$ and for all $X \subseteq \mathcal{V}(\mathcal{G})$, we have $d^+_X(\mathcal{G}, z) \leq d^+_X(\mathcal{G}', z)$.*

PROOF. This is immediate since $E(\mathcal{G}) \subseteq E(\mathcal{G}')$. $\square$

Our aim in this section will be to define an equivalence relation on $IS(\mathcal{G})$ (see Definition 4.5.10), using the closure operator Γ'. More precisely, we will say that A and B in $IS(\mathcal{G})$ are $\mathcal{G}'$-equivalent if and only if $\Gamma'(A) = \Gamma'(B)$. Then, we will show that each equivalence class of this relation admits a maximal element (see Proposition 4.5.13), and finally we will prove that the poset obtained by ordering by inclusion these maximal elements is a lattice (see Corollary 4.5.17), but not a sub-lattice of $(IS(\mathcal{G}), \subseteq)$ in general (see Figure 4.26).

Definition 4.5.10. Let $A, B \in IS(\mathcal{G})$, we say that A and B are $\mathcal{G}'$-equivalent, denoted by $A \sim_{\mathcal{G}'} B$, if and only if $\Gamma'(A) = \Gamma'(B)$.

We now begin the study of this equivalence relation.

Proposition 4.5.11. *For all $A \in IS(\mathcal{G})$, we have $\Gamma'(A) \in IS(\mathcal{G}')$.*

Proof. For all $z \in A$, we have $\theta(z) \leq d_A^+(\mathcal{G}, z) \leq d_A^+(\mathcal{G}', z)$, hence A is a germ in $\mathcal{G}'$ and we conclude thanks to Lemma 4.5.8. □

The following Lemma shows how, in a certain sense, Γ' "dominate" Γ.

Lemma 4.5.12. *For all $X \subseteq \mathcal{V}(\mathcal{G}) = \mathcal{V}(\mathcal{G}')$, $\Gamma'(\Gamma(X)) = \Gamma'(X)$.*

Proof. We have $X \subseteq \Gamma(X)$, so that $\Gamma'(X) \subseteq \Gamma'(\Gamma(X))$.

In order to prove that the reverse inclusion holds, we consider the non-decreasing sequences of sets $(J_i)_{i \geq 0}$ and $(J'_i)_{i \geq 0}$ respectively associated with $\Gamma(X)$ and $\Gamma'(X)$. We will show by induction that $J_i \subseteq J'_i$ for all $i \geq 0$. Since $J_0 = X = J'_0$, the case $i = 0$ is obvious. Let i be such that the property is true, and fix $z \in J_{i+1} \setminus J_i$, we have

$$\theta(z) < d_{J_i}^+(\mathcal{G}, z) \leq d_{J'_i}^+(\mathcal{G}, z) \leq d_{J'_i}^+(\mathcal{G}', z).$$

Therefore, $z \in J'_{i+1}$, thus $J_{i+1} \subseteq J'_{i+1}$. Hence, we have $\Gamma(X) \subseteq \Gamma'(X)$, so that

$$\Gamma'(\Gamma(X)) \subseteq \Gamma'(\Gamma'(X)) = \Gamma'(X).$$

By double inclusion, we have $\Gamma'(\Gamma(X)) = \Gamma'(X)$ and this concludes the proof. □

We now prove that each equivalence class of $\sim_{\mathcal{G}'}$ admits a maximal element.

Proposition 4.5.13. *Consider $Q \subseteq IS(\mathcal{G})$ an equivalence class of $\sim_{\mathcal{G}'}$, then Q has a maximal element for the inclusion, which is given by the join in $(IS(\mathcal{G}), \subseteq)$ of all the elements of Q.*

Proof. Set $X := \bigcup_{A \in Q} A$, thanks to Proposition 4.4.3 we have that $\Gamma(X)$ is the join of Q in $(IS(\mathcal{G}), \subseteq)$. Therefore, if $\Gamma(X) \in Q$, then $\Gamma(X)$ is maximal in Q. It remains to prove that $\Gamma(X) \in Q$. Fix $A \in Q$, we have $A \subseteq X$, hence $A \subseteq \Gamma(X)$ and $\Gamma'(A) \subseteq \Gamma'(\Gamma(X))$.

We now prove that the reverse inclusion holds. Let $B \in Q$, we have $\Gamma'(A) = \Gamma'(B)$, so that $B \subseteq \Gamma'(A)$, and this is true for all $B \in Q$. Then, we have $X \subseteq \Gamma'(A)$, so that

$$\Gamma'(X) \subseteq \Gamma'(\Gamma'(A)) = \Gamma'(A).$$

Finally, thanks to Lemma 4.5.12 we have $\Gamma'(\Gamma(X)) \subseteq \Gamma'(A)$. Consequently, $\Gamma(X)$ is in Q, and it is maximal in Q. This ends the proof. □

Definition 4.5.14. We denote by $IS(\mathcal{G}, \mathcal{G}') \subseteq IS(\mathcal{G})$ the set constituted of the maximal elements of the equivalence classes of $\sim_{\mathcal{G}'}$.

Proposition 4.5.15. *Let $A, B \in IS(\mathcal{G}, \mathcal{G}')$, we have $A \subseteq B$ if and only if $\Gamma'(A) \subseteq \Gamma'(B)$.*

Proof. Clearly, $A \subseteq B$ implies $\Gamma'(A) \subseteq \Gamma'(B)$.

We now show the converse implication. Consider $B' = \Gamma(B \cup A) \in IS(\mathcal{G})$, we have $B \subseteq B'$, thus $\Gamma'(B) \subseteq \Gamma'(B')$. We also have that $B \cup A \subseteq \Gamma'(B)$, so that $\Gamma'(B \cup A) \subseteq \Gamma'(B)$, and thanks to Lemma 4.5.12 we have $\Gamma'(B \cup A) = \Gamma'(\Gamma(B \cup A)) = \Gamma'(B')$. As a consequence, $\Gamma'(B') = \Gamma'(B)$, and by maximality of B we have $\Gamma(B \cup A) = B$, so that $B \cup A \subseteq B$, hence $A \subseteq B$, and this ends the proof. □

Proposition 4.5.15 implies that the two posets $(IS(\mathcal{G}, \mathcal{G}'), \subseteq)$ and $(\Gamma'(IS(\mathcal{G})), \subseteq)$ are isomorphic. Consequently, proving that $(\Gamma'(IS(\mathcal{G})), \subseteq)$ is a lattice implies that $(IS(\mathcal{G}, \mathcal{G}'), \subseteq)$ is a lattice.

Theorem 4.5.16. *The poset $(\Gamma'(IS(\mathcal{G})), \subseteq)$ is a lattice.*

PROOF. Let X' be a subset of $\Gamma'(IS(\mathcal{G}))$, $X \subseteq IS(\mathcal{G},\mathcal{G}')$ be such that $\Gamma'(X) = X'$ and denote by Y the union of all the elements of X. Consider $\Gamma(Y) \in IS(\mathcal{G})$, for all $A \in X$ we have $A \subseteq Y$ so that $\Gamma'(A) \subseteq \Gamma'(Y) = \Gamma'(\Gamma(Y))$. Thus, $\Gamma'(\Gamma(Y))$ is an upper bound of X' in $(\Gamma'(IS(\mathcal{G})), \subseteq)$.

We now prove that $\Gamma'(\Gamma(Y))$ is the supremum of X'. Let M be an upper bound of X' in $(\Gamma'(IS(\mathcal{G})), \subseteq)$. By definition, we have $\Gamma'(A) \subseteq M$ for all A in X. Therefore, $A \subseteq M$ for all $A \in X$, hence $Y \subseteq M$, so that

$$\Gamma'(\Gamma(Y)) = \Gamma'(Y) \subseteq \Gamma'(M) = M.$$

This shows that $\Gamma'(\Gamma(Y))$ is the supremum of X' in $(\Gamma'(IS(\mathcal{G})), \subseteq)$. In order to complete the proof, remark that both $\emptyset$ and $\mathcal{V}(\mathcal{G}) = \mathcal{V}(\mathcal{G}')$ are in $\Gamma'(IS(\mathcal{G}))$. Thus, $(\Gamma'(IS(\mathcal{G})), \subseteq)$ is bounded, hence $(\Gamma'(IS(\mathcal{G})), \subseteq)$ is a lattice. □

COROLLARY 4.5.17. *The poset* $(IS(\mathcal{G},\mathcal{G}'), \subseteq)$ *is a lattice.*

REMARK 4.5.18. Note that in general $(IS(\mathcal{G},\mathcal{G}'), \subseteq)$ is not a sub-lattice of $(IS(\mathcal{G}), \subseteq)$. Indeed, in the example represented on Figure 4.26, the join of $\{a\}$ and $\{b\}$ is $\{a, b, c\}$ in

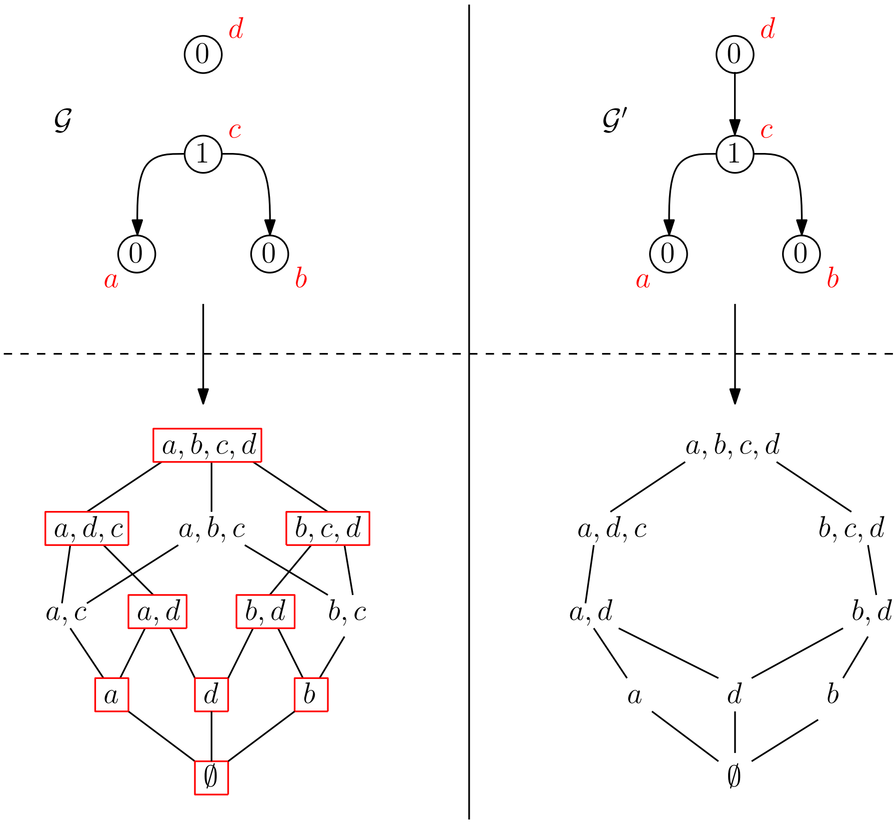


FIGURE 4.26. The red boxes correspond to the elements of $IS(\mathcal{G},\mathcal{G}')$.

$(IS(\mathcal{G}), \subseteq)$, but their join in $(IS(\mathcal{G},\mathcal{G}'), \subseteq)$ is $\{a, b, c, d\}$.

Now that we have an interesting sub-poset of $(IS(\mathcal{G}), \subseteq)$, the next natural step is to look for a way to identify the elements of $IS(\mathcal{G},\mathcal{G}')$ inside $IS(\mathcal{G})$. At this point, finding such a description in the general case remains an open problem. However, the following proposition is a partial but useful result in this direction.

PROPOSITION 4.5.19. *Let* $A \in IS(\mathcal{G}')$, *if* $A \in IS(\mathcal{G})$, *then* $A \in IS(\mathcal{G},\mathcal{G}')$.

PROOF. Since $\Gamma'(A) = A$, we have that A is maximal in its equivalence class, so it is in $IS(\mathcal{G},\mathcal{G}')$. □

Finally, we use the theory developed in this section to start the study of the problem of finding an extension of Cambrian lattices.

THEOREM 4.5.20. *Let W be a Coxeter group with root system $\Phi = \Phi^+ \cup \Phi^-$, $\mathcal{G}$ be a valued digraph well-assembled on Φ^+, $\overrightarrow{\Phi}$ be an orientation of Φ and $\mathcal{G}_{\overrightarrow{\Phi}}$ be the valued digraph associated with this orientation. If $A \subseteq \Phi^+$ is $\overrightarrow{\Phi}$-aligned, then A is in $IS(\mathcal{G}, \mathcal{G}_{\overrightarrow{\Phi}})$.*

PROOF. By definition, A is bi-closed, hence $A \in IS(\mathcal{G})$. Let $z \in \mathcal{V}(\mathcal{G})$, if $z \in A$ then $\theta(z) \leq d_A^+(\mathcal{G}, z)$. Consequently, we have $\theta(z) \leq d_A^+(\mathcal{G}, z) \leq d_A^+(\mathcal{G}_{\overrightarrow{\Phi}}, z)$.

If $z \notin A$, then assume by contradiction that $\theta(z) < d_A^+(\mathcal{G}_{\overrightarrow{\Phi}}, z)$. Since $\theta(z) \geq d_A^+(\mathcal{G}, z)$, there exists Φ' an oriented element of $\mathcal{M}_{>2}$ such that $\psi(\Phi') = (z, \alpha)$ and $s_\alpha(z) \in A$, however A is bi-closed and $z \notin A$, so that $\alpha \in A$. But A is $\overrightarrow{\Phi}$-aligned, thus $\Phi' \subseteq A$ and $z \in A$, and this is absurd, hence $\theta(z) \geq d_A^+(\mathcal{G}_{\overrightarrow{\Phi}}, z)$.

Therefore, $A \in IS(\mathcal{G}_{\overrightarrow{\Phi}})$, and thanks to Proposition 4.5.19, we have $A \in IS(\mathcal{G}, \mathcal{G}_{\overrightarrow{\Phi}})$, and this ends the proof. $\square$

Theorem 4.5.20 has the following immediate corollary, which shows that our construction extends Cambrian semi-lattices.

COROLLARY 4.5.21. *If W is a Coxeter group, and $\overrightarrow{\Phi}$ is a $\mathbf{c}$-orientation, then the Cambrian semi-lattice associated with $\mathbf{c}$ is a sub-poset of $(IS(\mathcal{G}, \mathcal{G}_{\overrightarrow{\Phi}}), \subseteq)$.*

CHAPTER 5

Description of Tamari lattices using valued digraphs

Introduction

In this chapter, we show how the theory developed in Chapter 4 can be used to study Tamari and Cambrian lattices. More precisely, we introduce a new valued digraph, denoted by $\mathcal{A}_n^{\uparrow}$, and we show, using combinatorial methods, that $(IS(\mathcal{A}_n^{\uparrow}), \subseteq)$ is isomorphic to the $(n+1)$-th Tamari lattice $\mathcal{T}_{n+1}$. Moreover, our proof allows us to provide a similar description of m-Tamari lattices, which are a generalisation of Tamari lattices.

5.1. Brief summary of previous results, and definition of Tamari and m-Tamari lattices using Dyck paths

5.1.1. How to obtain the valued digraph associated with $(A_{n-1}, \leq_R)$ using root systems. Consider the Coxeter group A_{n-1}, with generating set $S = \{s_1, \dots, s_{n-1}\}$. Consider the vector space $\mathbb{R}^n$ with canonical base $(e_i)_{1 \leq i \leq n}$. A root system of A_{n-1} is given by the set

$$\Phi = \{e_j - e_i \mid j \neq i,\ i, j \in [n]\},$$

with the bilinear form given by the ordinary scalar product on $\mathbb{R}^n$. In this case, we choose $\Delta = \{e_{i+1} - e_i \mid 1 \leq i < n\}$ as simple system (here, s_i corresponds to the simple root $e_{i+1} - e_i$). With this choice, the set of positive roots is $\Phi^+ = \{e_j - e_i \mid 1 \leq i < j \leq n\}$. Since A_{n-1} is simply-laced, to construct the (unique) well assembled on Φ^+ valued digraph $\mathcal{G}_W$ (see Section 4.3.5), we just need to identify, for each $(e_j - e_i) \in \Phi^+$, all the pairs of distinct roots $\{u, v\}$ such that

$$\{u, v\} \subseteq \Phi^+,\ e_j - e_i = au + bv, \text{ with } a > 0 \text{ and } b > 0. \tag{5.1}$$

We have that for all $k \in [n]$

$$\text{if } i < k < j, \text{ then } e_j - e_i = (e_j - e_k) + (e_k - e_i),$$

and it is easy to see that the pairs of the form $\{(e_j - e_k), (e_k - e_i)\}$ are the only ones satisfying Equation 5.1. Thus, the only arcs having $e_j - e_i$ as starting point in $\mathcal{G}_W$ are $(e_j - e_i, e_j - e_k)$ and $(e_j - e_i, e_k - e_i)$, for all $i < k < j$. As in Chapter 3, we represent $\mathcal{G}_W$ with a Ferrers

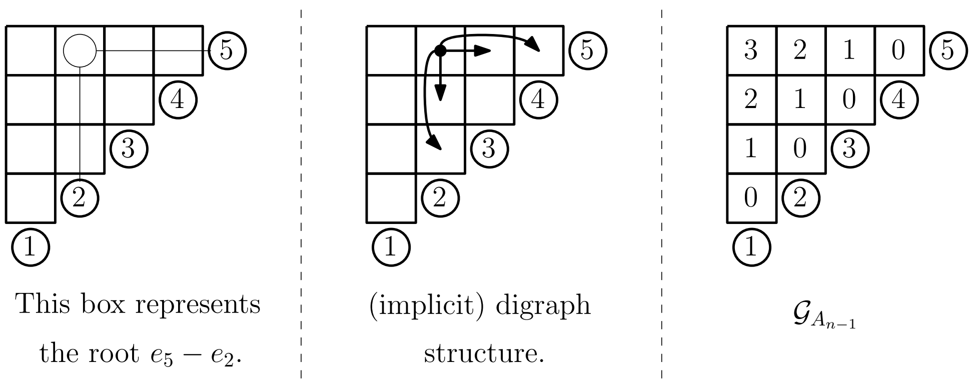


FIGURE 5.1. The case of A_4.

diagram of staircase shape as depicted in Figure 5.1. The left picture on Figure 5.1 explains how to associate a positive root to each box of the diagram, the center one depicts the digraph structure, and the right one represents the valuation. As claimed before, the obtained valued digraph is exactly the one constructed in Chapter 3.

5.1.2. Tamari and m-Tamari lattices. The n-th Tamari lattice can be defined in many ways, and the most adapted to our purpose is the one involving *Dyck paths*. A Dyck path of size n is a path on the square grid consisting of steps $(1,0)$ and $(0,1)$, starting at $(0,0)$ and ending at (n,n), and which never goes below the line of equation $x=y$. To each Dyck path of size n corresponds a word of length $2n$ having entries in $\{0,1\}$, where the i-th entry of the word is a 1 if the i-th step of the corresponding Dyck path is $(0,1)$, 0 otherwise. Consider a word D which corresponds to a Dyck path and denote by $D[i]$ the i-th letter of D. We say that a word D' of the form $D[j]D[j+1]\cdots D[j+2k]$ for some j and k is a *Dyck factor* of D if and only if D' is the word corresponding to a Dyck path of size k (see Figure 5.2 for an illustration of these notions). In what follows, we identify a Dyck path with its corresponding word. That is, for any Dyck path P, $P[i]$ will denote the i-th letter of its corresponding word.

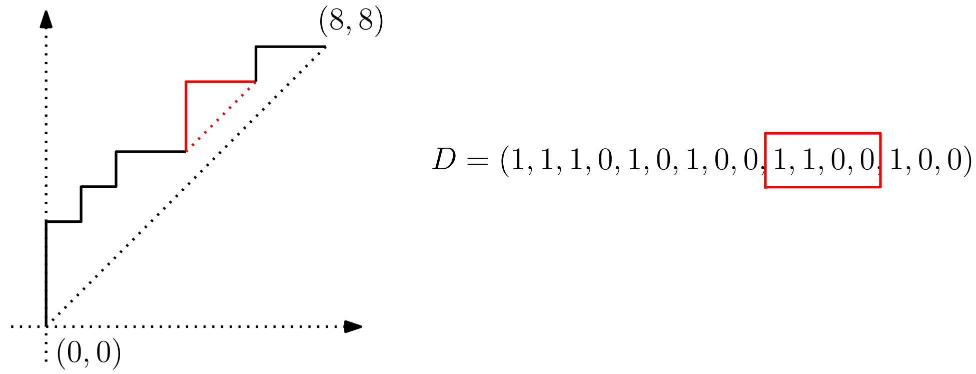


FIGURE 5.2. An example of Dyck path of size 8. In red, a dyck factor of size 2.

Let us denote by Dyck(n) the set of all the Dyck paths of size n. We define an order on Dyck(n) as follows. Let $D\in$ Dyck(n) which does not begin by n up-steps, and set $1\leq i<2n$ such that $D[i]=0$ and $D[i+1]=1$. It is clear on the graphical representation of D that there exists a minimal $k>0$ such that $D[i+1]\cdots D[i+2k]$ is a Dyck factor of D. We say that the Dyck path D', obtained by swapping the positions of $D[i]$ and this shortest Dyck factor, covers D (see Figure 5.3).

DEFINITION 5.1.1. The n-th Tamari lattice $\mathcal{T}_n$ is the poset obtained by considering the transitive and reflexive closure of this covering relation.

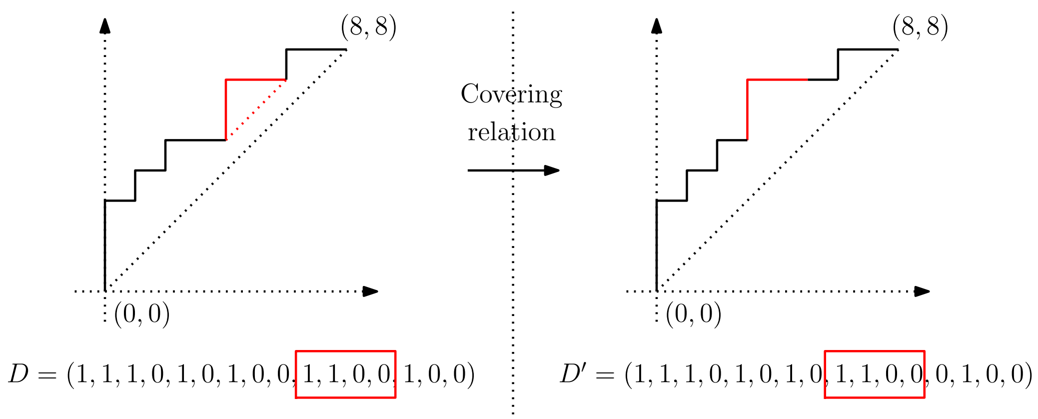


FIGURE 5.3. An example of covering relation.

The n-th m-Tamari lattice $\mathcal{T}_n^{(m)}$ is defined similarly, by replacing the Dyck paths of size n by *m-ballot paths* of size n. A m-ballot path of size n is a path on the square grid consisting of steps $(1,0)$ and $(0,1)$, starting at $(0,0)$, ending at (mn,n) and never going below the line of equation $x=my$. As in the case of Dyck paths, for each up-step i of a m-ballot path there exists a notion of shortest m-ballot factor which starts by the step i. This leads us to the following definition of n-th m-Tamari lattice.

DEFINITION 5.1.2. Set D and D' two m-ballot paths of size n. We say that D' covers D if and only if there exists in D a horizontal step i followed by an up-step, such that D' is obtained from D by swapping the positions of the step i and the shortest m-ballot factor of D that begins with the step $i+1$. The n-th m-Tamari Lattice $\mathcal{T}_n^{(m)}$ is the poset obtained by considering the reflexive and transitive closure of this covering relation.

5.2. From valued digraphs to Tamari and m-Tamari lattices

Let us consider the example $S_4 = A_3$. It is known (see [**R1**]) that the Cambrian lattice associated with $s_1s_2s_3$ is isomorphic to the Tamari lattice $\mathcal{T}_4$. In the previous chapter, we associated a valued digraph to the $s_1s_2s_3$-orientation and we checked that the resulting lattice is indeed the associated Cambrian lattice (see Figure 3.23). As it is depicted in Figure 5.4, this valued digraph can easily be represented using the staircase diagram, by adding arcs in each column.

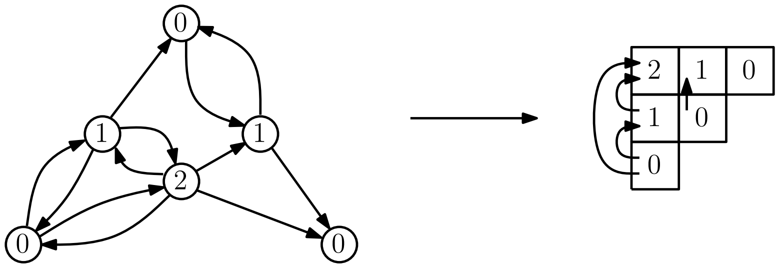


FIGURE 5.4. Two representations of the valued digraph associated with the $s_1s_2s_3$-orientation.

In what follows, we will generalize this construction and prove that each Tamari Lattice $\mathcal{T}_n$ can be described with a valued digraph of this form.

5.2.1. The valued digraph $\mathcal{A}_n^{\uparrow}$ and its connections with Dyck$(n+1)$. We denote by $\mathcal{A}_n = (G, \theta)$ the valued digraph associated with $(A_n, \leq_R)$. We recall that we have

$$\begin{aligned}
\mathcal{V}(\mathcal{A}_n) &:= \{(a,b) \mid 1 \leq a < b \leq n+1\},\\
E(\mathcal{A}_n) &:= \{((a,b),(a,d)) \mid b > d\} \cup \{((a,c),(b,c)) \mid a < b\},\\
\theta(a,b) &:= b - a - 1 \text{ for all } (a,b) \in \mathcal{V}(\mathcal{A}_n).
\end{aligned}$$

DEFINITION 5.2.1. We denote by $\mathcal{A}_n^{\uparrow} = (G', \theta')$ the valued digraph such that:

$$\begin{aligned}
\mathcal{V}(\mathcal{A}_n^{\uparrow}) &= \mathcal{V}(\mathcal{A}_n),\\
E(\mathcal{A}_n^{\uparrow}) &= E(A_n) \cup \{((a,b),(a,d)) \mid b < d\},\\
\theta'(z) &= \theta(z) \text{ for all } z \in \mathcal{V}(\mathcal{A}_n^{\uparrow}).
\end{aligned}$$

The construction of $\mathcal{A}_n^{\uparrow}$ from $\mathcal{A}_n$ is depicted in Figure 5.5, in the case $n = 3$. Since there is no ambiguity, in what follows we will simply denote by θ the valuation associated with both $\mathcal{A}_n$ and $\mathcal{A}_n^{\uparrow}$. Note that for all $\mathfrak{c} \in \mathcal{V}(\mathcal{A}_n^{\uparrow})$, $\theta(\mathfrak{c})$ is the number of boxes which are below $\mathfrak{c}$ and in the same column.

On Figure 5.6, we represent the lattices associated with $\mathcal{A}_2^{\uparrow}$ and $\mathcal{A}_3^{\uparrow}$, where the red boxes correspond to the elements of $IS(\mathcal{A}_n^{\uparrow})$. In the first case, we can recognize the Tamari lattice $\mathcal{T}_3$, and in the second $\mathcal{T}_4$.

In what follows, we will prove that the correspondence between the lattice $(IS(\mathcal{A}_n^{\uparrow}), \subseteq)$ and $\mathcal{T}_{n+1}$ observed for $n = 2$ and 3 is actually general. For this purpose, we will construct explicitly a poset isomorphism between these two lattices decomposing our study into three steps, that we now detail.

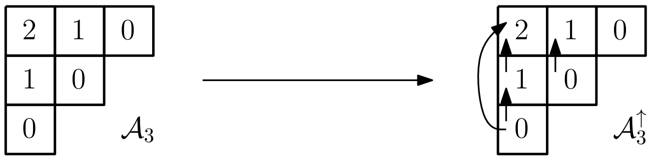


FIGURE 5.5. From $\mathcal{A}_3$ to $\mathcal{A}_3^{\uparrow}$.

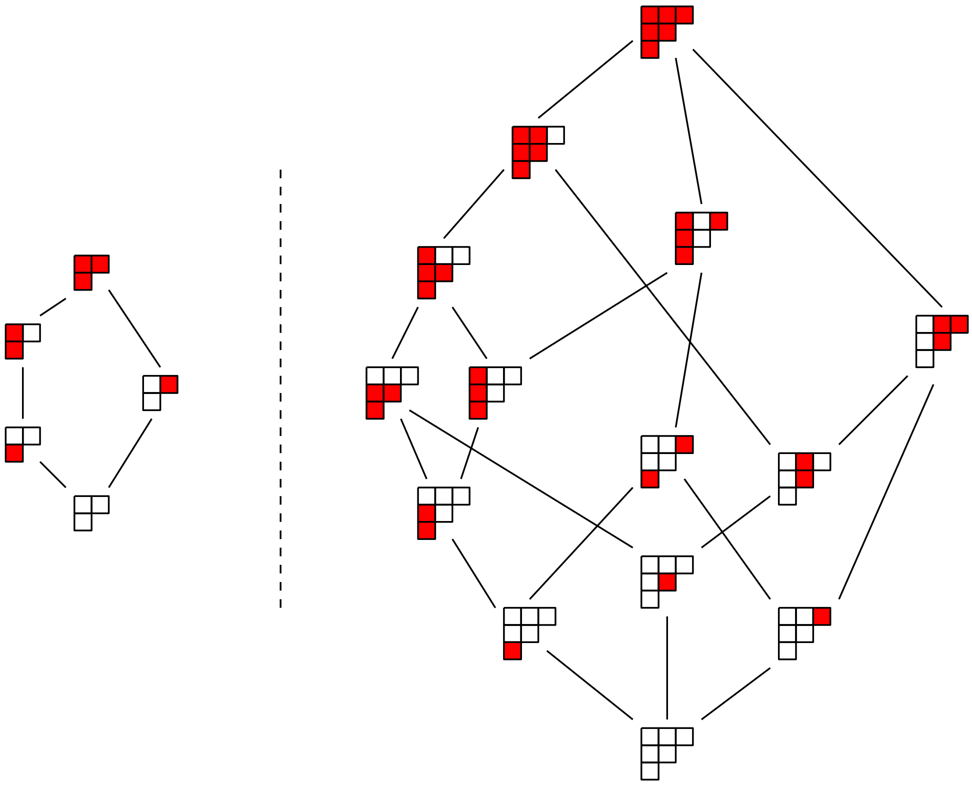


FIGURE 5.6. The lattices $(IS(\mathcal{A}_2^{\uparrow}), \subseteq)$ and $(IS(\mathcal{A}_3^{\uparrow}), \subseteq)$.

(Step 1) First, we associate each element of $\mathrm{Dyck}(n+1)$ with a sequence of integers called its *code*.
(Step 2) Then, do the same thing with the elements of $IS(\mathcal{A}_n^{\uparrow})$, associating them with sequences of integers that we also call codes.
(Step 3) Finally, we construct the isomorphism by defining a well-chosen bijection between codes of $\mathrm{Dyck}(n+1)$ and codes of $IS(\mathcal{A}_n^{\uparrow})$.

Let us summarize this planning on a diagram, depicted in Figure 5.7.

$$\begin{tikzcd}
\mathrm{Dyck}(n+1) \arrow[r, leftrightarrow, "\text{Step 1}"] & \{\text{Codes of } \mathrm{Dyck}(n+1)\} \arrow[d, leftrightarrow, "\text{Step 3}"] \\
IS(\mathcal{A}_n^{\uparrow}) \arrow[r, leftrightarrow, "\text{Step 2}"] & \{\text{Codes of } IS(\mathcal{A}_n^{\uparrow})\}
\end{tikzcd}$$

FIGURE 5.7.

Let us begin with defining the code of a Dyck path. Note that we clearly have a one-to-one correspondence between partitions $(\lambda_1, \ldots, \lambda_k)$ such that for all $1 \leq i \leq n$, $\lambda_i \leq n - i$ and $\mathrm{Dyck}(n+1)$, associating such a partition with the Dyck path given by its boundary, as depicted in Figure 5.8.

Thanks to this point of view, we are now able to accomplish the first step (see Figure 5.7).

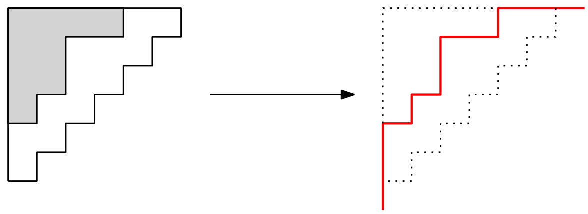


FIGURE 5.8. Dyck path of size 7 associated with $(4, 2, 2, 1)$.

DEFINITION 5.2.2. Let $D \in \mathrm{Dyck}(n)$ and $\lambda = (\lambda_1, \ldots, \lambda_k)$ be its associated partition. We define the code of D to be the finite sequence $(c_i(D))_{1 \leq i \leq n-1}$, where

$$c_i(D) = n - i - \lambda_i \text{ for all } i \leq n - 1,$$

with the convention that $\lambda_i = 0$ whenever $i > k$.

It is clear that two distinct Dyck paths have distinct codes. Before we move to the second step, let us give a useful characterization of the finite sequences which are the code of some Dyck paths.

LEMMA 5.2.3. *A sequence $(c_i)_{1 \leq i \leq n-1}$ is the code of a Dyck path of size n if and only if the following two conditions are satisfied:*

- *for all $i \in [n-1]$, $0 \leq c_i \leq n - i$;*
- *for all $i \in [n-1]$, $c_i - c_{i+1} \leq 1$.*

PROOF. Assume that $(c_i)_i$ is the code of a Dyck path of size n, and denote by $\lambda = (\lambda_i, \ldots, \lambda_k)$ the corresponding partition. It clearly satisfies the first condition. Moreover, we have

$$c_i - c_{i+1} = n - i - \lambda_i - (n - i - 1 - \lambda_{i+1}) = 1 - (\lambda_i - \lambda_{i+1}),$$

but λ is a non-increasing sequence, hence $c_i - c_{i+1} \leq 1$.

We now prove the converse. Let us consider $\mu = (\mu_1, \ldots, \mu_{n-1}$ where $\mu_i := n - i - c_i$ for all $i \in [n-1]$. Clearly, we have $\mu_i \leq n - i$. Furthermore, we have

$$\mu_i - \mu_{i+1} = n - i - c_i - (n - i - 1 - c_{i+1}) = 1 - (c_i - c_{i+1}) \geq 1 - 1 = 0.$$

Thus, μ is a non-increasing sequence, *i.e.* it is a partition. This concludes the proof. $\square$

We now begin with the study of the second step (Figure 5.7). Fortunately, there is a canonical sequence of integers associated with each element A of $IS(\mathcal{A}_n^{\uparrow})$, which is given by the number of boxes of A in each row of $\mathcal{V}(\mathcal{A}_n^{\uparrow})$. Let us formalize this remark in a definition.

DEFINITION 5.2.4. Let $A \in IS(\mathcal{A}_n^{\uparrow})$, we define the code of A to be the sequence $(c_i(A))_i$, where for all $i \in [n]$,

$$c_i(A) := |\{(a, i+1) \in V(\mathcal{A}_n^{\uparrow}) \mid (a, i+1) \in A\}|.$$

REMARK 5.2.5. At this point, it is not clear that we have a one-to-one correspondence between $IS(\mathcal{A}_n^{\uparrow})$ and the set of its codes, but this will follow from Proposition 5.2.7.

Our aim is now to construct a one-to-one correspondence between codes of $IS(\mathcal{A}_n^{\uparrow})$ and codes of $\mathrm{Dyck}(n+1)$. In order to do so, we need to understand a bit more what the elements of $IS(\mathcal{A}_n^{\uparrow})$ look like. In the following lemma, we formalize and prove the following two properties:

(1) for all $A \in IS(\mathcal{A}_n^{\uparrow})$ and $\mathfrak{c} \in \mathcal{V}(\mathcal{A}_n^{\uparrow})$, if $\mathfrak{c}$ is in A, then all the boxes which are in the same column and below $\mathfrak{c}$ are in A;
(2) for all $A \in IS(\mathcal{A}_n^{\uparrow})$ and $\mathfrak{c} \in \mathcal{V}(\mathcal{A}_n^{\uparrow})$, if there is a box in the same row and on the right of $\mathfrak{c}$ which is in A and if the box being just below $\mathfrak{c}$ is also in A, then $\mathfrak{c}$ is in A (see Figure 5.9 for a graphical representation of this situation).

LEMMA 5.2.6. *Let $A \in IS(\mathcal{A}_n^{\uparrow})$ and $(a, b) \in \mathcal{V}(\mathcal{A}_n^{\uparrow})$. Then, we have:*

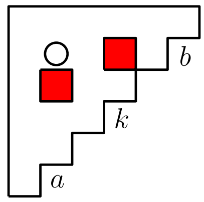


FIGURE 5.9. The red boxes are in A

(1) if $(a,b) \in A$, then $(a,k) \in A$ for all $a < k < b$;
(2) if $(a,b-1) \in A$ and there exists $a < q < b$ such that $(q,b) \in A$, then $(a,b) \in A$.

PROOF. We show the first point by induction on k, where k ranges between $a+1$ and $b-1$. Note that $\theta(a,a+1) = 0$ and $((a,a+1),(a,b)) \in E(\mathcal{A}_n^\uparrow)$, hence

$$\theta(a,a+1) < d_A^+(\mathcal{A}_n^\uparrow,(a,a+1)),$$

so that $(a,a+1) \in A$. Let $k < b-1$ be such that $(a,j) \in A$ for all $a+1 \leq j \leq k$. Then, we have $((a,k+1),(a,b)) \in E(\mathcal{A}_n^\uparrow)$, and $((a,k+1),(a,j)) \in E(\mathcal{A}_n^\uparrow)$ for all $a+1 \leq j \leq k$. Thus, we have

$$\theta(a,k+1) = k-a < d_A^+(\mathcal{A}_n^\uparrow,(a,k+1)),$$

so that $(a,k+1) \in A$. By induction, (1) is proved.

Let us now prove (2). Since $(a,b-1) \in A$, by (1) we have $(a,k) \in A$ for all $a < k < b-1$. Moreover, we also have

$$((a,b),(q,b)) \in E(\mathcal{A}_n^\uparrow) \text{ and } ((a,b),(a,k)) \in E(\mathcal{A}_n^\uparrow) \text{ for all } a < k \leq b-1,$$

so that $\theta(a,b) < d_A^+(\mathcal{A}_n^\uparrow,(a,b))$, hence $(a,b) \in A$ and this concludes the proof. ☐

Let us explain how one can associates the code of an element of $IS(\mathcal{A}_n^\uparrow)$ with the code of an element of $\mathrm{Dyck}(n+1)$.

PROPOSITION 5.2.7. *Let $A \in IS(\mathcal{A}_n^\uparrow)$ and $(d_i)_{1\leq i\leq n}$ be the sequence defined by $d_i := c_{n+1-i}(A)$. Then, $(d_i)_{1\leq i\leq n}$ is the code of a Dyck path of size $n+1$.*

PROOF. We prove this proposition using Lemma 5.2.3. First, note that we clearly have $d_i \leq n+1-i$, hence we just need to prove that $d_i - d_{i+1} \leq 1$. Equivalently, we have to show that $c_{i+1}(A) - c_i(A) \leq 1$ for all i. For that purpose, let us consider a box $(k,i+1) \in A$. If $1 \leq k < i$, then by Lemma 5.2.6 we have $(k,i) \in A$, and since $(i,i+1)$ is the only box in the $(i+1)$-th row which does not have any box below it, this immediately implies that $c_{i+1}(A) - c_i(A) \leq 1$, as required. ☐

DEFINITION 5.2.8. We denote by Ψ the application from $IS(\mathcal{A}_n^\uparrow)$ to $\mathrm{Dyck}(n+1)$ which associate each $A \in IS(\mathcal{A}_n^\uparrow)$ with the unique $D \in \mathrm{Dyck}(n+1)$ whose code is given by $(c_{n+1-i}(A))_{1\leq i\leq n}$.

PROPOSITION 5.2.9. *The application Ψ is a bijection.*

Since the proof of Proposition 5.2.9 is quite long and technical, we detail it separately in Section 5.3.1.

5.2.2. The Tamari lattice $\mathcal{T}_{n+1}$ and $(IS(\mathcal{A}_n^\uparrow),\subseteq)$ are isomorphic. In this section, we will prove that Ψ is more than just a bijection: it is a poset isomorphism between $(IS(\mathcal{A}_n^\uparrow),\subseteq)$ and $\mathcal{T}_{n+1}$. In order to prove this, we will prove that Ψ "respects" covering relations in both posets. That is, we will prove that A' covers A in $(IS(\mathcal{A}_n^\uparrow),\subseteq)$ if and only if $\Psi(A')$ covers $\Psi(A)$ in $\mathcal{T}_{n+1}$. Consequently, we need to understand covering relations in $(IS(\mathcal{A}_n^\uparrow),\subseteq)$ first.

DEFINITION 5.2.10. Let $A \in IS(\mathcal{A}_n^\uparrow)$ and $i \in [n]$. We define the following terms.

- We call the *height* of the column i of A the minimal integer k such that $(i,k) \notin A$.
- We say that the box (i,k) is *admissible above* A if and only if k is the height of the column i, and for all $j < i$ the height of the column j is different from k.

- A subset C of $\mathcal{V}(\mathcal{A}_n^{\uparrow})$ is called an *admissible column above* A if and only if there exists a box (i,k) admissible above A such that $C = \{(i,k),(i,k+1),\ldots,(i,q-1)\}$, where q is the height of the column k of A.

See Figure 5.10 for an illustration of these notions.

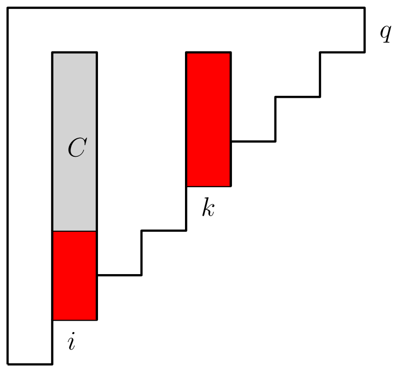


FIGURE 5.10. An element A in $IS(\mathcal{A}_n^{\uparrow})$ is depicted in red, while an admissible column above A is in grey.

Using the notion of admissible columns, we provide a complete characterization of covering relations in $(IS(\mathcal{A}_n^{\uparrow}),\subseteq)$ in the following proposition, whose proof is given in Section 5.3.2.

PROPOSITION 5.2.11. *Let $A, A' \in IS(\mathcal{A}_n^{\uparrow})$, then A' covers A in $(IS(\mathcal{A}_n^{\uparrow}),\subseteq)$ if and only if there exists C an admissible column above A such that $A' = A \cup C$.*

Since the definition of Ψ involves the notion of code of a Dyck path, we need to interpret covering relations in $\mathcal{T}_{n+1}$ in terms of the codes of the appearing Dyck paths. In what follows, we consider $D \in \text{Dyck}(n+1)$. Label by 1 the second up-step of D, 2 the third, and so on. Consider the sequence (d_i) defined by $d_i = c_{n-i}(D)$ and set $d_0 = d_{n+1} = 0$ (this is fundamental for our further remarks). Note that d_i gives the position of the up-step labelled by i of D, as depicted in Figure 5.11. Thanks to this graphical interpretation the following facts are obvious.

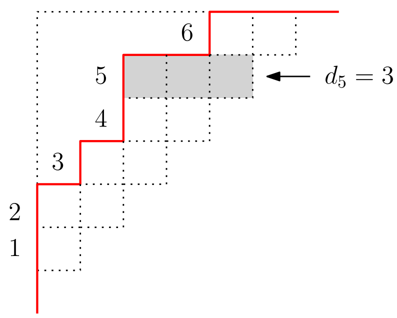


FIGURE 5.11.

- There is an horizontal step between the i-th and $(i+1)$-th up-step if and only if $d_i \geq d_{i+1}$.
- Consider the Dyck factor of size k of D which has the i-th up-step as first step. If k is minimal, then $d_{k+i} \leq d_i$ and for all j such that $i < j < k+i$, we have $d_j > d_i$.
- Conversely, if there exist two integers i and k such that $d_{k+i} \leq d_i$ and for all j such that $i < j < k+i$, we have $d_j > d_i$, then there exists a minimal Dyck factor of size k of D which has the i-th up-step as first step.

These remarks allow us to propose an alternative description of the covering relations in $\mathcal{T}_{n+1}$.

PROPOSITION 5.2.12. *Let $D, D' \in$ Dyck$(n+1)$ and denote by $(d_i)_{1\leq i\leq n}$ and $(d'_i)_{1\leq i\leq n}$ the sequences defined by $d_i = c_{n-i}(D)$ and $d'_i = c_{n-i}(D')$ with the convention that $d_0 = d'_0 = d_{n+1} = d'_{n+1} = 0$. We have that D' covers D in $\mathcal{T}_{n+1}$ if and only if there exist two integers $1 \leq i \leq n$ and $k > 0$ such that:*

(1) $d_i \leq d_{i-1}$;
(2) $d_{i+k} \leq d_i$ and for all $i < j < k+i$, $d_j > d_i$;
(3) for all $j \in [n+1]$, if $i \leq j < k+i$ then $d'_j = d_j + 1$, and $d_j = d'_j$ otherwise.

PROOF. Thanks to the graphical interpretation, if D' covers D then the three points of Proposition 5.2.12 are satisfied.

We now prove the converse: consider D and D' satisfying the three points. Point (2) implies that there is a minimal Dyck factor F of D having the i-th up-step of D as first step. Point (1) implies that the step which is just before the i-th up-step of D is an horizontal step. Finally, thanks to the point (3) and the graphical interpretation, we have that D' is obtained from D by swapping the positions of F and the horizontal step that precedes F. That is, D' covers D in $\mathcal{T}_{n+1}$. □

Finally, we have everything we need to state and prove the main theorem of this section.

THEOREM 5.2.13. *Let $A, A' \in IS(\mathcal{A}_n^{\uparrow})$. Then, A' covers A in $(IS(\mathcal{A}_n^{\uparrow}), \subseteq)$ if and only if $\Psi(A')$ covers $\Psi(A)$ in $\mathcal{T}_{n+1}$.*

PROOF. **Step 1:** assume that A' covers A in $(IS(\mathcal{A}_n^{\uparrow}), \subseteq)$ and denote by $(d_j)_j$ and $(d'_j)_j$ their respective codes. By Proposition 5.2.11 there exists C an admissible column above A such that $A' = A \cup C$. Denote by q the index of the column which contains C and let i and k be the minimal integers such that $(q, i) \in C$ and $(q, i+k) \notin C$ (see Figure 5.12).

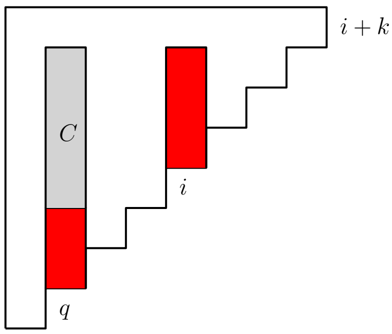


FIGURE 5.12.

We will prove that the sequences $(d_j)_j$ and $(d'_j)_j$ satisfy Points (1), (2) and (3) of Proposition 5.2.12.

- Point (1): we have to show that A has more boxes in row $i-1$ than in row i. Let us consider $(p, i) \in A$ and divide our study into three cases, depending on the value of p. If $p < q$, then by Lemma 5.2.6 $(p, i-1) \in A$. Since (q, i) is admissible above A, it is not in A. Finally, assume by contradiction $p > q$. Then, by construction the box just below (q, i) is in A, and (p, i) is in the same row on the right of (q, i). Thus, by Lemma 5.2.6 $(q, i) \in A$ which is absurd. Consequently, for each box of A in row i there is a corresponding box in A in row $i-1$, so that $d_i \leq d_{i-1}$.
- Point (2): let us first prove that $d_j > d_i$ for all $i < j < k+i$. Since this property is clearly true when $k = 1$, we assume that $k > 1$. Let us consider $(p, i) \in A$, we will prove by induction on j, where j ranges between i and $k+i$, that $(p, j) \in A$. Since $i+k$ is the height of column i of A, $(i, i+1) \in A$. Thus, the box just below $(p, i+1)$ is in A by definition and $(i, i+1)$ is on the right of $(p, i+1)$, hence $(p, i+1) \in A$. One can complete the induction repeating this argument, so that we have $d_j \leq d_i$ for all $i < j < k+i$. Finally, note that $i+k$ is the height of column i, so that $(i, j) \in A$ for

all $i < j < k+i$. However, $(i,i) \notin \mathcal{V}(\mathcal{A}_n^{\uparrow})$, so it is obviously not in A, and this implies that $d_j > d_i$, as required.

We still have to show that $d_{i+k} \le d_i$. By definition, all the boxes below $(q, i+k)$ are in A', but $(q, i+k) \notin A$. Thus, for all $p \geq q$, we have $(p, i+k) \notin A$. Moreover, for all p such that $p < q$ and $(p, i+k) \in A$, by Lemma 5.2.6 we have $(p, i) \in A$. Therefore, we have $d_i \geq d_{i+k}$.

- Point 3: this is clear, since $A' = A \cup C$ with C an admissible column.

Step 2: we now prove the converse. Let us denote by D and D' the Dyck paths $\Psi(A)$ and $\Psi(A')$, respectively, and assume that D' covers D in $\mathcal{T}_{n+1}$. Let us denote by $(d_j)_j$ and (d'_j) the sequences defined by $d_j = c_{n-j}(D)$ and $d'_j = c_{n-j}(D')$, by hypothesis $(d_j)_j$ and $(d'_j)_j$ satisfy the three conditions of Proposition 5.2.12. Let us keep the notations of this proposition, and let i and k be the two integers which appear in Proposition 5.2.12.

Our aim is now to prove that there exists an admissible column C above A such that $A' = A \cup C$. For that purpose, let us first prove that there is an admissible box above A in row i. Assume by contradiction that $(i-1, i)$ is in A, then for all $p < i-1$ such that $(p, i-1) \in A$, we have $(p, i) \in A$ by Lemma 5.2.6 (2). Thus, we have $d_i > d_{i-1}$ and this contradicts point (1) of Proposition 5.2.12, hence $(i-1, i) \notin A$. Therefore, there exists a box (q, i) which is admissible above A (note that we possibly have $(q, i) = (i-1, i)$).

Let us now show that $\{(q,i), (q, i+1), \ldots, (q, i+k-1)\}$ is admissible above A. Equivalently, we will show that the height of the colum i of A is $i+k$ by dividing our study into two cases.

- Case $k = 1$: assume by contradiction that $(i, i+1)$ is in A. Then, for all $p < i$ such that $(p, i) \in A$, we have $(p, i+1) \in A$, which implies $d_i < d_{i+1} = d_{i+k}$ and thus contradicts point (2) of Proposition 5.2.12. Therefore, $(i, i+1)$ is not in A so the height of the column i of A is $i+1$. Thus, $\{(q, i)\}$ is an admissible column above A.
- Case $k > 1$: we will show by induction on j, where j ranges strictly between i and $k+i$, that the following two properties are true:
  - $(i, j) \in A$;
  - for all $p < i$ such that $(p, i) \in A$, we have $(p, j) \in A$.

  First, note that for all $p < i$ such that $(p, i+1) \in A$ we have $(p, i) \in A$, and since $d_{i+1} > d_i$, we have $(i, i+1) \in A$. Thus, the properties are true for $j = i+1$.

  Let j be such that the properties are true and such that $j < k+i-1$. Assume by contradiction that $(i, j+1)$ is not in A. By induction hypothesis, all the boxes below $(i, j+1)$ are in A. Consequently, there is no box in the same row and on the right of $(i, j+1)$ by Lemma 5.2.6, hence for any box in A different from $(i, j+1)$ and in row $j+1$, there exists a box below it in row i which is also in A. Thus, we have $d_{j+1} \le d_i$ and this is absurd. We thus have $(i, j+1) \in A$. Let us now ow consider an integer $p < i$ such that $(p, i) \in A$. Then, by induction hypothesis all the boxes below $(p, j+1)$ are in A, and $(i, j+1)$ is on the right of (i, p). Thus, $(p, j+1)$ is in A and this concludes the induction.

  We can now prove that $i+k$ is the height of column i. Assume by contradiction that $(i, i+k) \in A$. Then, thanks to similar arguments as in (case $k = 1$), we have that $d_{i+k} > d_i$ contradicting point (2) of Proposition 5.2.12. Thus, $i+k$ is the height of column i. Therefore, the set $\{(q, i+1), \ldots, (q, i+k-1)\}$ is an admissible column above A.

In all cases, $C = \{(q, i+1), \ldots, (q, i+k-1)\}$ is admissible above A. Moreover, $A \cup C$ has the same code as A', so that $A' = A \cup C$ thanks to the fact that Ψ is a bijection. Thus, A' covers A in $(IS(\mathcal{A}_n^{\uparrow}), \subseteq)$, and this concludes the proof. □

This theorem has the following immediate corollary, which concludes this section.

COROLLARY 5.2.14. *The posets $\mathcal{T}_{n+1}$ and $(IS(\mathcal{A}_n^{\uparrow}), \subseteq)$ are isomorphic.*

5.2.3. A similar description of m-Tamari lattices. As it is mentioned in [**BMFPR**], the n-th m-Tamari lattice $\mathcal{T}_n^{(m)}$ can be seen as an interval in $\mathcal{T}_{mn}$. More precisely, let us consider the minimal m-ballot bath D of $\mathcal{T}_n^{(m)}$ and replace each up-step of D by a sequence of m consecutive up-step, as depicted in Figure 5.13.

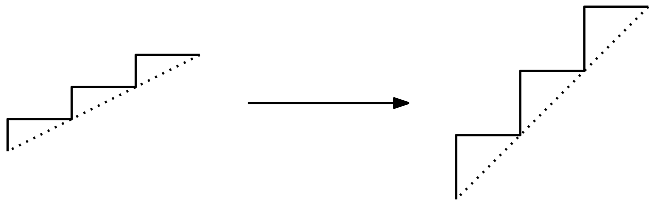


FIGURE 5.13.

Clearly, the resulting path D' is a Dyck path of size nm, and one can easily show that the sub-poset of $\mathcal{T}_{mn}$ made of all the paths bigger than D' is isomorphic to $\mathcal{T}_n^{(m)}$. With this property, it is clear that m-tamari lattices can be described using a valued digraph, as described in Figure 5.14.

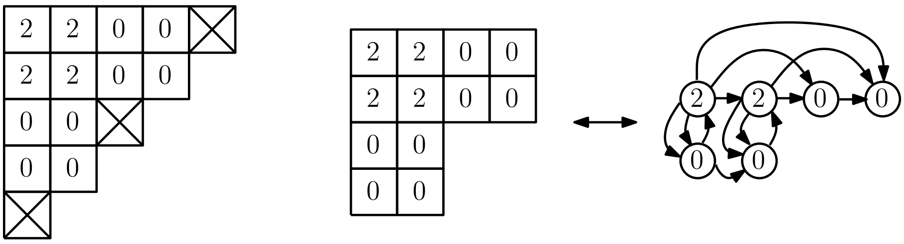


FIGURE 5.14. How to obtain a valued digraph associated with $\mathcal{T}_3^{(2)}$. The (non-simple) valued digraph on the right also gives a lattice isomorphic to $\mathcal{T}_3^{(2)}$.

5.3. Proofs

In this section, we provide the proofs of Proposition 5.2.9 and 5.2.11.

5.3.1. Proposition 5.2.9.

PROOF OF PROPOSITION 5.2.9. We will show this assertion recursively on n. First, note that it is obvious for $n = 1$. Let n be such that the property is true.

Step 1 (injectivity): let A and A' be in $IS(\mathcal{A}_{n+1}^{\uparrow})$, with respective codes $(c_i)_{1\leq i\leq n+1}$ and $(c'_i)_{1\leq i\leq n+1}$. Assume that $\Psi(A) = \Psi(A')$, then we have $c_i = c'_i$ for all $i \in [n+1]$. We will prove that $A = A'$. For this purpose, let us denote by $\widetilde{A}$ and $\widetilde{A}'$ the two subsets of $\mathcal{V}(\mathcal{A}_n^{\uparrow})$ defined by

$$\widetilde{A} := A \cap \mathcal{V}(\mathcal{A}_n^{\uparrow}),$$
$$\widetilde{A}' := A' \cap \mathcal{V}(\mathcal{A}_n^{\uparrow}).$$

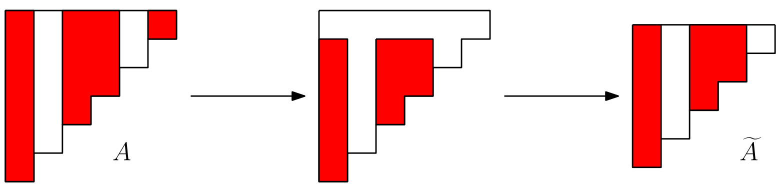


FIGURE 5.15. From A to $\widetilde{A}$.

We now show that both $\widetilde{A}$ and $\widetilde{A}'$ are in $IS(\mathcal{A}_n^{\uparrow})$ considering a box $\mathfrak{c}$ in $\mathcal{V}(\mathcal{A}_n^{\uparrow})$. If $\mathfrak{c} \in \widetilde{A}$, then $\mathfrak{c} \in A$. Consequently, all the boxes below $\mathfrak{c}$ are in A, so that they are also in $\widetilde{A}$. Thus, we have $\theta(\mathfrak{c}) \leq d^+_{\widetilde{A}}(\mathfrak{c})$. If $\mathfrak{c} \notin \widetilde{A}$, then $\mathfrak{c} \notin A$, so that we have $\theta(\mathfrak{c}) \geq d^+_A(\mathfrak{c}) \geq d^+_{\widetilde{A}}(\mathfrak{c})$ thanks to the fact that $\widetilde{A} \subseteq A$. This proves that $\widetilde{A} \in IS(\mathcal{A}^{\uparrow}_{n+1})$, and we prove similarly that $\widetilde{A}' \in IS(\mathcal{A}^{\uparrow}_{n+1})$. Clearly, the sequence $(c_i)_{1\leq i\leq n}$ is the code of both $\widetilde{A}$ and $\widetilde{A}'$, hence by induction hypothesis we have $\widetilde{A} = \widetilde{A}'$.

We are now able to prove that $A = A'$. Let us denote by $a_1 < a_2 < \ldots < a_k$ all the indices such that $(a_j, n+1) \in A$ and denote by a_{k+1} the integer $n+1$. Thanks to Lemma 5.2.6, a box which is both in A and in row $n+2$ is in the set

$$\{(a_1, n+2), (a_2, n+2), \ldots, (a_k, n+2), (a_{k+1}, n+2)\}.$$

Assume that $(a_i, n+2) \in A$, then for all $1 \leq j < i$ we have $(a_j, n+2) \in A$ thanks to Lemma 5.2.6 (2). This means that the boxes which are in A in row $n+2$ are all in the leftmost positions available (with respect to Lemma 5.2.6 (1)). In particular, this implies that A and A' are equal since they have same number of boxes in row $n+2$.

Step 2 (surjectivity): Let $D \in \mathrm{Dyck}(n+2)$ and $(c_i)_{1\leq i\leq n+1}$ be its code. Clearly, there exists $D' \in \mathrm{Dyck}(n+1)$ whose code is given by $(c_i)_{2\leq i\leq n+1}$. Thus, by induction hypothesis there exists $A' \in IS(\mathcal{A}^{\uparrow}_n)$ such that $\Psi(A') = D'$, and A' can also be seen as an element of $IS(\mathcal{A}^{\uparrow}_{n+1})$, as depicted in Figure 5.16. As in Step 1, let us denote by $a_1 < a_2 < \ldots < a_k$ all the indices

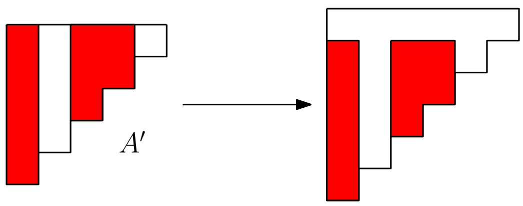


FIGURE 5.16.

such that $(a_j, n+1) \in A'$ and we set $a_{k+1} = n+1$. Let us denote by A the subset of $\mathcal{V}(\mathcal{A}^{\uparrow}_{n+1})$ defined by

$$A := A' \cup \{(a_1, n+2), (a_2, n+2), \ldots, (a_{c_1}, n+2)\}.$$

By construction, if A is in $IS(\mathcal{A}^{\uparrow}_{n+1})$, then $\Psi(A) = D$. We still have to prove that $A \in IS(\mathcal{A}^{\uparrow}_{n+1})$. In order to do so, we consider $\mathfrak{c} = (a, b) \in \mathcal{V}(\mathcal{A}^{\uparrow}_{n+1})$ and we split our study into four cases.

- If $\mathfrak{c} \in A'$, then $\theta(\mathfrak{c}) \leq d^+_{A'}(\mathfrak{c}) \leq d^+_A(\mathfrak{c})$ since $A' \subseteq A$.
- If $\mathfrak{c} \notin A$ and is not in the $(n+2)$-th row, then $d^+_A(\mathfrak{c}) = d^+_{A'}(\mathfrak{c})$. Thus, we have $\theta(\mathfrak{c}) \geq d^+_A(\mathfrak{c})$.
- If $\mathfrak{c} \in A$ and $b = n+2$, then all the boxes below $\mathfrak{c}$ are in A. Therefore, we have $\theta(\mathfrak{c}) \leq d^+_A(\mathfrak{c})$.
- If $\mathfrak{c} \notin A$ and $b = n+2$, then we have two sub-cases.
  - If all the boxes below $\mathfrak{c}$ are in A, then by construction of A from A', we have that all the boxes on the right of $\mathfrak{c}$ are not in A. Hence $d^+_A(\mathfrak{c}) = b - a - 1 = \theta(\mathfrak{c})$.
  - If there exists a box in the same column and below $\mathfrak{c}$ which is not in A, then we consider $\mathfrak{d} = (a, k)$ with k minimal such that $\mathfrak{d} \notin A$. By hypothesis, we have $k \leq n+1$, so that $\mathfrak{d} \notin A'$. Assume by contradiction that there exists an integer j such that $a < j < k$ and $(j, n+2) \in A$. Then, by construction of A we have $(j, k) \in A$, and thus we have $(j, k) \in A'$. However, by minimality of k all the boxes in the same column and below $\mathfrak{d}$ are in A', and (j, k) is in the same row and on the right of $\mathfrak{d}$. We thus have that $\mathfrak{d}$ is in A', which is absurd. Therefore, we have $(j, n+2) \notin A$ for all $a < j < k$. Thus, the following inequality holds:

$$\begin{aligned} d^+_A(\mathfrak{c}) &\leq |\{(a, a+1), (a, a+2), \ldots, (a, k-1), (k, n+2), (k+1, n+2), \ldots, (n+1, n+2)\}| \\ &\leq (k - 1 - a) + (n + 2 - k) = (n+2) - a - 1 = \theta(\mathfrak{c}). \end{aligned}$$

The conclusion of this case by case study is that A is in $IS(\mathcal{A}^{\uparrow}_{n+1})$, and this ends the proof. $\square$

5.3.2. Proof of Proposition 5.2.11. We split the proof of this Proposition into two distinct lemmas.

LEMMA 5.3.1. *Let $A, A' \in IS(\mathcal{A}_n^{\uparrow})$. If $A \neq A'$ and $A \subset A'$, then there exists C an admissible column above A which is included in A'.*

PROOF. By hypothesis, there exists $\mathfrak{d} = (a, b) \in A' \setminus A$. Let us denote by k the eight of the column b of A. By definition of the height, we have $(a, k) \in A' \setminus A$, so that there exists $i \leq k$ such that (i, k) is admissible above A. Then, all the boxes in the same column and below (i, k) are in A, hence they are in A'. Moreover, (a, k) is in A' and (a, k) is on the right of (i, k), hence $(i, k) \in A'$ by Lemma 5.2.6.

We now set q the height of the column k of A and we show by induction on j, $k \leq j < q$, that $(i, j) \in A'$. Notice that we already know that $(i, k) \in A'$. Fix j such that the property is true and such that $j + 1 < q$. We have that all the boxes below and in the same column as $(i, j + 1)$ are in A'. Moreover, by definition of q we have $(k, j + 1) \in A'$, and $(k, j + 1)$ is on the right and in the same row as $(i, j + 1)$. As a consequence, we have $(i, j + 1) \in A'$. This proves that

$$\{(i, k), (i, k + 1), \ldots, (i, q - 1)\} \subseteq A',$$

and this set is an admissible column above A by definition. □

We now prove a kind of converse implication to Lemma 5.3.1.

LEMMA 5.3.2. *Let $A \in IS(\mathcal{A}_n^{\uparrow})$ and C be an admissible column above A, then $A \cup C$ is in $IS(\mathcal{A}_n^{\uparrow})$.*

PROOF. Let $\mathfrak{c} \in V(\mathcal{A}_n^{\uparrow})$. If $\mathfrak{c} \in A$, then $\theta(\mathfrak{c}) \leq d^+_A(\mathfrak{c}) \leq d^+_{A \cup C}(\mathfrak{c})$. If $\mathfrak{c} \in C$, then all the boxes below and in the same column as $\mathfrak{c}$ are in $A \cup C$, hence $\theta(\mathfrak{c}) \leq d^+_{A \cup C}(\mathfrak{c})$. If $\mathfrak{c} \notin A \cup C$, then we have three cases.

- If $\mathfrak{c} = (a, b)$ is above and in the same column as C, then consider (a, q) the unique box which is in the column a and admissible above A, and denote by p the height of the column q in A. By construction, for all integers k such that $p \leq k < b$ we have $(a, k) \notin A \cup C$. Now consider (k, b) in A such that $a < k$, *i.e.* (k, b) is on the right and in the same row as $\mathfrak{c}$. Clearly, $k \neq q$ (otherwise p could not be the height of the column q in A). Assume by contradiction that $a < k < q$, then by Lemma 5.2.6, (k, q) is in A, but (a, q) is admissible in A, thus all the boxes below (a, q) are in A and there is an arc from (a, q) to (k, q), hence $\theta(a, q) < d^+_A(a, q)$, and this contradicts the fact that $A \in IS(\mathcal{A}_n^{\uparrow})$. Therefore we have $k > q$. Assume, again by contradiction, that $q < k < p$, then we similarly have that $(k, p) \in A$, but all the boxes below and in the same column as (q, p) are in A, and there is an arc from (q, p) to (k, p), hence (q, p) is in A, and this contradicts the fact that the height of the column q is p. As a consequence $k \geq p$, and we have the following inequality:

$$\begin{aligned} d^+_{A \cup C}(a, b) &\leq |\{(a, a + 1), \ldots, (a, p - 1), (p, b), \ldots, (b - 1, b)\}| \\ &\leq (p - a - 1) + (b - p) = b - a - 1 = \theta(a, b). \end{aligned}$$

- If there exists $\mathfrak{d} = (q, b) \in C$ which is on the right and on the same row as $\mathfrak{c} = (a, b)$, then we have two sub-cases.
  - If there exists p such that $a < p < q$ and $(p, b) \in A$, then by Lemma 5.2.6 $(p, p+1)$ is in A. Assume by contradiction that (a, p) is in A, then all the boxes below and in the same column as $(a, p + 1)$ are in A, and $(p, p + 1)$ is a box on the right and in the same row as $(a, p + 1)$ which is in A, thus $(a, p + 1)$ is in A. By induction, we show that $(a, p + 2)$ is in A, and so on, until we reach (a, b), and thus (a, b) is in A, which is absurd. Therefore, if there exists p (minimal) such that $a < p < q$

and $(p,b) \in A$, then the following inequality holds:

$$\begin{aligned} d^+_{A\cup C}(a,b) &\leq |\{(a,a+1),\ldots,(a,p-1),(p,b),\ldots,(b-1,b)\}| \\ &\leq (p-1-a)+(b-p) = \theta(a,b). \end{aligned}$$

– If for all p such that $(p,b) \in A \cup C$, we have $p \geq q$. Assume by contradiction that (a,q) is in A and consider $(a,q+1)$: we have two possibilities, either $(q,q+1) \in A$ and this implies that $(a,q+1)$ is also in A (since all the boxes below it are in A), or $(q,q+1) \notin A$ and then $(a,q+1) \in A$ (since $(q,q+1)$ is admissible above A by definition). Using the same type of arguments as in the previous case, we show by induction that (a,b) is also in A, and this is a contradiction. Consequently, we have

$$\begin{aligned} d^+_{A\cup C}(a,b) &\leq |\{(a,a+1),\ldots,(a,q-1),(q,b),\ldots,(b-1,b)\}| \\ &\leq \theta(a,b). \end{aligned}$$

- In all the other cases, $d^+_A(\mathfrak{c}) = d^+_{A\cup C}(\mathfrak{c})$ so $\theta(\mathfrak{c}) \geq d^+_{A\cup C}(\mathfrak{c})$.

The conclusion of this case by case study is that $A \cup C$ is in $IS(\mathcal{A}^{\uparrow}_n)$, which ends the proof. □

Eventually, Proposition 5.2.11 follows immediately from Lemma 5.3.1 and 5.3.2.

CHAPTER 6

Application to tableaux combinatorics

Introduction

In this chapter we present the problem that initially motivated the introduction of the notions developed in this thesis, namely the combinatorics of tableaux. More precisely, we will generalize some of the results from [**FGRS**] and [**EG**] relative to the combinatorial properties of balanced tableaux to a wider class of tableaux.

6.1. Valued digraphs and type of a tableau

In this section we explain how the construction made in Chapter 3 leads to a new classification of tableaux, associating to each tableau a combinatorial object that we called its *type*. Note that both sets of standard and balanced tableaux of a given shape can be seen as special classes in our classification.

6.1.1. Type of a tableau. In Chapter 3, we constructed various examples of valued digraphs using suitable notions of hooks. In this chapter, we go back to the usual notion of hook, as used in Section 2.3.1 to study weak order on A_{n-1}.

Definition 6.1.1. Let S be a diagram and $\mathfrak{c} = (a, b)$ be a box of S. We define the following sets,

$$L_S(a,b) := \{(k,b) \mid k \geq a, (k,b) \in S\}, \; A_S(a,b) := \{(a,k) \mid k > b, (a,k) \in S\}, \tag{6.1}$$

$$\text{and } H_S(a,b) := A_{a,b} \biguplus L_{a,b}, \tag{6.2}$$

respectively called the *leg*, the *arm* and the *hook* based on $\mathfrak{c}$. We denote by $l_S(a,b)$, $\mathtt{a}_S(a,b)$, and $h_S(a,b)$ their respective cardinalities.

With this notion of hook, we can associate a digraph with each diagram.

Definition 6.1.2. Let S be a diagram, we denote by G_S the simple acyclic digraph defined by

$$\mathcal{V}(G_S) := S \text{ and } E(G_S) := \{(\mathfrak{c}, \mathfrak{d}) \mid \mathfrak{c} \neq \mathfrak{d} \text{ and } \mathfrak{d} \in H_S(\mathfrak{c})\}.$$

Eventually, we are now able to associate a valued digraph to each tableau of a given shape.

Definition 6.1.3. Let S be a diagram, a valued digraph whose underlying digraph is G_S is called a *type* of shape S. We denote by $\mathrm{Type}(S)$ the set of all the types of shape S, and the shape of the underlying diagram of any type $\mathcal{T}$ is denoted by $\mathrm{Sh}(\mathcal{T})$. Let $T = (t_\mathfrak{c})_{\mathfrak{c} \in S} \in \mathrm{Tab}_S$, then the type of T is the type $\mathcal{T} = (G_S, \theta) \in \mathrm{Type}(S)$ whose valuation is given by

$$\theta(\mathfrak{c}) = |\{\mathfrak{d} \in H_S(\mathfrak{c}) \mid t_\mathfrak{d} < t_\mathfrak{c}\}| \text{ for all } \mathfrak{c} \in S.$$

We denote by $\mathrm{Tab}_S(\mathcal{T})$ the set of all tableaux of shape S whose type is $\mathcal{T}$. When there is no ambiguity, we simply denote this set by $\mathrm{Tab}(\mathcal{T})$.

Notice that this definition contains the definition of standard Young tableaux. Let λ be a partition of n and consider the type $\mathcal{S}t_\lambda = (G_\lambda, \theta)$ where for all $\mathfrak{c} \in \lambda$ we have $\theta(\mathfrak{c}) = 0$. Then, it is clear that $\mathrm{Tab}(\mathcal{S}t_\lambda) = \mathrm{SYT}(\lambda)$. Our definition also contains that of balanced tableaux, and we will detail this in Section 6.2.

$$\begin{array}{c} 4 \end{array} \qquad \begin{array}{cc} 1 & 0 \end{array} \qquad \begin{array}{cccccc} 10 & 7 & 6 & 4 & 3 & 2 \\ 8 & 5 & 4 & 2 & 1 & \\ 7 & 4 & 3 & 1 & 0 & \\ 4 & 1 & 0 & & & \\ 1 & & & & & \\ 0 & & & & & \end{array} \qquad \begin{array}{cccccc} 5 & 4 & 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 & \\ 4 & 3 & 2 & 1 & 0 & \\ 2 & 1 & 0 & & & \\ 0 & & & & & \\ 0 & & & & & \end{array}$$

$$\begin{array}{cc} 1 & 0 \\ 0 & \end{array}$$

FIGURE 6.1. The types of the tableaux in Figure 1.

6.1.2. Some general results about types. In this section, we focus on some general properties of the notion of types. Let us first explain how one can construct each tableaux of a given type using the peeling process (see Chapter 3, Definition 3.1.3).

PROPOSITION 6.1.4. *Let S be a diagram, $\mathcal{T} = (G_S, \theta)$ be a type of shape S and $L = [\mathfrak{c}_1, \ldots, \mathfrak{c}_n] \in PS(\mathcal{T})$. Then, the tableau $T_L := (t_\mathfrak{c})_{\mathfrak{c} \in S}$ defined by*

$$t_{\mathfrak{c}_i} = i \text{ for all } i \in [n],$$

is of type $\mathcal{T}$, and all tableaux of type $\mathcal{T}$ can be obtained by this way.

PROOF. Thanks to Proposition 4.2.3 in Chapter 4, this is immediate. □

COROLLARY 6.1.5. *Let S be a diagram and $\mathcal{T} = (G_S, \theta)$ be a type of shape S, then* $\mathrm{Tab}(\mathcal{T}) \neq \emptyset$.

We give an example to visualise a dynamic construction of the tableau T_L associated with a peeling sequence L.

EXAMPLE 6.1.6. Consider the type $\mathcal{T}$ on the top-left of Figure 6.2 and the peeling sequence $L = [(1,3); (1,1); (1,2); (2,2); (2,1)]$ (since we represent Ferrers diagram with the English convention, we use the matrix coordinates for each box). The types on the top of the figure are the types obtained after each iteration of the peeling process.

$\mathcal{T} = \mathcal{P}^0$	$\mathcal{P}^1$	$\mathcal{P}^2$	$\mathcal{P}^3$	$\mathcal{P}^4$	$\mathcal{P}^5$
$\begin{array}{ccc} 2 & 1 & 0 \\ 1 & 0 & \end{array}$	$\begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array}$	$\begin{array}{cc} 0 & \\ 1 & 0 \end{array}$	$\begin{array}{cc} 1 & 0 \end{array}$	$\begin{array}{c} 0 \end{array}$	
$\begin{array}{ccc} \square & \square & \square \\ \square & \square & \end{array}$	$\begin{array}{ccc} \square & \square & 1 \\ \square & \square & \end{array}$	$\begin{array}{ccc} \square & 2 & 1 \\ \square & \square & \end{array}$	$\begin{array}{ccc} 3 & 2 & 1 \\ \square & \square & \end{array}$	$\begin{array}{ccc} 3 & 2 & 1 \\ \square & 4 & \end{array}$	$\begin{array}{ccc} 3 & 2 & 1 \\ 5 & 4 & \end{array}$ T_L

FIGURE 6.2.

Now that we have a classification of all the tableaux of a given shape according to their type and a way to construct all tableaux in a given class, the following natural question arises.

QUESTION 6.1.7. For any type $\mathcal{T}$, is it possible to find a formula to compute $|\mathrm{Tab}(\mathcal{T})|$?

Even if the general case seems to be quite difficult, we have some basic properties in that direction that we now detail.

First, note that if $\lambda = (n)$ or $\lambda = (1^n)$, then for any $\mathcal{T} \in \mathrm{Type}(\lambda)$ there exists a unique tableau of type $\mathcal{T}$. This is clear, since at each iteration of the peeling process there is only one $(i,j) \in \lambda$ which is erasable. This basic fact leads us to our first enumerative proposition, generalizing Lemma 3.2 from **[EG]**.

PROPOSITION 6.1.8. *Let k and p be two integers and $\mathcal{T} = (G_\lambda, \theta)$ be a type of shape $\lambda = (k, 1^p)$. Then, we have*

$$|\mathrm{Tab}(\mathcal{T})| = f^\lambda.$$

PROOF. First, note that for any tableau $T = (t_{\mathfrak{c}})_{\mathfrak{c}\in\lambda}$ of type $\mathcal{T}$, we have $t_{1,1} = n - \theta_{1,1}$ by definition. Thus, if we set $S := \lambda \setminus \{(1,1)\}$ (see Figure 6.3) and $\mathcal{T}' := (G_S, \theta)$, then we have

$$|\mathrm{Tab}_\lambda(\mathcal{T})| = |\mathrm{Tab}_S(\mathcal{T}')|.$$

Moreover, when we perform the peeling process on $\mathcal{T}'$, the only thing we have to chose at each

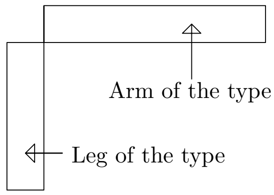


FIGURE 6.3.

step is an element in the leg or in the arm of S and this is independent of the choice of $\mathcal{T}$. This concludes the proof. $\square$

This first result might lead us to think that there should be a simple way to answer Question 6.1.7, such as a general "hook-length formula", but a quick verification shows that the situation seems to be considerably more complicated. For example, the number of tableaux of type

0	0	0
2	0	
0		

is 11 which does not divide $6! = 720$. Nevertheless, we have a sort of probabilist result.

PROPOSITION 6.1.9. *Let S be a diagram and* $\mathrm{Type}(S)$ *the set of all types of shape S, if we choose uniformly a type $\mathcal{T}$ in* $\mathrm{Type}(S)$, *then the expected value for* $|\mathrm{Tab}_S(\mathcal{T})|$ *is*

$$\frac{n!}{\prod_{\mathfrak{c}\in S} h_S(\mathfrak{c})}.$$

PROOF. Set $H_S = \prod_{\mathfrak{c}\in\lambda} h_S(\mathfrak{c})$, it is clear that the number of types of shape S is precisely H_S. Then, because of the uniform choice, the probability for a type $\mathcal{T}$ to be chosen is exactly $\frac{1}{H_S}$. Thus, the expected value for $|\mathrm{Tab}(\mathcal{T})|$ is

$$\frac{\sum_{\mathcal{T}\in\mathrm{Type}(S)} |\mathrm{Tab}_S(\mathcal{T})|}{H_S},$$

and the numerator clearly equals $n!$. The result follows. $\square$

This last proposition leads us to the following natural question.

QUESTION 6.1.10. Is it possible to find an explicit formula for the variance ?

This last question is open, and it seems that the value of the variance heavily depends on the shape of the considered diagram.

6.2. Types and reduced decompositions

As it has been mentioned in the previous section, it seems to be difficult to find a general formula to compute the value of $|\mathrm{Tab}(\mathcal{T})|$ for any given type $\mathcal{T}$. However, in the rest of this chapter we will exhibit, for any partition λ, a family of types of shape λ whose corresponding tableau are enumerated by f^λ. For that purpose, we will adapt the approach of **[EG]** and **[FGRS]** to our terminology (see Section 6.2.2), and then generalize their results to a wider class of tableaux (Section 6.3).

6.2.1. Balanced tableaux. In this section we recall the definition of balanced tableaux and we briefly explain some of the results of [**FGRS**] and [**EG**]. Let us begin with the definition of balanced tableaux, which is a straightforward reformulation of the one given in [**FGRS**] and [**EG**] using our terminology.

Definition 6.2.1. Let S be a diagram and denote by $\mathcal{T}_S^{bal} = (G_S, \theta)$ the type of shape S such that

$$\theta(\mathfrak{c}) := \mathsf{a}_S(\mathfrak{c}) \text{ for all } \mathfrak{c} \in S.$$

The element of $\mathrm{Tab}(\mathcal{T}_S^{bal})$ are called the *balanced tableaux* of shape S. When S is the Ferrers diagram of a partition λ, we denote by $\mathrm{Bal}(\lambda)$ the set of the balanced tableaux of shape λ.

In [**EG**] the authors proved the following result about combinatorics of balanced tableaux.

Theorem 6.2.2 ([**EG**], Theorem 2.2). *Let λ be a partition of n. Then, we have*

$$|\mathrm{Bal}(\lambda)| = |SYT(\lambda)| = f^{\lambda}.$$

The original proof is quite involved, and an alternative one is given in [**FGRS**], which we now detail. The main ingredient is the notion of vexillary permutations.

Definition 6.2.3. Let $\sigma \in S_n$, we denote by $(d_i(\sigma))_i$ and $(g_i(\sigma))_i$ the finite sequences defined by

- $d_i(\sigma) := |\{j > i \mid \sigma(j) < \sigma(i)\}|$,
- $g_i(\sigma) := |\{j < i \mid \sigma(j) > \sigma(i)\}|$.

We denote by $\mu(\sigma)$ and $\lambda(\sigma)$ the partitions obtained by rearranging in a nonincreasing order the sequences $(d_i)_i$ and $(g_i)_i$, respectively. We say that σ is *vexillary* if and only if $\lambda(\sigma) = \mu'(\sigma)$.

In [**S2**], Stanley proved the following result using symmetric functions, giving an explicit formula to compute the number of reduced decompositions of any vexillary permutation.

Theorem 6.2.4 (Stanley, [**S2**]). *Let $\sigma \in S_n$, if σ is vexillary then*

$$|\mathrm{Red}(\sigma)| = f^{\lambda(\sigma)}.$$

Let us now explain the proof of Theorem 6.2.2 that can be found in [**FGRS**], using Theorem 6.2.4 as fundamental tool. The first step consists in associating a diagram to each permutation.

Definition 6.2.5. Let $\sigma \in S_n$, the *Rothe diagram* $\mathbf{D}(\sigma)$ of σ is the subset of $[n] \times [n]$ defined by

$$\mathbf{D}(\sigma) := \{(a, \sigma(b)) \in [n] \times [n] \mid a < b \text{ and } \sigma(a) > \sigma(b)\}.$$

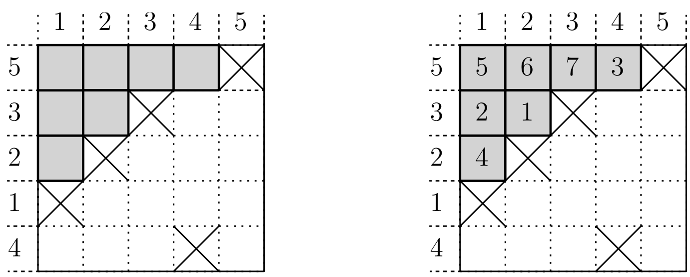


Figure 6.4. Rothe diagram of the permutation $\sigma = [5, 3, 2, 1, 4]$ (on the left) and a balanced tableau of shape $\mathbf{D}(\sigma)$ (on the right).

As it is shown in [**FGRS**], the balanced tableaux whose shape is the Rothe diagram of a given permutation σ (also called *balanced labellings* of $\mathbf{D}(\sigma)$) are intimately related to the reduced decompositions of σ.

THEOREM 6.2.6 ([**FGRS**], Theorem 2.4). *Let $\sigma \in S_n$ and $\mathbf{D}(\sigma)$ be its Rothe diagram. Then, there is a bijection between the set of the reduced decompositions of σ and the set of the balanced tableaux of shape $\mathbf{D}(\sigma)$.*

Let us briefly explain how this bijection is constructed. By definition of the Rothe diagram of a permutation $\sigma \in S_n$, it is clear that

$$(a, \sigma(b)) \in \mathbf{D}(\sigma) \text{ if and only if } (a, b) \in \mathrm{Inv}(\sigma).$$

Thus, a balanced tableaux corresponds to an ordering $[(a_1, b_1), \ldots, (a_{\ell(\sigma)}, b_{\ell(\sigma)})]$ of the inversions of σ. Furthermore, the authors proved in [**FGRS**] that $\{(a_1, b_1), \ldots, (a_i, b_i)\}$ is the inversion set of a permutation $\sigma_i \in S_n$ for all $1 \leq i \leq \ell(\sigma)$. Therefore, we have

$$Id \lhd_R \sigma_1 \lhd_R \sigma_2 \lhd_R \ldots \lhd_R \sigma_{\ell(\sigma)} = \sigma,$$

i.e., a balanced tableau of shape $\mathbf{D}(\sigma)$ corresponds to a maximal chain from Id to σ in $(S_n, \leq_R)$. Thus, it corresponds to a reduced decomposition of σ, and it is proved in [**FGRS**] that this correspondence is in bijective.

Eventually, this correspondence leads to a proof of Theorem 6.2.2.

THEOREM 6.2.7 ([**FGRS**], Theorem 3.4). *Let λ be a partition of n. Then, there exists a vexillary permutation $\sigma \in S_k$ for some $k \in \mathbb{N}$ such that*

- $\lambda(\sigma) = \lambda$;
- *the shape of $\mathbf{D}(\sigma)$ is λ (up-to the deletion of some empty columns).*

Combining Theorem 6.2.4 and Theorem 6.2.7, Theorem 6.2.2 follows immediately.

6.2.2. A similar approach using valued digraphs. We introduced in Chapter 3 an interpretation of weak order on A_{n-1} using a valued digraph, and we detail in this section how this description lead to an interpretation of the reduced decomposition of any permutation σ in terms of tableaux. First, recall that the valued digraph $\mathcal{A} = (G, \theta)$ defined by

$$\begin{aligned} \mathcal{V}(\mathcal{A}) &:= \lambda_n = (n-1, n-2, \ldots, 2, 1), \\ E(\mathcal{A}) &:= \{(\mathfrak{c}, \mathfrak{d}) \mid \mathfrak{c} \neq \mathfrak{d} \text{ and } \mathfrak{d} \in H_{\lambda_n}(\mathfrak{c})\}, \\ \theta(a, b) &:= b - a - 1 \text{ for all } (a, b) \in \lambda_n, \end{aligned}$$

provides a combinatorial description of $(A_{n-1}, \leq_R)$. That is, we have

$$IS(\mathcal{A}) = \{\mathrm{Inv}(\sigma) \mid \sigma \in S_n\}. \tag{6.3}$$

This description naturally leads to associate each permutation with a type.

DEFINITION 6.2.8. Let $\sigma \in S_n$, we denote by $\mathcal{T}_\sigma = (G_\sigma, \theta_\sigma)$ the type such that

- $\mathrm{Sh}(\mathcal{T}_\sigma) = \mathrm{Sh}(\mathrm{Inv}(\sigma))$ where $\mathrm{Inv}(\sigma)$ is seen as an element of $IS(\mathcal{A})$, *i.e.* a sub diagram of λ_n;
- for all $\mathfrak{c} \in \mathrm{Inv}(\sigma)$, $\theta_\sigma(\mathfrak{c}) = \theta(\mathfrak{c})$.

Since there is no ambiguity, we will usually denote by θ the valuation associated with $\mathcal{T}_\sigma$.

PROPOSITION 6.2.9. *Let $\sigma \in S_n$, then we have*

$$|\mathrm{Tab}(\mathcal{T}_\sigma)| = |\mathrm{Red}(\sigma)|.$$

PROOF. The proof is straightforward: by Proposition 6.1.4 we have $|\mathrm{Tab}(\mathcal{T}_\sigma)| = |PS(\mathcal{T}_\sigma)|$, and thanks to (6.3) we have a one-to-one correspondence between $PS(\mathcal{T}_\sigma)$ and the set of maximal chains from Id to σ in $(A_{n-1}, \leq_R)$. However, there is a one-to-one correspondence between these maximal chains and reduced decompositions of σ. This ends the proof. $\square$

The benefit of this interpretation are twofold. First, it allows us to easily construct all the reduced decompositions of any permutation using the peeling process, as depicted on Figure 6.5. Moreover, it allows us to provide an interpretation of the reduced decompositions of any permutation in terms of tableaux, as represented on Figure 6.6.

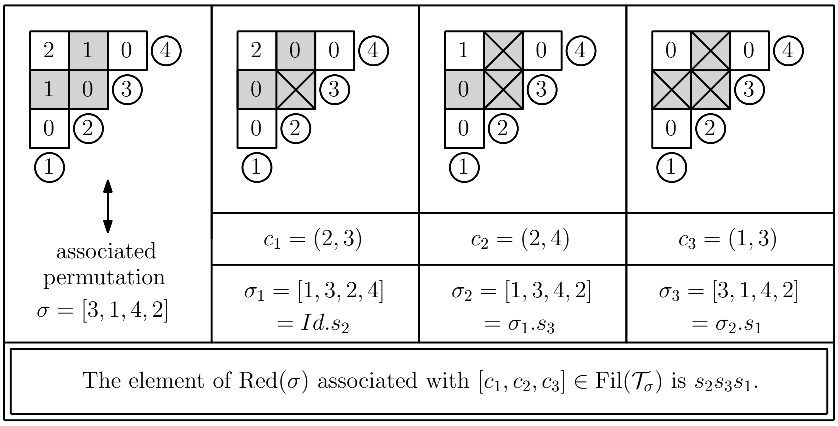


FIGURE 6.5.

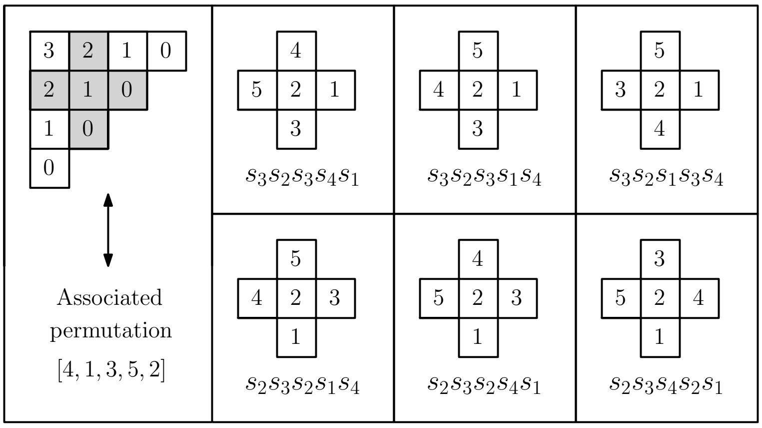


FIGURE 6.6. The set $\mathrm{Red}([4,1,3,5,2])$ seen as a set of tableaux.

Note that these two points are also true for the tableaux of [**FGRS**]: for any permutation $\sigma \in S_n$, the balanced tableaux of shape $\mathbf{D}(\sigma)$ already provide an interpretation of $\mathrm{Red}(\sigma)$ in terms of tableaux, and one can easily construct all such balanced tableaux using the peeling process. However, the benefit of our description lies on two points:

- our description provides a description of the weak order on S_n, *i.e.* it is straightforward to check whenever two permutations are comparable using their corresponding type;
- as it will be detailed in the next section, the type associated with a tableau allows us to use combinatorial techniques to construct bijections between sets of tableaux.

6.3. Study of vexillary permutations using their type

Our aim in this section is to generalize Theorem 6.2.2. That is, we will associate each vexillary permutation σ with a type $\mathcal{T}_\sigma^E$ whose shape is $\lambda(\sigma)$ and such that

$$|\mathrm{Tab}(\mathcal{T}_\sigma^E)| = |\mathrm{Red}(\sigma)| = f^{\lambda(\sigma)}. \tag{6.4}$$

For that purpose, we need to introduce new tools. Indeed, our starting point will be the type $\mathcal{T}_\sigma$, but this type is generally not of shape $\lambda(\sigma)$, so that we need to find a way to modify its shape without changing its combinatorial properties. This will be done in Section 6.3.2, using the transformation introduced in the next section.

6.3.1. A transformation on type. Let us begin this section with introducing two notations.

DEFINITION 6.3.1. Let S be a diagram and "a" (resp. "b") be a row (resp. a column) of S. We denote by $S{\downarrow_a}$ (resp. $\overrightarrow{S}^b$) the diagram obtained by swapping rows a and $a+1$ (resp. columns b and $b+1$) of S.

Definition 6.3.2. Let T be a tableau of shape S and a (resp. b) a row (resp. a column) of S. We denote by $T{\downarrow_a}$ (resp. $\overrightarrow{T}^b$) the tableau of shape $S{\downarrow_a}$ (resp. $\overrightarrow{S}^b$) obtained from T by exchanging rows a and $a+1$ (resp. columns b and $b+1$).

Let us consider a type $\mathcal{T}$ of shape S and let a be the index of a row of S. In general, the set

$$A = \{T{\downarrow_a} \ \mid T \in \mathrm{Tab}(\mathcal{T})\}$$

does not correspond to a class of our classification. That is, in general there is no type $\mathcal{T}'$ of shape $S{\downarrow_a}$ such that $A = \mathrm{Tab}(\mathcal{T}')$. However, we will prove in the sequel of this section that such a type $\mathcal{T}'$ exists in a specific case.

Definition 6.3.3. Let S be a diagram, $\mathcal{T} = (G_S, \theta) \in \mathrm{Type}(S)$ and "a" be the index of a row of $\mathcal{T}$. We say that the row a is *dominant* if and only if

- for all $(a, y) \in \mathbb{N} \times \mathbb{N}$, if $(a, y) \in S$, then $(a+1, y) \in S$;
- for all $(a, y) \in S$ we have $\theta(a, y) > \theta(a+1, y)$.

We have a similar definition of *dominant column* (see Figure 6.7) for a graphical representation of these two notions).

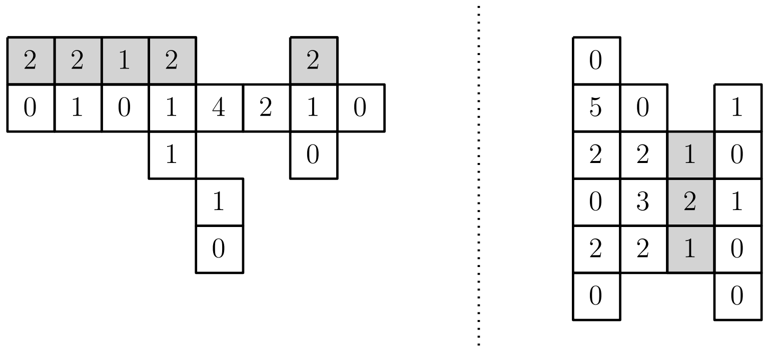


Figure 6.7. A dominant row (on the left) and a dominant column (on the right).

Before moving to the combinatorial study of the types having a dominant row or column, let us introduce one last notation.

Definition 6.3.4. Let $\mathcal{T}$ be a type of shape S and "a" be the index of a dominant row of $\mathcal{T}$. We denote by $\mathcal{T}{\downarrow_a}$ the type of shape $S{\downarrow_a}$ such that obtained from $\mathcal{T}$ by first decreasing by one all the integers in the row a of $\mathcal{T}$, then by swapping rows a and $a+1$ (resp. columns b and $b+1$) of $\mathcal{T}$, and keeping all other entries unchanged (see Figure 6.8).

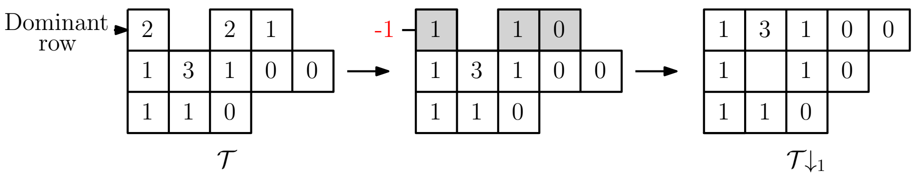


Figure 6.8.

Our aim is not to prove that for any dominant row a of a type $\mathcal{T}$ we have

$$\{T{\downarrow_a} \ \mid T \in \mathrm{Tab}(\mathcal{T})\} = \mathrm{Tab}(\mathcal{T}{\downarrow_a}). \tag{6.5}$$

For that purpose, we first prove a technical lemma.

Lemma 6.3.5. *Let $\mathcal{T} = (G_S, \theta) \in \mathrm{Type}(S)$ and a be the index of a dominant row of $\mathcal{T}$. Then, for any $T = (t_c) \in \mathrm{Tab}(\mathcal{T})$, we have $t_{a,y} > t_{a+1,y}$ for all $(a, y) \in S$.*

PROOF. Let $T = (t_{(x,y)})_{(x,y)\in S} \in \mathrm{Tab}(\mathcal{T})$, and assume by contradiction that the lemma is not true and consider y maximal such that $t_{(a,y)} \leq t_{(a+1,y)}$. Let $\mathfrak{c} \in H_S(a,y) \setminus \{(a,y)\}$ such that $t_{\mathfrak{c}} < t_{(a,y)}$, and let us split our study into two cases.

- If $\mathfrak{c} \in L_S(a,y)$, then we have that $\mathfrak{c} \in H_S(a+1,y)$ and $t_{\mathfrak{c}} < t_{(a,y)} \leq t_{(a+1,y)}$.
- If $\mathfrak{c} \in A_S(a,y)$, then there exists $z > y$ such that $\mathfrak{c} = (a,z)$, and we have by maximality of y
$$t_{(a+1,y)} \geq t_{(a,y)} > t_{(a,z)} > t_{(a+1,z)}.$$

This is enough to show that $\theta(a,y) \leq \theta(a+1,y)$, and this contradicts the fact that a is dominant. This concludes the proof. □

We now prove that (6.5) holds.

PROPOSITION 6.3.6 (Exchange property). *Let $\mathcal{T}$ be a type and "a" (resp. b) be a dominant row (resp. column) of $\mathcal{T}$. Then, the map $T \mapsto T{\downarrow_a}$ (resp. $T \mapsto \overrightarrow{T}^b$) is a bijection between $Tab(\mathcal{T})$ and $Tab(\mathcal{T}{\downarrow_a})$ (resp.* $\mathrm{Tab}(\overrightarrow{\mathcal{T}}^b)$*).*

PROOF. Let $T \in \mathrm{Tab}(\mathcal{T})$, and denote by $\mathcal{T}' = (G_{S\downarrow_a}, \theta')$ the type of the tableau $T' := T{\downarrow_a} = T' = (t'_{x,y})$. We will prove that $\mathcal{T}' = \mathcal{T}{\downarrow_a}$.

Let (x,y) be a box of $S{\downarrow_a}$ and let us define the following set
$$H_{x,y}(T) := \{t_{a,b} \mid (a,b) \in H_{x,y}(\mathrm{Sh}(T))\ \}.$$

We split our study into three cases.

- If $x \notin \{a, a+1\}$, then we have $H_{x,y}(T) = H_{x,y}(T')$, so that $\theta'(x,y) = \theta(x,y)$.
- If $x = a$, then we have $H_{a,y}(T') = H_{a+1,y}(T) \cup \{t_{a,y}\}$. However, by Lemma 6.3.5 we have $t'_{a,y} = t_{a+1,y} < t_{a,y}$, so that $\theta'(a,y) = \theta(a+1,y)$.
- If $x = a+1$, then we have $H_{a+1,y}(T') = H_{a,y}(T) \setminus \{t_{a+1,y}\}$, so that $\theta'_{a+1,y} = \theta_{a,y} - 1$.

Then, we have $\mathcal{T}' = \mathcal{T}{\downarrow_a}$, hence $T \mapsto T{\downarrow_a}$ send an element of $\mathrm{Tab}_S(\mathcal{T})$ to an element of $\mathrm{Tab}_{S\downarrow_a}(\mathcal{T}{\downarrow_a})$. Similar arguments show that $T \mapsto T{\downarrow_a}$ also sends an element of $\mathrm{Tab}_{S\downarrow_a}(\mathcal{T}{\downarrow_a})$ to an element of $\mathrm{Tab}_S(\mathcal{T})$ and $\downarrow_a$ is an involution so is bijective. This concludes the proof for rows. The proof of the same property for columns is similar. □

We finish this section with a useful definition.

DEFINITION 6.3.7. Let $\mathcal{T}$ be a type of shape S and a be the index of a row of $\mathcal{T}$. The row a is called *dethroned* if and only if

- for all $(a,y) \in \mathbb{N} \times \mathbb{N}$, if $(a,y) \in S$, then $(a-1,y) \in S$;
- for all $(a,y) \in S$ we have $\theta(a-1,y) \leq \theta(a,y)$.

We have a similar notion of *dethroned column.*

Obviously, if a is a dominant row of $\mathcal{T}$, then $a+1$ is a dethroned line of $\mathcal{T}{\downarrow_a}$ and viceversa. The same holds for dominant columns. If $a+1$ is a dethroned line of $\mathcal{T}$, we denote by $\mathcal{T}{\uparrow_{a+1}}$ the type $\mathcal{T}'$ such that $\mathcal{T}'{\downarrow_a} = \mathcal{T}$.

6.3.2. The exchange algorithm. In this section, we explain how one can turn the type $\mathcal{T}_\sigma$ (where $\sigma \in S_n$ is vexillary) into a type of shape $\lambda(\sigma)$ using recursively Proposition 6.3.6 on lines an column.

DEFINITION 6.3.8 (Line-exchange algorithm). Let $\mathcal{T}$ be a type of shape S, the *line-exchange algorithm* is the algorithm described below.

(1) Erase all the empty rows of $\mathcal{T}$.
(2) Set $i := 1$.
    (a) If i is a dominant row of $\mathcal{T}$, then set $\mathcal{T} := \mathcal{T}{\downarrow_i}$ and go back to step (2). Otherwise, go to step (2-b).
    (b) If there is no row below i, then the algorithm stops. Otherwise, set $i := i+1$ and go back to step (2-a).

We denote by $\mathcal{T}^L$ the type obtained after we perform the line-exchange algorithm.

There is an obvious analogous *column-exchange algorithm*, and we denote by $\mathcal{T}^C$ the type obtained after we perform this algorithm on a type $\mathcal{T}$.

LEMMA 6.3.9. *For any type $\mathcal{T}$, we have*

$$|\mathrm{Tab}(\mathcal{T})| = |\mathrm{Tab}(\mathcal{T}^L)| = |\mathrm{Tab}(\mathcal{T}^C)|.$$

PROOF. It is clear by Proposition 6.3.6. □

DEFINITION 6.3.10. For all type $\mathcal{T}$, we denote by $\mathcal{T}^E$ the type $(\mathcal{T}^L)^C$ obtained by first performing the line exchange algorithm on $\mathcal{T}$, and then performing the column exchange algorithm on $\mathcal{T}^L$.

We now state the main result of this section, whose proof is detailed in Section 6.3.3

THEOREM 6.3.11. *Let $\sigma \in S_n$ be a vexillary permutation and $\mathcal{T}_\sigma$ be its associated type. Then, we have:*

(1) $|\mathrm{Tab}(\mathcal{T}_\sigma^E)| = f^{\lambda(\sigma)} = f^{\lambda(\sigma)'}$;
(2) The shape of $\mathcal{T}^E$ is $\lambda(\sigma)'$.

3	3	0	0	0	0
3	3	0	0		
3	3	0	0		
0	0				
0	0				
0	0				

FIGURE 6.9. This is the type $\mathcal{T}_\sigma^E$ obtained considering the vexillary permutation $\sigma = [4, 8, 9, 5, 7, 6, 1, 3, 2]$

6.3.3. Proof of Theorem 6.3.11.

The first step of the proof consists in a characterization of vexillary permutations using their associated type.

DEFINITION 6.3.12. Let $\sigma \in S_n$, we denote by $(l_i(\mathcal{T}_\sigma))_i$ and $(c_i(\mathcal{T}_\sigma))_i$ the sequences defined by

$$l_i(\mathcal{T}_\sigma) := |\{j \mid (j, i) \in \mathrm{Inv}(\sigma)\}|,$$
$$c_i(\mathcal{T}_\sigma) := |\{j \mid (i, j) \in \mathrm{Inv}(\sigma)\}|.$$

The following lemma is immediate by Definition 6.2.3.

LEMMA 6.3.13. *Let $\sigma \in S_n$, then the partition obtained by rearranging the sequence $(l_i(\mathcal{T}_\sigma))_i$ (resp. $(c_i(\mathcal{T}_\sigma))_i$) in a non-increasing order is $\mu(\sigma)$ (resp. $\lambda(\sigma)$).*

Let us now consider $\sigma \in S_n$, we begin with putting the diagram $\mathrm{Inv}(\sigma)$ in a grid as depicted on Figure 6.10. We first push all the boxes of $\mathrm{Inv}(\sigma)$ against the Y-axes, and we then push all the boxes against the X-axes, obtaining by this way a Ferrers diagram (see Figure 6.11).

The partition obtained after this *YX-process* is $\mu(\sigma)$. Indeed, after we packed all the boxes against the Y-axes we obtain a diagram whose rows are left-justified, and row i contains exactly $l_i(\mathcal{T}_\sigma)$ boxes. Therefore, when we push everything on the X-axes, we are just rearranging these rows in a non-increasing order. Thus, thanks to Lemma 6.3.13 the resulting diagram is precisely $\mu(\sigma)$. Clearly, if we first stack on the X and then on the Y-axes (this process is called the *XY-process*), then the resulting partition is precisely $\lambda(\sigma)'$.

The following proposition is an immediate consequence of the observation made in the previous paragraph.

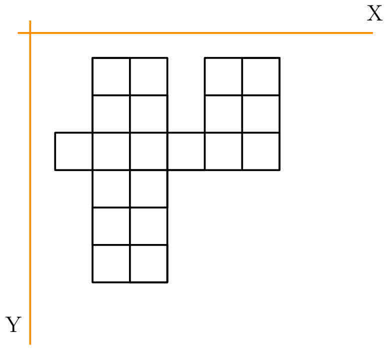


FIGURE 6.10. Diagram associated with $\sigma = [7, 8, 4, 5, 1, 2, 6, 9, 3] \in S_9$

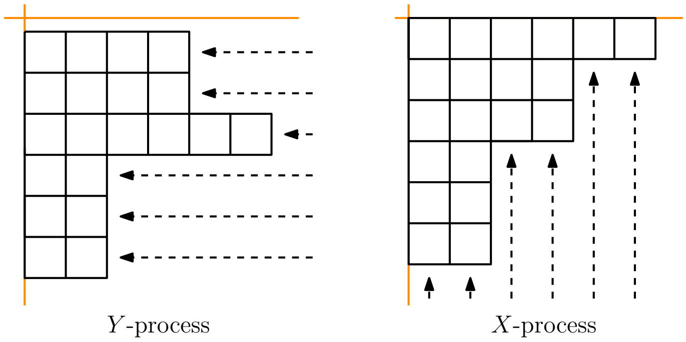


FIGURE 6.11.

PROPOSITION 6.3.14. *Let $\sigma \in S_n$. Then, σ is vexillary if and only if the partitions obtained after we perform the XY-process and YX-process on* $\mathrm{Inv}(\sigma)$ *are the same.*

We now prove an intermediate lemma.

LEMMA 6.3.15. *Let $\sigma \in S_n$ be a vexillary permutation and i, j be two integers. Then, we have the following two properties.*

- *If $l_i(\mathcal{T}_\sigma) \leq l_j(\mathcal{T}_\sigma)$, then we have that for all $(i, a) \in \mathbb{N} \times \mathbb{N}$,*
$$\textit{if } (a, i) \in \mathrm{Sh}(\mathcal{T}_\sigma), \textit{ then } (a, j) \in \mathrm{Sh}(\mathcal{T}_\sigma).$$
- *If $c_i(\mathcal{T}_\sigma) \leq c_j(\mathcal{T}_\sigma)$, then we have that for all $(a, i) \in \mathbb{N} \times \mathbb{N}$,*
$$\textit{if } (i, a) \in \mathrm{Sh}(\mathcal{T}_\sigma), \textit{ then } (j, a) \in \mathrm{Sh}(\mathcal{T}_\sigma).$$

PROOF. We prove this lemma only for lines since the proof for columns is similar, and we simply denote by l_i the integer $l_i(\mathcal{T}_\sigma)$.

We denote by n the number of non-empty rows in the diagram $\mathrm{Sh}(\mathcal{T}_\sigma)$ and we set $i_1, \ldots, i_n$ a sequence of indices such that:

- $l_{i_k} \neq 0$ for all $k \in [n]$;
- the sequence $(l_{i_1}, \ldots, l_{i_n})$ is non-increasing.

We will prove by induction on $k \in [n]$ that the property holds for the row i_k. First, notice that we have $l_{i_1} \geq l_{i_q}$ for all $1 < q \leq n$. Let us fix such a q, and consider a box $\mathfrak{c} = (i_q, p) \in \mathrm{Sh}(\mathcal{T}_\sigma)$.

Assume by contradiction that $(i_1, p) \in \mathrm{Sh}(\mathcal{T}_\sigma)$, then we have the configuration depicted on Figure 6.12. Therefore, if we push the boxes against the X-axes, then in the first row there must be strictly more than l_{i_1} boxes as represented on Figure 6.13. Thus, there are strictly more than l_{i_1} boxes in the first row of the partition obtained after we perform the XY-process on $\mathrm{Sh}(\mathcal{T}_\sigma)$. However, by maximality of l_{i_1}, the first row of the partition obtained we perform the YX-process on $\mathrm{Sh}(\mathcal{T}_\sigma)$ contains l_{i_1} boxes. Thus, $\lambda(\sigma) \neq \mu(\sigma)'$, and this contradicts the fact that σ is vexillary. Consequently, we have $(i_1, p) \in \mathrm{Sh}(\mathcal{T}_\sigma)$ and the lemma is true for row i_1.

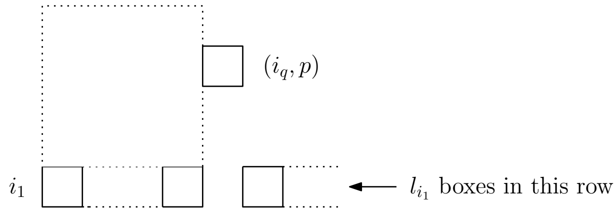


FIGURE 6.12. Configuration when $(i_q, p) \in \mathrm{Sh}(\mathcal{T}_\sigma)$ and $(i_1, p) \notin \mathrm{Sh}(\mathcal{T}_\sigma)$

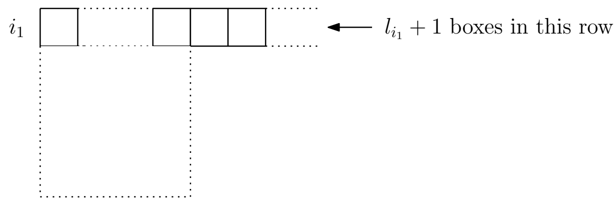


FIGURE 6.13. Configuration after we pushed the boxes of $\mathrm{Sh}(\mathcal{T}_\sigma)$ against the X-axes.

Let k be such that the lemma is true for rows $i_1, \ldots, i_k$, and let $\lambda(\sigma) = (\lambda_1, \ldots, \lambda_m)$. By induction, if we delete rows $i_1, \ldots, i_k$ in $\mathrm{Sh}(\mathcal{T}_\sigma)$ and then perform the XY or YX stacking process on the obtained diagram, then the resulting partition is $(\lambda_{k+1}, \ldots, \lambda_m)$ in both cases. Then, the same argument as for i_1 proves that the lemma holds for row i_{k+1}, and this ends the proof. □

Eventually, we can now provide a proof of Theorem 6.3.11.

PROOF OF THEOREM 6.3.11. **Point (1):** this is an immediate consequence of Theorem 6.2.4 together with Proposition 6.2.9 and Proposition 6.3.6.

Point (2): first, note that by definition of the line-exchange algorithm, we have

$$\text{for all type } \mathcal{T} \text{ of a given shape } S, \text{ we have } (\mathcal{T}^L)^L = \mathcal{T}^L \text{ and } (\mathcal{T}^C)^C = \mathcal{T}^C. \tag{6.6}$$

Let us denote by θ the valuation associated with $\mathcal{T}^L_\sigma$ and by l_i the number of boxes in row i of $\mathrm{Sh}(\mathcal{T}^L_\sigma)$. Assume by contradiction that there exists an integer k such that $l_k < l_{k+1}$. Then, thanks to Lemma 6.3.15 we have that $\mathrm{Sh}(\mathcal{T}^L_{sigma})$ is as represented on Figure 6.14. However, by

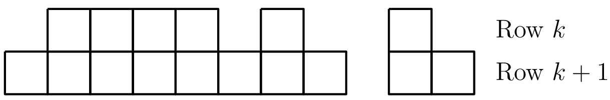


FIGURE 6.14. Rows k and $k+1$ of $\mathrm{Sh}(\mathcal{T}^L_\sigma)$.

construction for all $(a, k) \in \mathrm{Sh}(\mathcal{T}^L_\sigma)$ we have $\theta(a, k) > \theta(a+1, k)$. Therefore, if we perform the line-exchange algorithm on $\mathcal{T}^L_\sigma$, then these two rows are exchanged, so that we have

$$(\mathcal{T}^L_\sigma)^L \neq \mathcal{T}^L_\sigma,$$

contradicting (6.6). Thus, the sequence $(l_i)_i$ is non-increasing. Using a similar argument, we prove that the sequence $(c_i)_i$ is non-increasing, where c_i is the number of boxes in column i of $\mathrm{Sh}(\mathcal{T}^E_\sigma) = \mathrm{Sh}((\mathcal{T}^L_\sigma)^C)$.

Eventually, the same arguments as for the proof of Lemma 6.3.15 prove that $\mathrm{Sh}(\mathcal{T}^E_\sigma)$ is a partition, which is necessarily equal to $\lambda(\sigma)'$. □

6.3.4. Link with balanced tableaux. Let us now explain how the construction made in the previous sections can be use to provide an alternative (but not fundamentally different from the one in [**FGRS**]) proof of Theorem 6.2.2. Let λ be a partition of an integer n that we identify with its Ferrers diagram. A box $\mathfrak{c} = (a, b)$ of λ is called a *corner* of λ if and only if there is no boxes on the right and below $\mathfrak{c}$, *i.e.* both $(a+1, b)$ and $(a, b+1)$ are not in λ.

Let $\mathfrak{c} = (a,b)$ be a corner of λ such that $k = \lambda_a + \lambda'_b - 1$ is maximal (such a corner is not necessarily unique). Then, we can place λ in the staircase partition λ_{k+1} as shown on Figure 6.15.

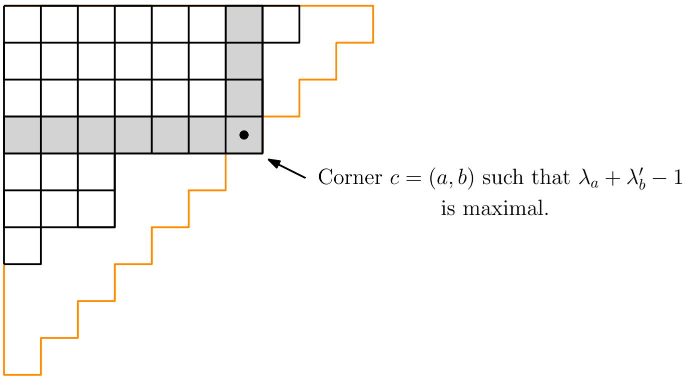


FIGURE 6.15.

Let us look at the corners (u,v) of λ which are on the diagonal boundary of the staircase partition. For each such corner, we set $R_{(u,v)} = \{(x,y) \in \lambda \mid x \leq u,\ y \leq v\}$ and we consider the union R of the $R_{(u,v)}$. Then we let each connected component of $\lambda \setminus R$ fall in the staircase tableau as shown on Figure 6.16. We repeat the same procedure for each connected component of the resulting diagram, while it is possible. At the end, we get a sub-diagram λ_{k+1}, which we denote by $S(\lambda)$.

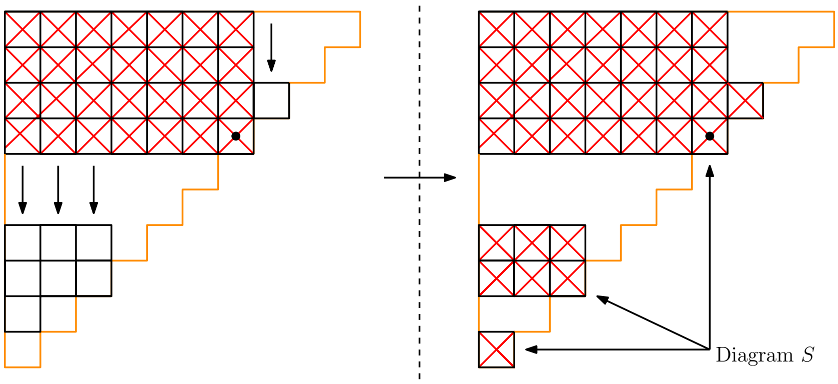


FIGURE 6.16.

LEMMA 6.3.16. *There exists $\sigma_\lambda \in S_{k+1}$ such that $\mathrm{Sh}(\mathcal{T}_{\sigma(\lambda)} = S(\lambda)$. Moreover, σ_λ is vexillary and $\lambda(\sigma)' = \lambda$.*

PROOF. We denote by θ the valuation of the valued digraph $\mathcal{A}$ associated with $(S_{k+1}, \leq_R)$. We will prove that $S(\lambda) \in IS(\mathcal{A})$. Let $z = (x,y) \in \lambda_{k+1}$.

- If $z \notin S(\lambda)$, then for all $z' \in H_z(\lambda_{k+1}) \cap S(\lambda)$, z' is in the same column and strictly below z. Moreover, by definition θ_z equals the number of boxes strictly below z. Thus, we have $\theta_z \geq |H_z(\lambda_{k+1}) \cap S(\lambda)| = d^+_{S(\lambda)}(\mathcal{A}, z)$;
- If $z = (x,y) \in S(\lambda)$, then by construction of $S(\lambda)$ there exists $z' = (x', y') \in S(\lambda)$ such that: $x' \geq x$, $y' \geq y$, $\theta_{z'} = 0$, and

$$\text{for all } x \leq u \leq x' \text{ and } y \leq v \leq y',\ \text{we have } (u,v) \in S(\lambda).$$

  Thus, we have $\theta_z = (x' - x) + (y' - y) \leq |H_z(S(\lambda))| - 1 = d^+_{S(\lambda)}(\mathcal{A}, z)$.

Therefore, by Definition 4.1.5 we have $S(\lambda) \in IS(\mathcal{A})$, so that there exists σ_λ such that $\mathrm{Sh}(\mathcal{T}_{\sigma_\lambda}) = S_\lambda$. Moreover, if we perform the stacking process on $S(\lambda)$, it is clear that both XY and YX processes end with the partition $\lambda(\sigma)$. Thus, σ_λ is vexillary and this concludes the proof. $\square$

PROPOSITION 6.3.17. *Let σ_λ be the permutation whose inversion set is $S(\lambda)$. Then, we have*

$$\mathrm{Tab}(\mathcal{T}^L_{\sigma(\lambda)}) = \mathrm{Bal}(\lambda).$$

PROOF. First, note that we have $\mathrm{Sh}(\mathcal{T}^L_{\sigma(\lambda)}) = \lambda$ by construction of $S(\lambda)$. We denote by θ and θ' the valuations associated with the types $\mathcal{T}_{\sigma(\lambda)}$ and $\mathcal{T}^L_{\sigma(\lambda)}$, respectively. Let $\mathfrak{c} = (a, b) \in S(\lambda)$ such that all boxes in the same row and on the right of $\mathfrak{c}$ are not in $S(\lambda)$. Then, by construction of $S(\lambda)$ and by definition of θ we have:

- $\theta(\mathfrak{c})$ equals the number of indices $k < a$ such that row k of $S(\lambda)$ contains more boxes than row a;
- for all $\mathfrak{d} = (a, d) \in S(\lambda)$, we have $\theta(\mathfrak{d}) = \theta(\mathfrak{c}) + d - b$.

Therefore, when we perform the line-exchange algorithm on $\mathcal{T}_{\sigma(\lambda)}$ we have that the row a of $\mathcal{T}_{\sigma(\lambda)}$ is swapped with exactly $\theta(\mathfrak{c})$ rows below it. Thus, we have that for all $\mathfrak{d} \in \lambda$,

$$\theta'(\mathfrak{d}) = a_\lambda(\mathfrak{d}),$$

hence the elements of $\mathrm{Tab}(\mathcal{T}_{\sigma(\lambda)})$ are the balanced tableaux of shape λ, and reciprocally. This ends the proof. □

The previous proposition together with Proposition 6.3.6 immediately imply Theorem 6.2.2.

6.3.5. An equivalence relation between vexillary permutations. At this point, a natural question arises: given two vexillary permutation σ and ω, when do we have $\mathcal{T}^E_\sigma = \mathcal{T}^E_\omega$? In this section we answer this section by exhibiting an equivalence relation $\sim_v$ on the set of vexillary permutations with the property that, for any two vexillary permutations $\sigma \in S_n$ and $\omega \in S_m$, $\mathcal{T}^E_\sigma = \mathcal{T}^E_\omega$ if and only if $\sigma \sim_v \omega$.

We first introduce a useful notation. Let $\sigma \in S_n$, $p \leq n$ be the lowest integer such that $\sigma(p) \neq p$ and $q \leq n$ be the biggest integer such that $\sigma(q) \neq q$. We define

$$\overline{\sigma} := [\sigma(p) - (p-1)\ ;\ \sigma(p+1) - (p-1)\ ;\ \dots\ ;\ \sigma(q) - (p-1)].$$

Note that $\overline{\sigma}$ is an element of $S_{n+1-p-q}$ because of the choice of p and q.

DEFINITION 6.3.18. We say that $\sigma \sim_v \omega$ if and only if $\overline{\sigma} = \overline{\omega}$.

THEOREM 6.3.19. *Let σ and ω be two vexillary permutations, then $\mathcal{T}^E_\sigma = \mathcal{T}^E_\omega$ if and only if $\sigma \sim_v \omega$.*

PROOF. **Step 1:** we begin with giving a combinatorial interpretation for the relation $\sim_v$. Let $\sigma \in S_n$ and $\omega \in S_m$ be such that $\sigma \sim_v \omega$. Without loss of generality, we can assume that n is larger than m. Then, we can see ω as a permutation of S_n by adding $(n-m)$ fixed points at the end of ω. Let us denote by p_σ (resp. p_ω) the smallest integer such that $\sigma(p_\sigma) \neq p_\sigma$ (resp. $\omega(p_\omega) \neq p_\omega$). By definition, we have $\mathrm{Inv}(\overline{\sigma}) = \mathrm{Inv}(\overline{\omega})$ and

$$\mathrm{Inv}(\sigma) = \{(x + (p_\sigma - 1), y + (p_\sigma - 1)) \mid (x, y) \in \mathrm{Inv}(\overline{\sigma})\}.$$

Thus, if we look at $\mathcal{T}_\sigma$ and $\mathcal{T}_\omega$, we have the situation described on Figure 6.17. Then, we have

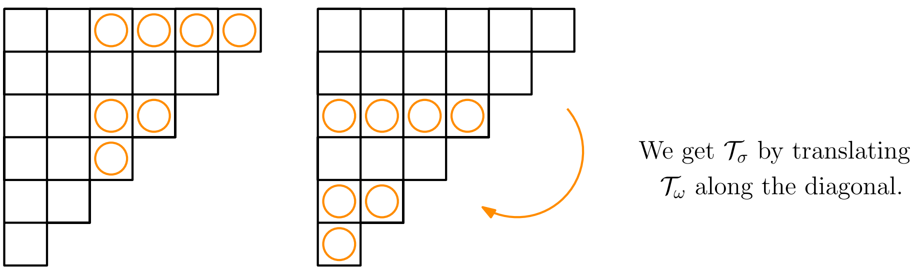


FIGURE 6.17. The types $\mathcal{T}_\sigma$ and $\mathcal{T}_\omega$ seen as subsets of λ_n.

$\mathcal{T}^E_\sigma = \mathcal{T}^E_\omega$ by construction.

Step 2: in order to prove the converse implication, we define two algorithms: one which reverses the Line-exchange Algorithm and another which reverse the Column-exchange Algorithm. Since these two algorithms are similar, we only give the definition of the algorithm on lines. Let $\mathcal{T}$ be a type of shape S, the *reverse line-exchange algorithm* is the algorithm described below.

(1) Erase all the empty rows of $\mathcal{T}$.
(2) Set $i := r$, where r is the number of non-empty rows of $\mathcal{T}$.
   (a) If i is a dethroned row of $\mathcal{T}$, then set $\mathcal{T} := \mathcal{T}\uparrow_i$ and go back to step (2). Otherwise, go to step (2-b).
   (b) If there is no row above i, then the algorithm stops. Otherwise, set $i := i - 1$ and go back to step (2-a).

We denote by ${}^L\mathcal{T}$ the type obtained after we perform the reverse line-exchange algorithm. We also denote by $\overline{\mathcal{T}}$ the type obtained from $\mathcal{T}$ deleting empty rows. In general, the type ${}^L(\mathcal{T}^L)$ is different from the type $\overline{\mathcal{T}}$. However, it is clear that ${}^L(\mathcal{T}^L) = \overline{\mathcal{T}}$ whenever there is no dethroned lines in $\overline{\mathcal{T}}$, and by construction there is no dethroned lines in $\overline{\mathcal{T}_\sigma}$ for any $\sigma \in S_n$.

Therefore, for all vexillary permutations $\sigma, \omega \in S_n$, if $\mathcal{T}_\sigma^E = \mathcal{T}_\omega^E$ then we have $\overline{\mathcal{T}_\sigma} = \overline{\mathcal{T}_\omega}$, and this implies that $\mathcal{T}_\sigma = \mathcal{T}_\omega$. Then, we have $\mathrm{Inv}(\overline{\sigma}) = \mathrm{Inv}(\overline{\omega})$ implying that $\overline{\sigma} = \overline{\omega}$, *i.e.* $\omega \sim_v \sigma$. This concludes the proof. □

6.3.6. Partial fillings of tableaux of type $\mathcal{T}_\sigma^E$. In this section, we generalize an implicit result in [**FGRS**] about combinatorics of balanced tableaux to our construction. That is, we will enumerate for any vexillary permutation $\sigma \in S_n$ the number of tableaux of type $\mathcal{T}_\sigma^E$ such that the integers $1, 2, \ldots, k$ appear at a given fixed position. We begin with introducing a useful notation.

Definition 6.3.20. Let $\sigma \in S_m$ be a vexillary permutation such that $\lambda(\sigma) \vdash n$ and $U = [z_1, \ldots, z_k]$ be a sequence of boxes of the Ferrers diagram of $\lambda(\sigma)$. We denote by $N_{\sigma,U}$ the set defined by

$$N_{\sigma,U} := \{T = (t_\mathfrak{c})_{\mathfrak{c}\in\lambda} \in \mathrm{Tab}(\mathcal{T}_\sigma^E) \mid t_{z_i} = i \text{ for all } i \in [k]\}.$$

Theorem 6.3.21. *Let $\sigma \in S_m$ be a vexillary permutation such that $\lambda(\sigma) \vdash n$ and $U = [z_1, \ldots, z_k]$ be a sequence of boxes of the Ferrers diagram of $\lambda(\sigma)$. Then, we have that either $N_{\sigma,U}$ is empty, or there exists $\omega \in S_m$ such that*

$$|N_{\sigma,U}| = |\mathrm{Red}(\omega)|.$$

Proof. Let us assume that $N_{\sigma,U}$ is not empty. By construction, there exists a bijection Ψ between the set of boxes of $\mathcal{T}_\sigma$ and the inversion set of σ such that for any sequence $L = [x_1, \ldots, x_n]$, we have $L \in PS(\mathcal{T}_\sigma^E)$ if and only if

$$[\Psi(x_1), \Psi(x_2), \ldots, \Psi(x_n)] \in PS(\mathcal{T}_\sigma),$$

Furthermore, thanks to Proposition 6.1.4 there is one-to-one correspondence between $N_{\sigma,U}$ and the set

$$P_U := \{[x_1, \ldots, x_n] \in PS(\mathcal{T}_\sigma^E) \mid x_i = z_i \text{ for all } i \in [k]\}.$$

Thus, $\Psi(\{z_1, \ldots, z_k\})$ is the inversion set of a permutation $\tau \in S_m$, and thanks to Corollary 3.3.10 we have a one-to-one correspondence between $N_{\sigma,U}$ and the set of the maximal chains from τ to σ in $(S_m, \leq_R)$. Thanks to [**BB**] (Prop. 3.1.6, page 69), we have that the set of maximal chains from τ to σ in $(S_m, \leq_R)$ is in one-to-one correspondence with $\mathrm{Red}(\tau^{-1}\omega)$, and this concludes the proof. □

This theorem applies in particular to balanced tableaux. Moreover, this result is constructive: the permutation ω can be computed. We cannot provide a systematic description of the associate permutation, however we have the following combinatorial result.

THEOREM 6.3.22. *Let $\sigma \in S_m$ be a vexillary permutation such that $\lambda(\sigma) \vdash n$ and $U = [z_1, \ldots, z_k]$ be a sequence of boxes of the Ferrers diagram of λ. If there exists a partition $\mu \vdash (n-k)$ such that the resulting partition of the XY and YX processes applied on the diagram $\lambda(\sigma) \setminus \{z_1, \ldots, z_k\}$ is μ, then $|N_{\sigma,U}| = f^\mu$.*

PROOF. We keep the notations introduced in the proof of Theorem 6.3.21. Let τ be the permutation whose inversion set is given by $\{\Psi(x_1), \ldots, \Psi(x_k)\}$. Since $\omega = \tau^{-1}\sigma$ with $\tau \leq_R \sigma$, we have

$$\mathrm{Inv}(\omega) = \tau^{-1}(\mathrm{Inv}(\sigma) \setminus \mathrm{Inv}(\tau)) = \tau^{-1}(\mathrm{Sh}(\mathcal{T}_\sigma) \setminus \{\Psi(x_1), \ldots, \Psi(x_k)\}).$$

Thus, if we denote by l_i (resp. c_i) the number of boxes in row (resp. column) i of $\lambda(\sigma) \setminus \{z_1, \ldots, z_k\}$, we have that the sequences $(l_i)_i$ and $(g_i(\omega))_i$ (resp. $(c_i)_i$ and $(r_i(\omega))_i$) are equal up to re-ordering. Therefore, ω is vexillary and $\mu(\omega) = \mu$. This concludes the proof. □

Notice that the results of the current section apply to balanced tableaux.

DEFINITION 6.3.23. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n and $(a, b), (c, d) \in \lambda$, we say that (a, b) and (c, d) are in the same *block* if and only if $\lambda_a = \lambda_c$. Let B be a block of λ and let i be the minimal integer such that $(i, \lambda_i) \in B$, then the box (i, λ_i) is called the *upper right corner* of B.

Let T be a balanced tableau of shape $\lambda \vdash n$. By definition, we have that the integer 1 appears in the upper right corner of a block (see Figure 6.18).

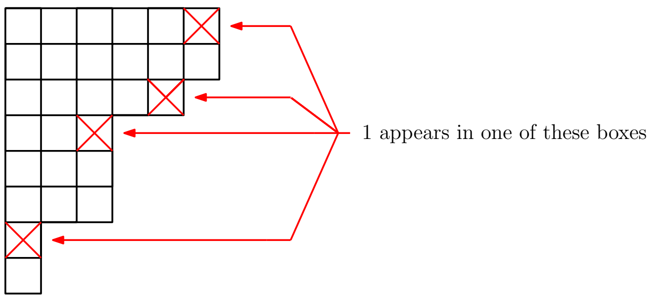


FIGURE 6.18.

Then, thanks to Theorem 6.3.22 we immediately have the following result.

PROPOSITION 6.3.24. *Let B be a block of $\lambda \vdash n$ and $\mathfrak{c}$ be the upper right corner of B. Then, the number of balanced tableaux of shape λ such that 1 is in the box $\mathfrak{c}$ equals the number of standard tableaux of shape λ^-, where $\lambda^- \vdash n-1$ is obtained from λ by suppressing the corner of the block B.*

6.4. Acknowledgements

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From valued digraphs to complete lattices: a new approach of weak order on Coxeter groups.

Abstract: Weak order on a Coxeter group W is a partial order on W appearing in many areas of algebraic combinatorics. In this thesis, we propose a new general model for the study of the weak order and other related partially ordered sets (also called "posets"), and we explore various algebraic and combinatorial consequences of this construction.
We begin with studying a restricted version of this model in Chapter 3. More precisely, we explain how one can associate a poset to any simple acyclic digraph together with a valuation on its vertices (also called "valued digraph"). We then prove that these posets are complete meet semi-lattices in general, complete lattices when the underlying digraph is finite, and we give an explicit formula to compute the value of their Möbius functions. Then, we show that the weak order on Coxeter groups of type A, B and $\widetilde{A}$, the flag weak order, and the up-set (resp. down-set) lattices of any finite poset can be described within this theory. This description naturally leads to associate a quasi-symmetric function to any element of A_n and $\widetilde{A_n}$, and we demonstrate that this function is in fact the corresponding Stanley symmetric function.
In Chapter 4 we introduce the main results of this thesis. Indeed, we introduce in this chapter the generalization of the construction made in Chapter 3 to the case of any valued digraph, that is without the simplicity and acyclicity condition. Furthermore, this new definition allows us to get rid of some constraints of the definition of Chapter 3, allowing us to associate a complete lattice to each valued digraph. In particular, the meet semi-lattices of Chapter 3 are naturally extended into complete lattices. This leads us to the study of some conjectures of Dyer about the properties of an extension of the weak order on any Coxeter group (among other things, these extensions would be complete lattices). Then, using our formalism we construct several extensions of the weak order having a lot of the properties conjecturally attached to Dyer's extensions, and we prove that each one of our extensions contains Dyer's extension as a sub-poset. We make the conjecture that one of this extension coincide with the one of Dyer, and we provide tools in order to test this conjecture.
Finally, we study various consequences of out theory: we provide extensions of Cambrian semi-lattices into complete lattices (end of Chapter 4), we construct a new combinatorial model for Tamari and m-Tamari lattices (Chapter 5), and we finish with an application to tableaux combinatorics (Chapter 6).

Keywords: Coxeter groups; Root systems; Weak order; Digraphs; lattices; Cambrian semi-lattices.

Des graphes orientés aux treillis complets : une nouvelle approche de l'ordre faible sur les groupes de Coxeter

Résumé: L'ordre faible sur un groupe de Coxeter W est un ordre partiel sur les éléments de W, intervenant dans de nombreux domaines de la combinatoire algébrique. Dans cette thèse, on propose un nouveau modèle général pour l'étude de cet ordre ainsi que d'autres ensembles ordonnés affiliés, et on explore diverses conséquences aussi bien algébriques que combinatoires de cette construction.

On commence, dans le chapitre 3, par étudier une version restreinte de ce modèle. Plus précisément, on explique comment on peut associer un ensemble ordonné (aussi appelé "poset") à tout graphe orienté, simple, acyclique et muni d'une valutation sur ses sommets (aussi appelé "graphe valué"). On montre ensuite que ces posets sont en général des semi-treillis inférieurs, des treillis quand le graphe est fini, et on donne une formule explicite pour les valeurs de leurs fonctions de Möbius. On prouve ensuite que l'ordre faible sur les groupes de Coxeter de type A, B et $\widetilde{A}$, le "flag weak order" ainsi que le treillis des idéaux supérieurs et inférieurs de tout poset fini peuvent être décrit avec notre modèle. Cette description amène naturellement à associer une série quasi-symétrique à chaque élément de A_n et $\widetilde{A_n}$, et on montre que cette série est en fait la série de Stanley associée.

On présente dans le chapitre 4 les résultats centraux de la thèse, en effet on y introduit la généralisation de la construction faite au chapitre précédent au cas de tout graphe valué, c'est à dire sans condition d'acyclicité et de simplicité. On s'affranchit également de certaines contraintes imposées par la définition du chapitre 3, ce qui nous permet d'associer à tout graphe valué un treillis complet, et non plus un semi-treillis. En particulier, les semi-treillis du chapitre 3 se retrouvent naturellement plongés dans un treillis complet. Ceci nous amène à nous intéresser à des conjectures de Dyer portant sur l'étude d'une extension de l'ordre faible sur tout groupe de Coxeter (entre autres, il est conjecturé que ces extensions sont des treillis complets). On construit alors, à l'aide de notre formalisme, des extensions de l'ordre faible ayant beaucoup des propriétés conjecturalement attachées aux extensions de Dyer, et contenant ces dernières comme sous-poset. On conjecture que l'une de ces extensions coïncide avec celle de Dyer, et on fournit des outils pour le tester.

Finalement, on étudie diverses conséquences de notre théorie : la construction d'extensions des semi-treillis cambriens (fin du chapitre 4), la construction d'un nouveau modèle combinatoire pour les treillis de Tamari et m-Tamari (chapitre 5), et enfin on propose une application à la combinatoire des tableaux (chapitre 6).

Mots clés: Groupe de Coxeter; Système de racines; Ordre faible; Graphes orientés; Treillis; Semi-treillis cambriens.

Image en couverture : Graphes valués associés à deux treillis cambriens en type A.

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