Westfälische
Wilhelms-Universität
Münster

Fachbereich 10
Mathematik und
Informatik

From the quartic analogue of the Kontsevich model to the Hermitian 2-Matrix Model

partially based on arXiv:1906.04600 with H. Grosse and R. Wulkenhaar

Alexander Hock

Motivation

The **Kontsevich Model**

$$\mathcal{Z}^{Kont} = \int_{H_N} \mathcal{D}\phi e^{-N\mathrm{Tr}(E\phi^2 + \frac{\lambda}{3}\phi^3)},$$

where external matrix $E \in M(N, \mathbb{C})$ hermitian and $\lambda \in \mathbb{C}$, has connection to a lot of areas in mathematics and physics

- counting large maps
- computing intersection number of moduli space
- 2D quantum gravity
- computing volumes of hyperbolic spaces
- constructing quantum field theory on a noncommutative space

$\vdots$

Motivation

What is with the **Quartic** analogue of the **Kontsevich Model**

$$\mathcal{Z} = \int_{H_N} \mathcal{D}\phi e^{-N\mathrm{Tr}(E\phi^2 + \frac{\lambda}{4}\phi^4)},$$

where external matrix $E \in M(N, \mathbb{C})$ hermitian and $\lambda \in \mathbb{C}$? Does it have connections to

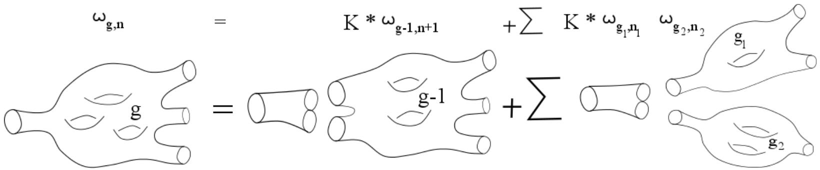
- counting large maps?
- computing intersection number of moduli space?
- 2D quantum gravity?
- computing volumes of hyperbolic spaces?
- constructing quantum field theory on a noncommutative space!

$\vdots$

Outline

1. Topological Recursion (TR) of the Kontsevich Model and hermitian 1-Matrix Model
2. Reminder of the hermitian 2-Matrix Model
   - The spectral curve with genus zero assumption
   - Mixed-boundary correlators
   - General loop equations
3. The Quartic Model
   - Definition of the (generalized) correlation functions
   - Perturbative expansion into Feynman graphs
   - The spectral curve
   - General loop equations
   - Some results
4. Work for the future

Topological Recursion



taken from wikipedia.org/wiki/Topological_ recursion

TR of the Kontsevich Model

$$\mathcal{Z}^{Kont} = \int_{H_N} \mathcal{D}\phi e^{-N\mathrm{Tr}(E\phi^2 + \frac{\lambda}{3}\phi^3)},$$

Let e_k be the distinct eigenvalues of E with multiplicity r_k. Then the correlation functions are defined for pairwise distinct e_{p_i} as connected correlation functions

$$\langle \phi_{p_1p_1}\phi_{p_2p_2}\dots\phi_{p_bp_b}\rangle_c$$

The aim is to compute all correlation functions which is done by TR
The **spectral curve** (genus zero assumption) is

$$\begin{aligned} x(z) =& z^2 - t_1, \\ y(z) =& z - \frac{\lambda}{N}\sum_k \frac{r_k}{x'(\varepsilon_k)(\varepsilon_k - z)} \end{aligned}$$

with $e_k^2 = x(\varepsilon_k)$, i.e. $\varepsilon_k = \sqrt{e_k^2 + t_1}$, and $t_1 = -\frac{\lambda}{N}\sum_k \frac{r_k}{2\varepsilon_k}$.

The correlation functions are special points on the spectral curve of the meromorphic forms $\omega_{g,b}$

$$x'(\varepsilon_{p_1})..x'(\varepsilon_{p_b})\langle\phi_{p_1p_1}\phi_{p_2p_2}...\phi_{p_bp_b}\rangle_c = \sum_g N^{2-2g-b}\omega_{g,b}(\varepsilon_{p_1},..,\varepsilon_{p_b}) - \frac{\delta_{g,0}\delta_{b,2}x'(\varepsilon_{p_1})x'(\varepsilon_{p_2})}{(x(\varepsilon_{p_1})-(x(\varepsilon_{p_2}))^2} - \delta_{g,0}\delta_{b,1}x'(\varepsilon_{p_1})e_{p_1}.$$

Then, **TR computes all correlation functions** for $2g+b-2>1$ [Eynard,Orantin '07]

$$\omega_{g,b+1}(z_0,...,z_b) = \operatorname*{Res}_{z\to 0}\left[K(z_0,z)\,dz\Big(\omega_{g-1,b+2}(z,-z,J) + \sum_{\substack{h+h'=g\\ I\uplus I'=J}}' \omega_{h,|I|+1}(z,I)\omega_{h',|I'|+1}(-z,I')\Big)\right],$$

where $K(z_0,z) = \frac{\lambda}{2(z^2-z_0^2)(y(z)-y(-z))}$ and the sum $\sum'$ excludes $(h,I)=(0,\emptyset)$ and $(h,I)=(g,J)$, and

$$\omega_{0,1}(z) = x'(z)y(-z)$$
$$\omega_{0,2}(z_1,z_2) = \frac{1}{(z_1-z_2)^2}.$$

We have one branch point at $z=0$ and the global Galois-involution $\sigma(z)=-z$

TR of the Hermitian 1-Matrix Model

The **partition function** is

$$\mathcal{Z}^{1M} = \int_{H_N} \mathcal{D}M e^{-N\mathrm{Tr}\,V(M)}$$

with polynomial potenial $V(x) = \sum_{i=2}^{d} t_i x^i$ with $t_i \in \mathbb{R}$.
The correlation function are (formally) defined

$$\Big\langle \mathrm{Tr}\frac{1}{x_1 - M}\mathrm{Tr}\frac{1}{x_2 - M}\ldots\mathrm{Tr}\frac{1}{x_b - M}\Big\rangle_c.$$

Also here, TR computes all correlation functions with the **spectral curve** (genus zero assummption)

$$x(z) = \alpha + \gamma\Big(z + \frac{1}{z}\Big),$$

$$y(z) = \sum_{k=0}^{d-1} u_k z^k$$

with $V'(x(z)) = \sum_{k=0}^{d-1} u_k (z^k + z^{-k})$

The relation between the correlation function and the meromorphic forms is

$$x_1'(z_1)..x_b'(z_b)\Big\langle \mathrm{Tr}\frac{1}{x_1(z_1)-M}\mathrm{Tr}\frac{1}{x_2(z_2)-M}...\mathrm{Tr}\frac{1}{x_b(z_b)-M}\Big\rangle_c$$
$$=\sum_g N^{2-2g-b}\omega_{g,b}(z_1,..,z_b)-\frac{\delta_{g,0}\delta_{b,2}x_1'(z_1)x_2'(z_2)}{(x_1(z_1)-x_2(z_2))^2}+\frac{1}{2}\delta_{g,0}\delta_{b,1}x_1'(z_1)V'(x_1(z_1))$$

Also here, **TR computes all correlation functions** for $2g+b-2>1$ [Eynard,Orantin '07]

$$\omega_{g,b+1}(z_0,...,z_b)=\operatorname*{Res}_{z\to\pm 1}\Big[K(z_0,z)\,dz\Big(\omega_{g-1,b+2}(z,1/z,J)+\sum_{\substack{h+h'=g\\ I\uplus I'=J}}' \omega_{h,|I|+1}(z,I)\omega_{h',|I'|+1}(1/z,I')\Big)\Big],$$

where $K(z_0,z)=\frac{\frac{1}{z_0-z}-\frac{1}{z_0-1/z}}{2x'(z)(y(z)-y(-z))}$ and the sum $\sum'$ excludes $(h,I)=(0,\emptyset)$ and $(h,I)=(g,J)$, and

$$\omega_{0,1}(z)=x'(z)y(1/z)$$
$$\omega_{0,2}(z_1,z_2)=\frac{1}{(z_1-z_2)^2}.$$

We have branch points at $z=\pm 1$ and the global Galois-involution $\sigma(z)=1/z$

Hermitian 2-Matrix Model

The **partition function** is

$$\mathcal{Z}^{2M}=\int_{H_N}\mathcal{D}M_1\mathcal{D}M_2 e^{-N\mathrm{Tr}\left[V_1(M_1)+V_2(M_2)-M_1M_2\right]}$$

with potenials $V_1(x)=\sum_{i=2}^{d_1}t_ix^i$ and $V_2(x)=\sum_{i=2}^{d_2}\tilde{t}_ix^i$ with $t_i,\tilde{t}_j\in\mathbb{R}$.
The correlation functions are

$$\left\langle\prod_{i=1}^{l}\mathrm{Tr}\left(\prod_{j=1}^{k_i}\frac{1}{x_{i,j}-M_1}\frac{1}{y_{i,j}-M_2}\right)\prod_{j=1}^{m}\mathrm{Tr}\frac{1}{x_j-M_1}\prod_{s=1}^{n}\mathrm{Tr}\frac{1}{y_s-M_2}\right\rangle_c$$

for $n+m$ non-mixed (one color) boundaries and l mixed boundaries with $2k_i$ color changes

The correlation functions $\left\langle \prod_{j=1}^{m} \mathrm{Tr}\frac{1}{x_j - M_1} \right\rangle_c$ (all boundaries with the same color) are computed by TR [Chekov, Eynard, Orantin '06]. After genus zero assumption is the **spectral curve** [Eynard '02]

$$x(p) = \gamma p + \sum_{i=0}^{d_2} \frac{\alpha_i}{p^i}$$
$$y(p) = \frac{\gamma}{p} + \sum_{j=0}^{d_1} \beta_j p^j .$$

In particular, for equal potentials $V_1 = V_2$ follows $\beta_j = \alpha_j \rightarrow$ **x(p)=y(1/p)**, which is the global Galois-involution of the hermitian 1-Matrix Model.
The function $y(p)$ is of the same form as in the hermitian 1-Matrix model.

The number of branch points $(dx(p) = 0)$ depends on the potential

The correlation functions defined on the spectral curve are [Eynard, Orantin '08]

$$\sum_g N^{-2g} H^{(g)}_{k_1,..,k_l;m;n}(S_1,S_2,..,S_l;p_1,..,p_m;q_1,..,q_n) = \Big\langle \prod_{i=1}^{l} \big(N\delta_{k_i,1} + \mathrm{Tr}\frac{1}{S_i}\big) \prod_{j=1}^{m} \mathrm{Tr}\frac{1}{x(p_j)-M_1} \prod_{s=1}^{n} \mathrm{Tr}\frac{1}{y(q_s)-M_2} \Big\rangle_c$$
$$+ \frac{\delta_{l,0}\delta_{m,2}\delta_{n,0}}{(x(p_1)-x(p_2))^2} + \frac{\delta_{l,0}\delta_{m,0}\delta_{n,2}}{(y(q_1)-y(q_2))^2} + \delta_{l,0}\delta_{m,1}\delta_{n,0}(y(p_1)-V_1'(x(p_1))) + \delta_{l,0}\delta_{m,0}\delta_{n,1}(x(q_1)-V_2'(y(q_1)))$$

with

$$\mathrm{Tr}\frac{1}{S_i} = \mathrm{Tr}\frac{1}{x(p_{i,1})-M_1}\frac{1}{y(q_{i,1})-M_2}\cdot\cdot\frac{1}{y(q_{i,k_i})-M_2}$$

and the set $S_i = [p_{i,1}, q_{i,1}, .., q_{i,k_i}]$.

The solutions of $H^{(g)}_{...}$ are given by an **extension of Topological Recursion**

Examples of solutions [Eynard, Orantin...]

- All $H^{(g)}_{0;m;0}$ and $H^{(g)}_{0;0;n}$ are given by TR
- The spehere with one-colored boundary is $H^{(0)}_{0;1;0}(p) = V_1'(x(p)) - y(p)$
- The sphere with one bicolored boundary is

$$H^{(0)}_{1;0;0}(\{p,q\}) = \frac{E(x(p),y(q))}{(x(p)-x(q))(y(q)-y(p))},$$

where $E(x(p),y(q)) = -t_{d_1}\prod_i (x(p)-x(\tilde{q}^i))$ and $y(q) = y(\tilde{q}^i)$

- All $H^{(g)}_{0;m;n}$ and $H^{(g)}_{1;m;n}$ are computed in a closed (triangular) system by knowing $H^{(g)}_{0;m;0}$
- The general result of $H^{(g)}_{k_1,..,k_l;m;n}$ needs recursively (in Euler-characteristic) all previous results

Loop equation in the 2-Matrix [Eynard, Orantin '08]

$$
\begin{aligned}
&(y(p_{1,1}) - y(q_{1,k_1}) - Pol_{x(p_{1,1})} V_1'(x(p_{1,1}))) H^{(g)}_{k_1,\dots,k_l;m;n}(S_1, S_2, \dots, S_l; p_1, \dots, p_m; q_1, \dots, q_n) = \\
&\textstyle\sum_{h=1}^{g} H^{(h)}_{0;1;0}(p_{1,1}) H^{(g-h)}_{k_1,\dots,k_l;m;n}(S_1, S_2, \dots, S_l; p_1, \dots, p_m; q_1, \dots, q_n) \\
&+ \textstyle\sum_h \sum_{A \bigcup B = \{2,\dots,l\}} \sum_{I,J} H^{(h)}_{k_1,\mathbf{k_A};|I|;|J|}(S_1, \mathbf{S_A}; \mathbf{p_I}; \mathbf{q_J}) H^{(g-h)}_{\mathbf{k_B};m-|I|+1;n-|J|}(\mathbf{S_B}; p_{1,1}\mathbf{p_{M/I}}; \mathbf{q_N}/\mathbf{J}) \\
&+ \textstyle\sum_h \sum_{A \bigcup B = \{2,\dots,l\}} \sum_{\alpha=2}^{k_1} \sum_{I,J} H^{(h)}_{k_1-\alpha+1,\mathbf{k_B};m-|I|;n-|J|}(\{p_{1,\alpha}, q_{1,\alpha}, \dots p_{1,k_1}, q_{1,k_1}\}, \mathbf{S_B}; \mathbf{p_{M/I}}; \mathbf{q_{N/J}}) \\
&\qquad \times \frac{H^{(g-h)}_{\alpha-1,\mathbf{k_A};|I|;|J|}(\{p_{1,1},q_{1,1},\dots p_{1,\alpha-1},q_{1,\alpha-1}\},\mathbf{S_A};\mathbf{p_I};\mathbf{q_J}) - H^{(g-h)}_{\alpha-1,\mathbf{k_A};|I|;|J|}(\{p_{1,\alpha},q_{1,1},\dots p_{1,\alpha-1},q_{1,\alpha-1}\},\mathbf{S_A};\mathbf{p_I};\mathbf{q_J})}{x(p_{1,\alpha})-x(p_{1,1})} \\
&+ \textstyle\sum_{i=2}^{l} \sum_{\alpha=1}^{k_i} \frac{1}{x(p_{i,\alpha})-x(p_{1,1})} \\
&\Big[H^{(g)}_{k_1+k_i,\mathbf{k_{L/\{1,i\}}};m;n}(\{S_1, p_{i,\alpha}, q_{i,\alpha}, p_{i,\alpha+1}, \dots, q_{i,k_i}, p_{i,1}, \dots, p_{i,\alpha-1}, q_{i,\alpha-1}\}, \mathbf{S_{L/\{1,i\}}}; \mathbf{p_M}; \mathbf{q_N}) \\
&- H^{(g)}_{k_1+k_i,\mathbf{k_{L/\{1,i\}}};m;n}(\{S_1, p_{i,\alpha}, q_{i,\alpha}, p_{i,\alpha+1}, \dots, q_{i,k_i}, p_{i,1}, \dots, p_{i,\alpha-1}, q_{i,\alpha-1}\}, \mathbf{S_{L/\{1,i\}}}; \mathbf{p_M}; \mathbf{q_N}) \Big|_{p_{1,1}:=p_{i,\alpha}} \Big] \\
&+ \textstyle\sum_{i=1}^{m} \partial_{p_i} \left[\frac{H_{\mathbf{k_L};m-1;n}(\mathbf{S_L};\mathbf{p_{M/\{i\}}};\mathbf{q_N})\big|_{p_{1,1}:=p_i}}{x(p_i)-x(p_{1,1})} \right] \\
&+ \textstyle\sum_{\alpha=2}^{k_1} \frac{1}{x(p_{1,\alpha})-x(p_{1,1})} \times \\
&\Big[H^{(g-1)}_{\alpha-1,k_1-\alpha+1,\mathbf{k_{L/\{1\}}};m;n}(\{p_{1,1}, q_{1,1}, \dots p_{1,\alpha-1}, q_{1,\alpha-1}\}, \{p_{1,\alpha}, q_{1,\alpha}, \dots p_{1,k_1}, q_{1,k_1}\}, \mathbf{S_{L/\{1\}}}; \mathbf{p_M}; \mathbf{q_N}) \\
&- H^{(g-1)}_{\alpha-1,k_1-\alpha+1,\mathbf{k_{L/\{1\}}};m;n}(\{p_{1,\alpha}, q_{1,1}, \dots p_{1,\alpha-1}, q_{1,\alpha-1}\}, \{p_{1,\alpha}, q_{1,\alpha}, \dots p_{1,k_1}, q_{1,k_1}\}, \mathbf{S_{L/\{1\}}}; \mathbf{p_M}; \mathbf{q_N}) \Big] \\
&+ H^{(g-1)}_{\mathbf{k_L};m+1;n}(\mathbf{S_K}; p_{1,1}, \mathbf{p_M}; \mathbf{q_N}).
\end{aligned}
$$

The Quartic Model

The **partition function** is

$$\mathcal{Z} = \int_{H_N} \mathcal{D}\phi e^{-N\mathrm{Tr}(E\phi^2 + \frac{\lambda}{4}\phi^4)},$$

with E be hermitian $N \times N$-matrix with $d \in \mathbb{N}$ (postive) distinct eigenvalues $(e_k)_{k=1,..,d}$, each of mutliplicity r_k and $\lambda \in \mathbb{R}_+$.

The correlation function for pairwise distinct eigenvalues $e_{p_i^j}$ is defined by

$$\left\langle \phi_{p_1^1 p_2^1} \phi_{p_2^1 p_3^1} .. \phi_{p_{N_1}^1 p_1^1} \phi_{p_1^2 p_2^2} .. \phi_{p_1^b p_2^b} .. \phi_{p_{N_b}^b p_1^b} \right\rangle_c$$

with b number of boundaries of length each of length N_i

Generalized Correlation Function

We define (formally) for pairwise distinct $e_{p_j^i}$ and e_{b_i} the **generalized correlation function** by

$$\frac{(-N)^m}{r_{b_1}\ldots r_{b_m}}\frac{\partial^m}{\partial e_{b_1}\partial e_{b_2}\ldots\partial e_{b_m}}\left\langle \phi_{p_1^1p_2^1}\phi_{p_2^1p_3^1}..\phi_{p_{N_1}^1p_1^1}\phi_{p_1^2p_2^2}..\phi_{p_1^bp_2^b}..\phi_{p_{N_b}^bp_1^b}\right\rangle_c$$

and say, it has $m+b$ boundaries.

Relation between Quartic Model $\leftrightarrow$ 2-Matrix Model

boundary labelled by b_i	$\leftrightarrow$	non-mixed boundary
boundaries labelled by p_i^j	$\leftrightarrow$	mixed boundary

Spectral Curve of the Quartic Model

Define the (genus zero) **spectral curve** by

$$\begin{aligned} x(z) &= R(z) \\ y(z) &= -R(-z), \end{aligned}$$

$$\text{with} \quad R(z) := z - \frac{\lambda}{N} \sum_k \frac{r_k}{R'(\varepsilon_k)(z+\varepsilon_k)}, \qquad e_p = R(\varepsilon_p).$$

In particular, $y(z)$ is of the same form as in the Kontsevich model

Both functions are related by **x(z)=-y(-z)** which is the global Galois-involution $\sigma(z) = -z$ of the Kontsevich model

The number of branch points $dx(z) = 0$ depends on the number of distinct eigenvalues of E

Correlation functions on the spectral curve

The correlation functions are special points on the spectral curve of the differential forms

$$\sum_g N^{2-2g-b}\mathcal{T}^{(g)}(\varepsilon_{b_1};\varepsilon_{b_2};..;\varepsilon_{b_m}|\varepsilon_{p_1^1},\varepsilon_{p_2^1},..,\varepsilon_{p^1_{N_1}}|\varepsilon_{p_1^2},..|\varepsilon_{p_1^b},..,\varepsilon_{p^b_{N_b}}|)$$
$$=\frac{(-N)^m}{r_{b_1}...r_{b_m}}\frac{\partial^m}{\partial e_{b_1}\partial e_{b_2}...\partial e_{b_m}}\Big\langle\phi_{p_1^1p_2^1}\phi_{p_2^1p_3^1}..\phi_{p^1_{N_1}p_1^1}\phi_{p_1^2p_2^2}..\phi_{p_1^bp_2^b}..\phi_{p^b_{N_b}p_1^b}\Big\rangle_c$$
$$+\frac{\delta_{m,2}\delta_{g,0}\delta_{b,0}}{(e_{b_1}-e_{b_2})^2}+\frac{\delta_{m,1}\delta_{g,0}\delta_{b,0}}{\lambda}\Big(e_{b_1}+\frac{\lambda}{N}\sum_k\frac{r_k}{e_k-e_{b_1}}\Big),$$

where $e_p=R(\varepsilon_p)$ and ε_p in the physical sheet, i.e. $\lim_{\lambda\to 0}\varepsilon_p=e_p$

Results

Theorem [H. Grosse, R. Wulkenhaar, AH '19]

For $x(z) = R(z)$ and $y(z) = -R(-z)$ with $R(z) = z - \frac{\lambda}{N}\sum_k \frac{r_k}{R'(\varepsilon_k)(z+\varepsilon_k)}$ and $e_p = R(\varepsilon_p)$, we find

$$\mathcal{T}^{(0)}(z) = y(z)$$

$$\mathcal{T}^{(0)}(|z,w|) = \frac{1}{y(w)-y(-z)}\prod_{i=1}^{d}\frac{x(w)-x(-\hat{z}^i)}{x(w)-x(\varepsilon_k)}$$

with $x(z) = x(\hat{z}^i)$ and $z \neq \hat{z}^i$.

Structurally the same as in the 2-Matrix model by identifying

$$\mathcal{T}^{(0)}(z) \quad \leftrightarrow \quad H^{(0)}_{0;1;0}(z)$$

$$\mathcal{T}^{(0)}(|z,-w|) \quad \leftrightarrow \quad H^{(0)}_{1;0;0}(\{z,w\})$$

Loop equations

$$
\begin{aligned}
&(R(w)-R(-z))\mathcal{T}^{(g)}(u_1;\ldots;u_m|z,w|\mathcal{J}|)-\frac{\lambda}{N}\sum_k r_k\frac{\mathcal{T}^{(g)}(u_1;\ldots;u_m|\varepsilon_k,w|\mathcal{J}|)}{R(\varepsilon_k)-R(z)}\\
=&\delta_{0,m}\delta_{0,|\mathcal{J}|}\delta_{g,0}-\lambda\Bigg\{\sum_{\substack{\mathcal{I}\uplus\mathcal{I}'=\mathcal{J},\,(\mathcal{K},\mathcal{I},h)\neq(\emptyset,\emptyset,0)\\ \mathcal{K}\uplus\mathcal{K}'=\{u_1;..;u_m\}\\ h+h'=g}}\mathcal{T}^{(h')}(\mathcal{K}'|z,w|\mathcal{I}'|)\mathcal{T}^{(h)}(\mathcal{K};z|\mathcal{I}|)\\
&+\sum_{\substack{\mathcal{I}\uplus\mathcal{I}'=\mathcal{J}\\ \mathcal{K}\uplus\mathcal{K}'=\{u_1;..;u_m\}\\ h+h'=g}}\mathcal{T}^{(h)}(\mathcal{K}|w|\mathcal{I}|)\frac{\mathcal{T}^{(h')}(\mathcal{K}'|z|\mathcal{I}'|)-\mathcal{T}^{(h')}(\mathcal{K}'|w|\mathcal{I}'|)}{R(w)-R(z)}+\sum_{i=1}^m\frac{\partial}{\partial R(u_i)}\frac{\mathcal{T}^{(g)}(u_1;..;\check{u}_i;..;u_m|u_i,w|\mathcal{J}|)}{R(u_i)-R(z)}\\
&+\sum_{\beta=2}^b\sum_{j=1}^{N_\beta}\frac{\mathcal{T}^{(g)}(u_1;\ldots;u_m|w,z,z_j^\beta,..,z^\beta_{N_\beta+j-1}|\mathcal{J}\backslash\{J^\beta\}|)-\mathcal{T}^{(g)}(u_1;\ldots;u_m|w,z_j^\beta,..,z^\beta_{N_\beta+j}|\mathcal{J}\backslash\{J^\beta\}|)}{R(z_j^\beta)-R(z)}\\
&+\frac{\mathcal{T}^{(g-1)}(u_1;\ldots;u_m|z|w|\mathcal{J}|)-\mathcal{T}^{(g-1)}(u_1;\ldots;u_m|w|w|\mathcal{J}|)}{R(w)-R(z)}+\mathcal{T}^{(g-1)}(u_1;\ldots;u_m;z|z,w|\mathcal{J}|)\Bigg\},
\end{aligned}
$$

$\rightarrow$ Structurally the same as in the 2-Matrix model

More Results

Algebraic/recursive equation for $N_1 > 2$, $g = 0$, $b = 1$

$$\mathcal{T}^{(0)}(|z_1, z_2, .., z_N|) = -\lambda \sum_{k=1}^{\frac{N-2}{2}} \frac{\mathcal{T}^{(0)}(|z_{2k+2}, .., z_N, z_1|)\mathcal{T}^{(0)}(|z_2, .., z_{2k+1}|) - \mathcal{T}^{(0)}(|z_{2k+1}, .., z_N|)\mathcal{T}^{(0)}(|z_1, z_2, .., z_{2k}|)}{(x(z_{2k+1}) - x(z_{p_1}))(x(z_{p_2}) - x(z_{p_N}))},$$

Equal to the algebraic equation of 2-Matrix model, genus $g = 0$, one mixed-boundary

Two Boundaries [J. Branahl, R. Wulkenhaar, AH '20]

$$\mathcal{T}^{(0)}(z;w) = \frac{1}{x'(z)x'(w)}\left(\frac{1}{(z-w)^2} + \frac{1}{(z+w)^2}\right)$$

$$\begin{aligned}\mathcal{T}^{(0)}(v|z,w|) = \lambda\mathcal{T}^{(0)}(|z,w|)\frac{\partial}{\partial x(v)}\bigg\{&\frac{1}{x'(z)(y(z)-y(-w))}\left(\frac{1}{v+z}+\frac{1}{v-z}\right)\\ &+\frac{1}{(x(z)-x(v))(y(v)-y(-w))}\\ &-\sum_{j=1}^{d}\frac{1}{x'(\hat{w}^j)(x(z)-x(-\hat{w}^j))}\left(\frac{1}{v+\hat{w}^j}+\frac{1}{v-\hat{w}^j}\right)\bigg\}\end{aligned}$$

$\rightarrow$ Both solutions are **linear combinations** of the corresponding solutions in the 2-Matrix model

$\mathcal{T}^{(0)}(z;w) = \frac{1}{2}H^{(0)}_{0;2;0}(z;w) - H^{(0)}_{0;1;1}(z;-w) + \frac{1}{2}H^{(0)}_{0;0;2}(-z;-w)$
$\mathcal{T}^{(0)}(v|z,w|) = H^{(0)}_{1;1;0}(\{z,-w\};v) - H^{(0)}_{1;0;1}(\{z,-w\};-v)$

Pair of pants

$$\mathcal{T}^{(0)}(z;v;u) = \frac{\lambda\partial^3}{\partial x(z)\partial x(v)\partial x(u)}\Big[\frac{1}{x'(u)y'(-u)(z+u)}\Big(\frac{1}{v+u}+\frac{1}{v-u}\Big)$$
$$+\frac{1}{x'(v)y'(-v)(z+v)}\Big(\frac{1}{u+v}+\frac{1}{u-v}\Big)$$
$$+\sum_{i=1}^{2d}\frac{1}{y'(-\beta_i)x''(\beta_i)(z-\beta_i)}\Big(\frac{1}{v+\beta_i}+\frac{1}{v-\beta_i}\Big)\Big(\frac{1}{u+\beta_i}+\frac{1}{u-\beta_i}\Big)\Big],$$

where β_i are the branch points $x'(\beta_i)=0$.

Again, $\mathcal{T}^{(0)}(z;v;u)$ is a **linear combination** of the corresponding solutions in the 2-Matrix model $H^{(0)}_{0;3;0}$, $H^{(0)}_{0;2;1}$, $H^{(0)}_{0;1;2}$ and $H^{(0)}_{0;0;3}$

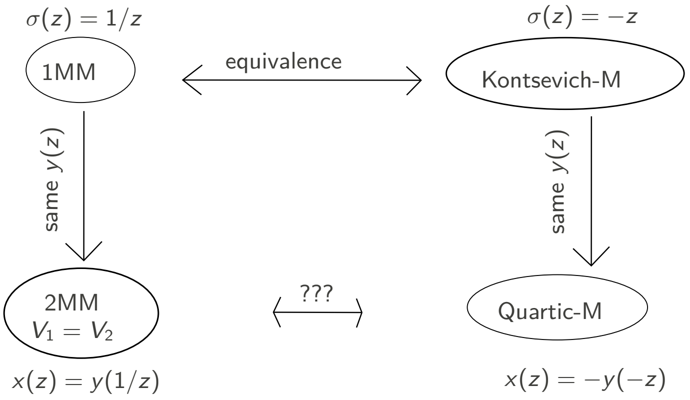
Genus $g = 1$ disc

$$
\begin{aligned}
x'(z)\mathcal{T}_1^{(1)}(z) = &-\frac{\lambda}{8(x'(0))^2 z^3} + \frac{\lambda x''(0)}{16(x'(0))^3 z^2} - \sum_{i=1}^{2d} \frac{\lambda}{8\beta_i^2 x''(\beta_i) y'(\beta_i)(z-\beta_i)^2} \\
&+ \lambda \sum_{i=1}^{2d} \Big\{ -\frac{1}{8y'(\beta_i)x''(\beta_i)(z-\beta_i)^4} + \frac{x'''(\beta_i)}{24(x''(\beta_i))^2 y'(\beta_i)(z-\beta_i)^3} \\
&+ \frac{y'''(\beta_i)}{48x''(\beta_i)(y'(\beta_i))^2(z-\beta_i)^2} - \frac{x'''(\beta_i)y''(\beta_i)}{48(x''(\beta_i))^2(y'(\beta_i))^2(z-\beta_i)^2} \\
&+ \frac{x''''(\beta_i)}{48(x''(\beta_i))^2 y'(\beta_i)(z-\beta_i)^2} - \frac{(x'''(\beta_i))^2}{48(x''(\beta_i))^3 y'(\beta_i)(z-\beta_i)^2} \Big\},
\end{aligned}
$$

where the last three lines are the general solution of **TR** for a genus zero spectral curve

The first line differs from **TR**

Summary: Relations between the Models



Work for the future

- Analyzing the linear combination the corrlation functions are built of

From mathematics perspective

- Study of more general spectral curves (beside the genus zero assumption)
- What is the corresponding moduli space?
- Does the previous procedure apply to other models with global Galois-involutions?

From physics perspective

- Performing the conitnuum limit $d \to \infty$ with infinitly many distinct eigenvalues $e_k \to$ construction of a qunatum field theory
  - Feynman graph expansion yields iterative integrals
  - For $e_\infty \to \infty$ renormalization gets necessary (UV-limit)
  - For instance $(D = 2)$, $e_k = \frac{k}{N}$ and $r_k = 1$ yields $R(z) = z + \lambda \log(1 + z)$
- In $D = 4$, the spectral dimension reduces to $D_\lambda = 4 - 2\frac{\arcsin(\lambda\pi)}{\pi}$ with $R(z) = z\,{}_2F_1(\alpha_\lambda, 1 - \alpha_\lambda; 2| - z)$ and $\alpha_\lambda = \frac{\arcsin(\lambda\pi)}{\pi}$
- Constructing generating functions of iterated integrals

Thank You