Des graphes orientés aux treillis complets : une nouvelle approche de l'ordre faible sur les groupes de Coxeter.

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ICJ - Lyon 1

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Plan

1. The initial motivation: reduced decompositions and balanced tableaux
2. Coxeter groups, weak order and valued digraphs
3. Root systems and Dyer's conjectures
4. Perspectives

reduced decompositions of a permutation

- The symmetric group S_n is generated by simple transpositions $s_i = (i, i+1)$, $i \in [n-1]$.

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- Each permutation σ can be written as a product $s_{i_1} \cdots s_{i_k} = \sigma$ of simple transpositions. When k is minimal, we say that the word $s_{i_1} \cdots s_{i_k}$ is a reduced decomposition of σ.

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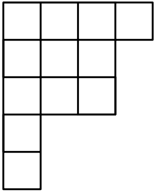
- Combinatorics of reduced decompositions is related to two families of tableaux: standard and balanced tableaux.

Reduced decompositions and standard tableaux

Definition

A partition of the integer n is a non-increasing sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \dots$ such that $\sum_i \lambda_i = n$.



Ferrers diagram of the partition $\lambda = (4, 3, 3, 1, 1)$.

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3	7	8	
4	9	11	
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12			

Standard tableaux of shape $\lambda = (4, 3, 3, 1, 1)$.

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Theorem, Stanley (1984)

The number of reduced decompositions of the permutation $[n, n-1, \ldots, 2, 1]$ is equal to the number of standard tableaux of shape $(n-1, n-2, \ldots, 2, 1)$.

Balanced tableaux (Edelman-Greene, 1987)

Definition of balanced tableaux

Set $T = (t_{\mathfrak{c}})_{\mathfrak{c}\in\lambda}$ a tableau of shape λ. T is a balanced tableau if and only if for all boxes $\mathfrak{c} \in \lambda$ we have $|\{z \in H_{\mathfrak{c}}(\lambda) \mid t_z < t_{\mathfrak{c}}\}| = a_{\mathfrak{c}}$, where $a_{\mathfrak{c}}$ is the number of boxes on the right of $\mathfrak{c}$ in λ, and $H_{\mathfrak{c}}(\lambda)$ is the hook based on $\mathfrak{c}$ in λ.

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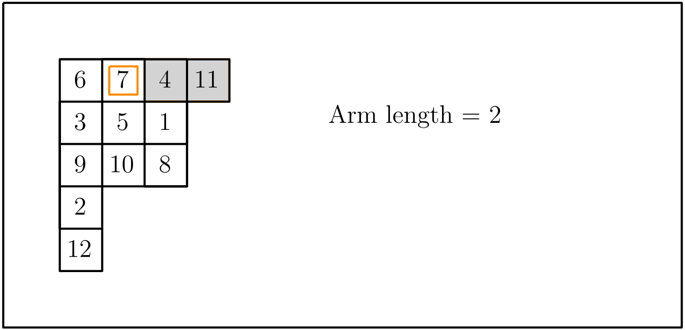

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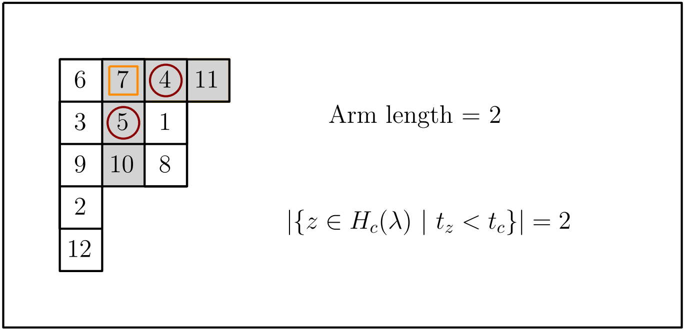

Link between balanced tableaux, standard tableaux and reduced decompositions

Theorem, Edelman and Greene (1987)

There exists a one-to-one correspondence between balanced tableaux of shape $\lambda_n = (n-1, n-2, \ldots, 1)$ and reduced decompositions of $[n, n-1, \ldots, 1]$. In particular, this implies that $|\mathrm{Bal}(\lambda_n)| = |\mathrm{SYT}(\lambda_n)|$.

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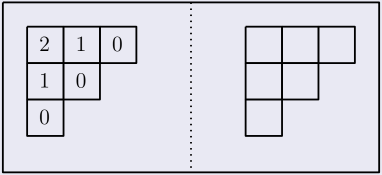
For each partition λ, we have $|\mathrm{Bal}(\lambda)| = |\mathrm{SYT}(\lambda)|$.

- The proof of this theorem is quite involved, and uses reduced decompositions.
- Can we find a more direct bijection between $\mathrm{Bal}(\lambda)$ and $\mathrm{SYT}(\lambda)$?

How to construct balanced tableaux

Filling algorithm



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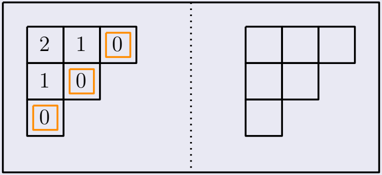

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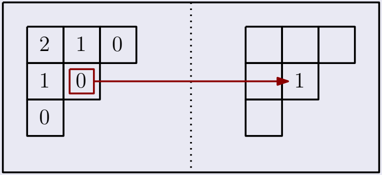

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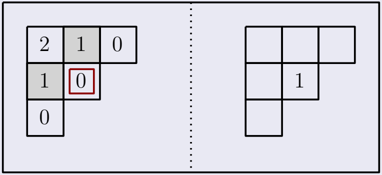

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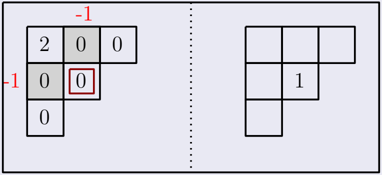

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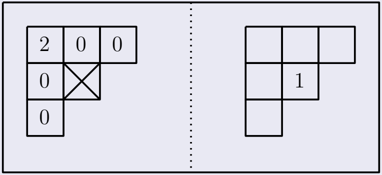

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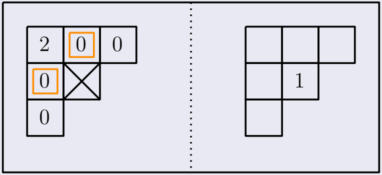

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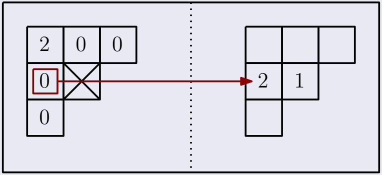

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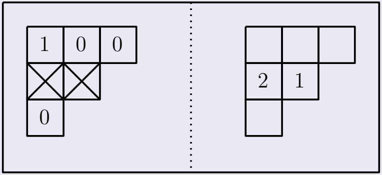

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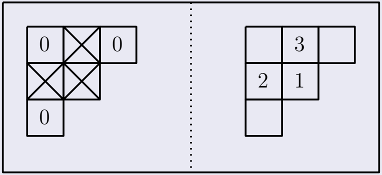

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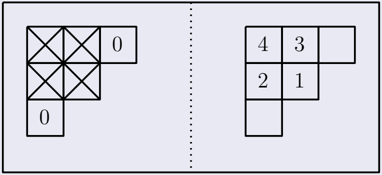

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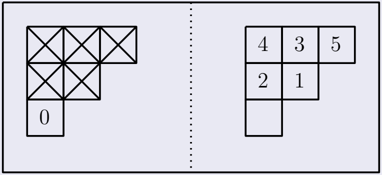

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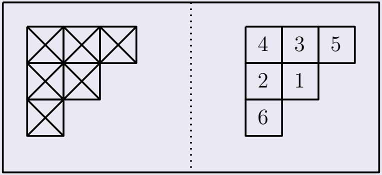

- The resulting tableau is a balanced tableau. Furthermore, each balanced tableau can be obtained by this way.

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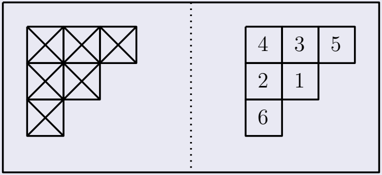

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- Despite many tests, this algorithm did not lead to a “direct” bijective proof of the result of Edelman and Greene.

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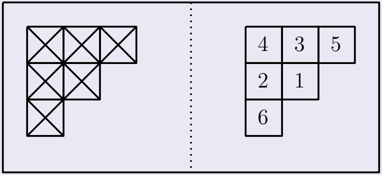

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- Despite many tests, this algorithm did not lead to a "direct" bijective proof of the result of Edelman and Greene.
- However, it allows us to generalize the definition of balanced tableaux and to generalize the result of Edelman and Greene.

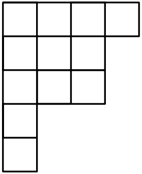
Generalizing the concept of balanced tableau

Definition

Let λ be a partition of n. A type $\mathcal{T}$ of shape λ is a filling of λ with integers $\theta(\mathfrak{c})$ satisfying the inequality

$$\text{for all } \mathfrak{c} \in \lambda, \; 0 \leq \theta(\mathfrak{c}) \leq |H_{\mathfrak{c}}(\lambda)| - 1.$$

3	3	2	0
1	2	1	
4	0	0	
1			
0			



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0			

Filling algorithm →

6	10	5	3
7	12	9	
11	2	4	
8			
1			

- We denote by $\mathrm{Tab}(\mathcal{T})$ the set of all tableaux coming from a type $\mathcal{T}$.

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Filling algorithm $\longrightarrow$

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- We denote by $\mathrm{Tab}(\mathcal{T})$ the set of all tableaux coming from a type $\mathcal{T}$.
- All tableaux are classified according to their type. Balanced and standard tableaux are special classes of this classification.

Generalizing the result of Edelman and Greene

Definition

Let $\sigma \in S_n$, the permutation σ is called vexillary if and only if σ is 2143-avoiding.

Theorem, Stanley (1984)

If σ is vexillary, then there exists a partition $\lambda(\sigma)$ such that

$$|\mathrm{Red}(\sigma)| = |\mathrm{SYT}(\lambda(\sigma))|.$$

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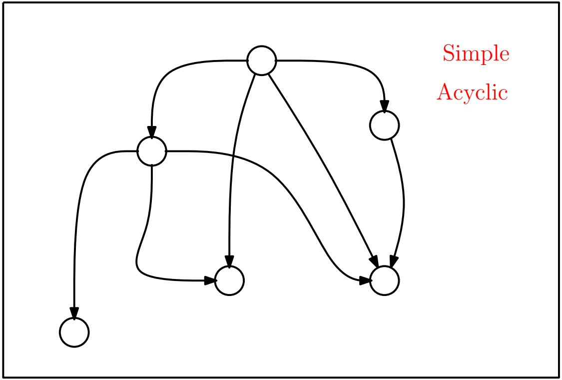
Theorem, V. (2013)

Let $\sigma \in S_n$, if σ is vexillary, then there exists a type $\mathcal{T}_\sigma$ of shape $\lambda(\sigma)$ such that

$$|\mathrm{Tab}(\mathcal{T}_\sigma)| = |\mathrm{Red}(\sigma)| = |\mathrm{SYT}(\lambda(\sigma))|.$$

A generalization of the filling algorithm to digraphs

$G = (V, E)$



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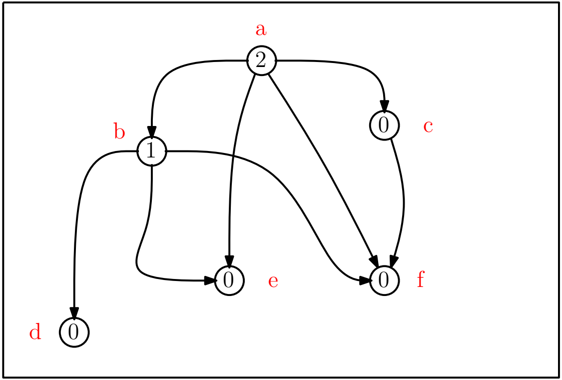
$G = (V, E)$	Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \le \theta(z) \le$ outdegree of z.



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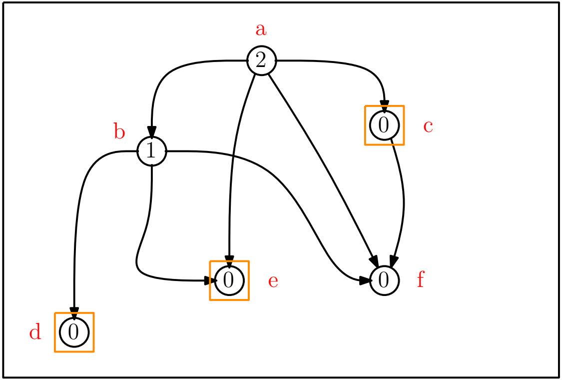

Step 1 :

Consider each $z \in V$ such that :
1) $\theta(z) = 0$
2) if $(y, z) \in E$, then $\theta(y) \neq 0$.

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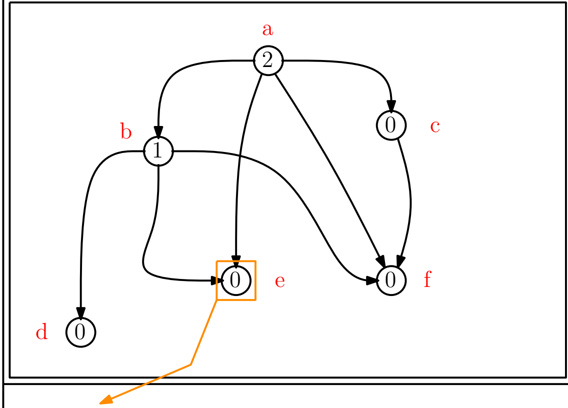

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Choose one of these vertices and add it to the list.

$L = [e]$

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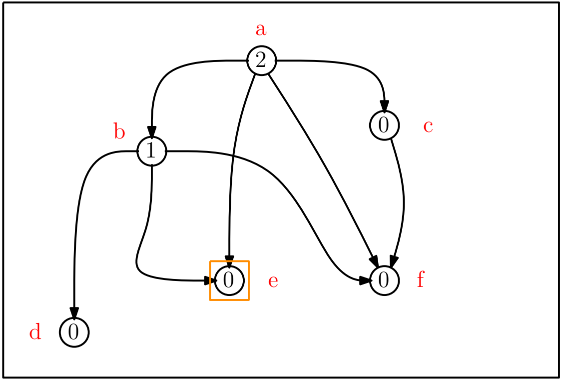

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Peeling process !

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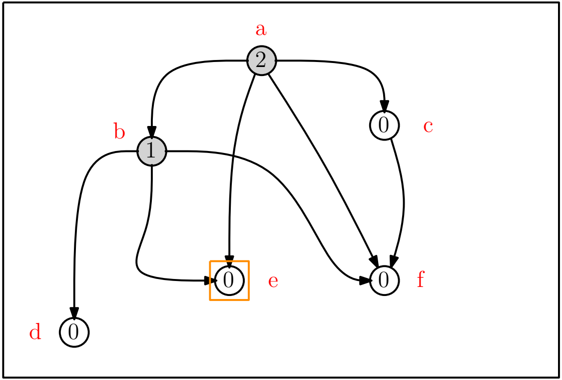

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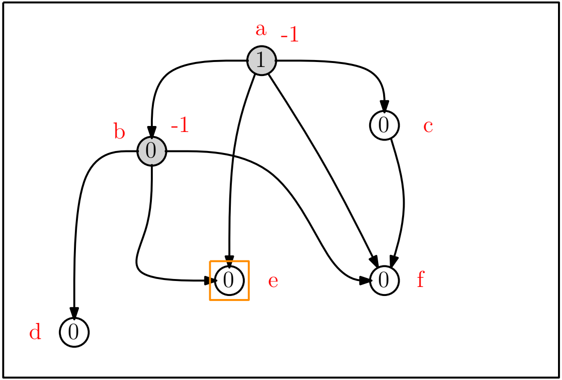

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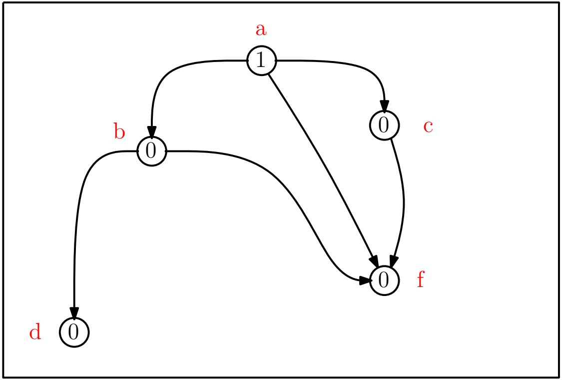

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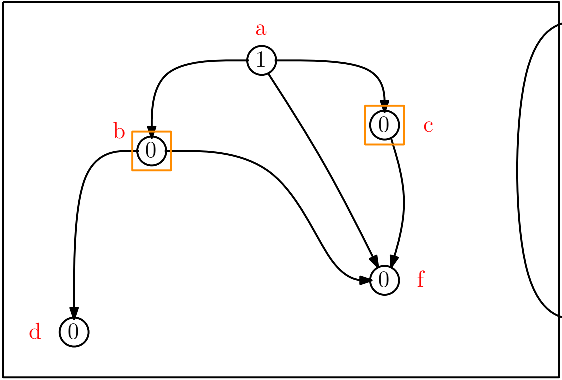

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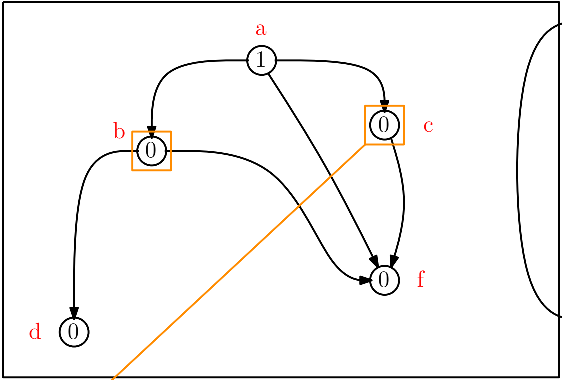

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$L = [e, c]$

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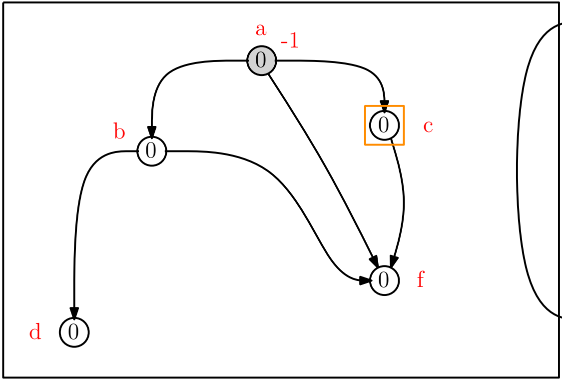

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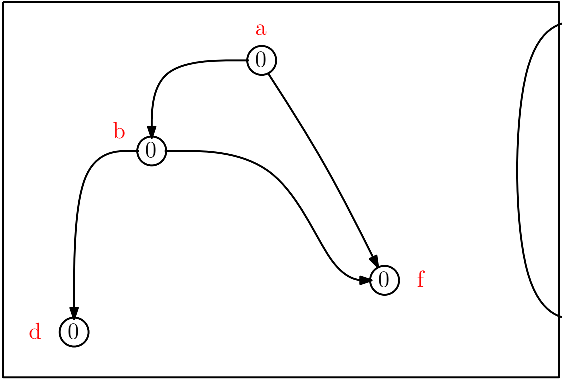

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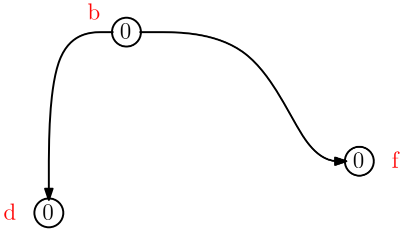

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2) if $(y, z) \in E$, then $\theta(y) \neq 0$.

Step 2 :

Choose one of these vertices and add it to the list.

Step 3 :

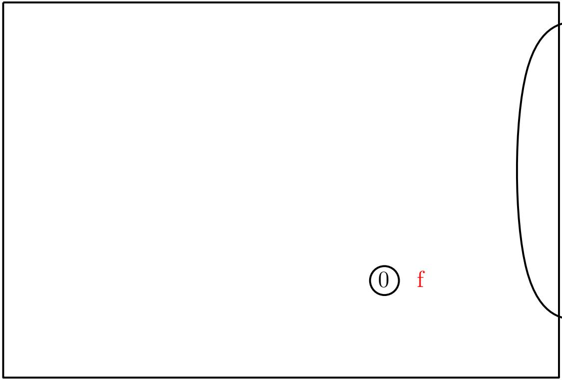
Peeling process !

$L = [e, c, a, b]$

A generalization of the filling algorithm to digraphs

$G = (V, E)$

Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta(z) \leq$ outdegree of z.



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Peeling process !

$L = [e, c, a, b, d, f]$ ← Such a sequence is called a *peeling sequence* of (G, θ).

Initial sections of a peeling sequence and definition of $IS(\mathcal{G})$

- Consider $L = [e, c, a, b, d, f]$ the previous peeling sequence. The initial sections of L are the following sets

$$L_0 = \emptyset,\ L_1 = \{e\},\ L_2 = \{e, c\}, \ldots,\ L_6 = \{e, c, a, b, d, f\}.$$

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Definition

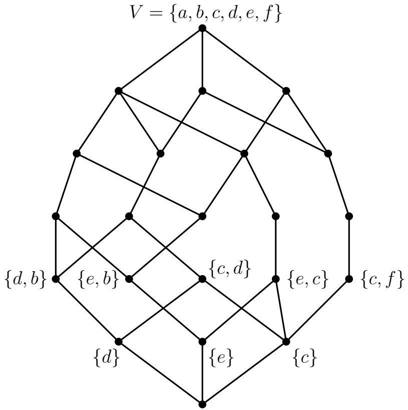
Let $\mathcal{G} = (G, \theta)$ be a pair of a simple acyclic digraph and a "compatible" valuation on its vertices (such a pair is called a simple acyclic valued digraph). We denote by $IS(\mathcal{G})$ the set constituted of all the initial sections of all the peeling sequences of $\mathcal{G}$.

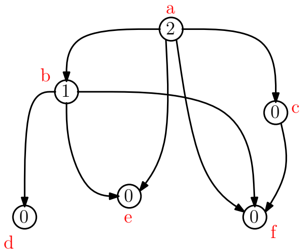
From now on, we will study the properties of the poset $(IS(\mathcal{G}), \subseteq)$.

Lattice structure of $(IS(\mathcal{G}), \subseteq)$

Theorem, (V, 2014)

Let $\mathcal{G} = (G, \theta)$ be a simple acyclic valued digraph. Then, the poset $(IS(\mathcal{G}), \subseteq)$ is a graded complete meet semi-lattice. Furthermore, if G is finite, then it is a complete lattice with $V(G)$ as maximal element.





Möbius function of $(IS(\mathcal{G}), \subseteq)$

Definition

For any locally finite poset $(P, \leq)$, the Möbius function of P is the function μ from $P \times P$ to $\mathbb{Z}$ recursively defined by:

- for all $x \in P$, $\mu(x, x) = 1$;
- for all $x, y \in P$, $\mu(x, y) = -\sum_{x \leq c < y} \mu(x, c)$.

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Theorem, V. (2014)

Let $A \in IS(\mathcal{G})$, we define:

- $\mathcal{N}(A) = \{z \in A \mid \theta(z) = 0\}$;
- $\mathcal{F}(A) = \{z \in A \mid A \setminus \{z\} \in IS(\mathcal{G})\}$.

We have the following two cases:

- if $\mathcal{N}(A) = \mathcal{F}(A)$, then $\mu(\emptyset, A) = (-1)^{|\mathcal{N}(A)|}$;
- otherwise, $\mu(\emptyset, A) = 0$.

Coxeter groups and weak order

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- Each $w \in W$ can be written as a product of a minimal number of elements of S. This minimal number is denoted by $\ell(w)$ and is called the length of w.
- We define the weak order $\leq_R$ on W as follows: $w \leq_R w'$ if and only if there exists $s_1, \ldots, s_k$ in S such that

$$w' = ws_1 \cdots s_k \text{ and } \ell(w') = \ell(w) + k.$$

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- Note that $(W, \leq_R)$ is a complete meet semi-lattice in general, a complete lattice when W is finite, and its Möbius function takes values into the set $\{-1, 0, 1\}$.

Toward the general case: root system and inversion sets

Let W be a Coxeter group of finite rank n and Φ be a root system of W. In particular, we have that:

- Φ is a discrete subset of $\mathbb{R}^n$ on which W acts;
- There exists a partition of Φ into two subsets Φ^+ and $\Phi^- = -\Phi^+$, separated by an hyperplane of $\mathbb{R}^n$.

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Let $w \in W$, the inversion set of w is defined by

$$\mathrm{Inv}(w) := \Phi^+ \cap w\left(\Phi^-\right).$$

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For all $w, w' \in W$, $w \leq_R w'$ if and only if $\mathrm{Inv}(w) \subseteq \mathrm{Inv}(w')$.

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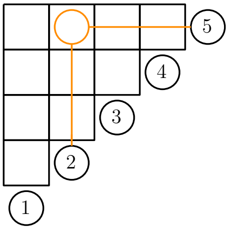
Idea

We are looking for a $\mathcal{G}$ such that $V(\mathcal{G}) = \Phi^+$ and the elements of $IS(\mathcal{G})$ are exactly the sets of the form $\mathrm{Inv}(w)$.

Weak order on S_n and valued digraph



This box represents the couple $(2, 5)$

One can easilly represent the set

$$\{(a, b) \mid 1 \leq a < b \leq n\}$$

as a staircase tableau.

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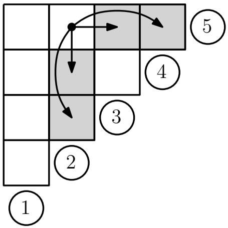

Hook based on the box $(2, 5)$

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We implement an (implicit) digraph structure on this diagram.

We say that there is an arc from c to d iff d is in the hook based on c.

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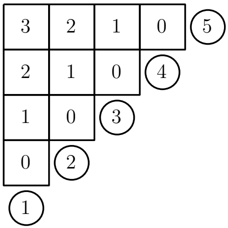

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We set $\theta(c) = \frac{\text{outdegree}(c)}{2}$.

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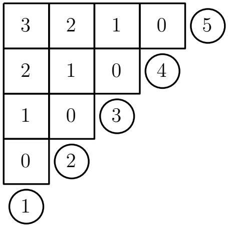

Denote by $\mathcal{A} = (G, \theta)$ the obtained pair.

We have $IS(\mathcal{A}) = \{\mathrm{Inv}(\sigma) \mid \sigma \in S_n\}$

Therefore, we have that $(S_n, \leq_R)$ and $(IS(\mathcal{A}), \subseteq)$ are isomorphic.

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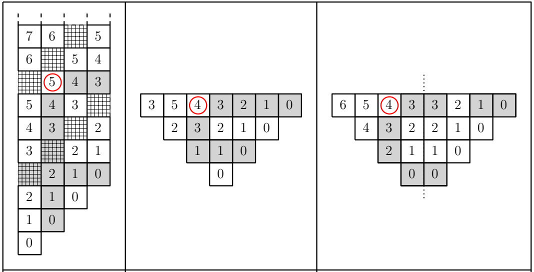
General result

Theorem, V. (2014-2015)

Each one of the following posets can be described with an explicit simple acyclic valued digraph.

- The weak order on Coxeter groups A_n, B_n, D_n, $I_2(n)$, E_6, E_7, E_8 and $\widetilde{A}_n$.
- The flag weak order on $\mathbb{Z}_r \wr S_n$ (Adin, Brenti and Roichman, 2011).
- The up-set and down-set lattice of any finite poset.



$(\widetilde{A}_4, \leq_R)$ | $(B_4, \leq_R)$ | $(D_5, \leq_R)$

Toward the general case?

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Proposition, V. (2014)

Let $\mathcal{G} = (G, \theta)$ be a pair of a simple acyclic digraph G together with a valuation θ on $V(G)$ such that $0 \leq \theta(z) \leq d^+(z)$ for all $z \in V(G)$, and $A \subseteq V(G)$. Then, $A \in IS(\mathcal{G})$ if and only if the following properties are satisfied:

- A is finite;
- for all $z \in A$, $\theta(z) \leq d_A^+(z)$;
- for all $z \in V(G) \setminus A$, $\theta(z) \geq d_A^+(z)$.

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Structure of $(IS_\infty(\mathcal{G}), \subseteq)$

Theorem, V. (2015)

Let $\mathcal{G} = (G, \theta)$ be a pair of a digraph G together with a valuation θ on $V(G)$ such that $0 \leq \theta(z) \leq d^+(z)$ for all $z \in V(G)$. Then, the poset $(IS_\infty(\mathcal{G}), \subseteq)$ is a complete lattice.

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- In this new context, there is no general equivalent of peeling sequences.
- The poset $(IS_\infty(\mathcal{G}), \subseteq)$ cannot describe the weak order on any Coxeter group, since weak order is not a complete lattice.
- However, there are some conjectures of Dyer, about an extension of the weak order into a complete lattice.

First Dyer's conjecture

Definition

Let $\alpha, \beta, \gamma \in \Phi^+$ such that $\gamma = a\alpha + b\beta$ with $a, b > 0$ and $A \subseteq \Phi^+$. We say that A is closed iff we have that if $\alpha, \beta \in A$, then $\gamma \in A$. We say that A is bi-closed iff both A and $\Phi^+ \setminus A$ are closed. We denote by $\mathcal{B}(\Phi^+)$ the set of the bi-closed sets of Φ^+.

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Theorem, Pilkington (2006)

The inversion sets of any Coxeter group W are exactly the finite bi-closed sets.

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Goal

We are now looking for a valued digraph $\mathcal{G}$ such that $IS_\infty(\mathcal{G}) = \mathcal{B}(\Phi^+)$.

Second Dyer's conjecture

Definition

Let $I = (\Phi^+, \preceq)$ be a total order. We say that I is a reflection ordering of Φ^+ if and only if for all $\alpha, \beta, \gamma \in \Phi^+$ such that $\gamma = a\alpha + b\beta$ with $a, b > 0$, we have either $\alpha \preceq \gamma \preceq \beta$ or $\beta \preceq \gamma \preceq \alpha$.

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Proposition (Folklore ?)

If W is finite, then there is a one-to-one correspondence between maximal chains and $(W, \leq_R)$ and reflection orderings of Φ^+.

Conjecture 2, Dyer (1993)

Let $\mathcal{C}$ be a chain of $(\mathcal{B}(\Phi^+), \subseteq)$. Then, there exists a reflection ordering I of Φ^+ such that $\mathcal{C}$ is included in the set of the initial sections of I, and $\mathcal{C}$ is maximal if and only if we have equality.

Observation

- Peeling sequences in the finite acyclic case satisfy exactly the condition of the second Dyer's conjecture.

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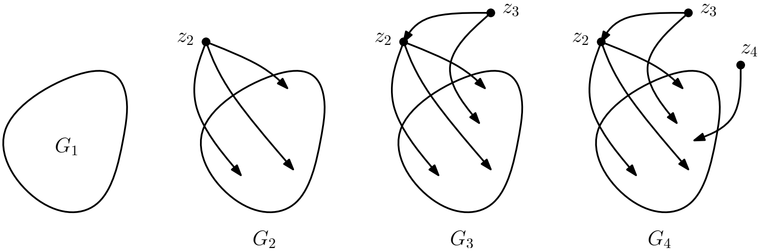
Answer

Yes! But not for all valued digraphs.

Projective valued digraphs

Theorem, V. (2015)

There exists a family of infinite valued digraphs $\mathcal{G}$, called projective, such that there exists a set $PS_\infty(\mathcal{G})$ of total orderings of $V(\mathcal{G})$ such that each chain of $(IS_\infty(\mathcal{G}), \subseteq)$ is included in the initial sections of an element of $PS_\infty(\mathcal{G})$, with equality if and only if the chain is maximal.



Projective valued digraph and Dyer's conjectures

Theorem, V. (2015)

There exists a projective valued digraph $\mathcal{G}$ such that:

1. $V(\mathcal{G}) = \Phi^+$, $\mathcal{B}(\Phi^+) \subseteq IS_\infty(\mathcal{G})$ and each reflection ordering of Φ^+ is in $PS_\infty(\mathcal{G})$.
2. The poset $(IS_\infty(\mathcal{G}), \subseteq)$ is an algebraic ortho-lattice.
3. We have that $\mathcal{B}(\Phi^+) = IS_\infty(\mathcal{G})$ iff $PS_\infty(\mathcal{G})$ is the set of the reflection orderings of Φ^+.

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- However, after many tests, I conjecture that at least one of them satisfies point $IS_\infty(\mathcal{G}) = \mathcal{B}(\Phi^+)$.

The Tamari lattice

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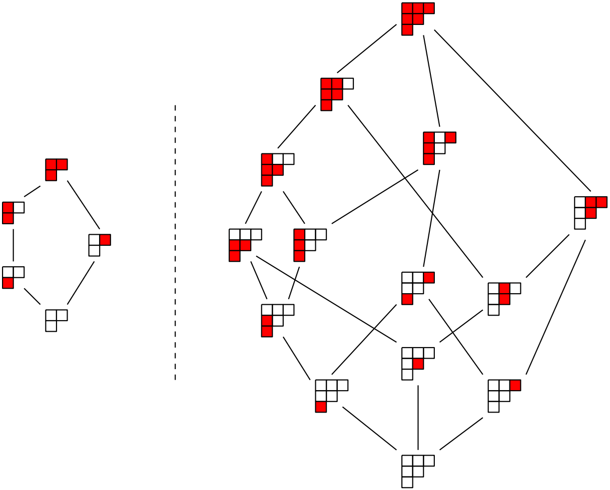

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Thank you for your attention!