Partial Duality of Hypermaps

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Outline

Partial Duality of Maps

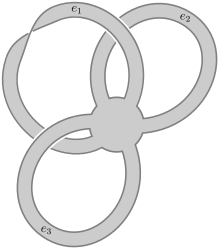
Ribbon Graphs

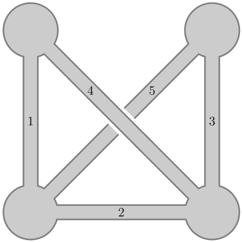
or cyclic graphs or combinatorial maps. . .

There are many equivalent definitions:

- (not necessarily orientable) surfaces equipped with a decomposition into vertices and edges,
- cellular embeddings of graphs,
- graphs + cyclic orders around vertices (orientable case),
- triple of permutations $(\sigma_0, \sigma_1, \sigma_2)$ such that $\sigma_0\sigma_1\sigma_2 = \text{id}$,
- etc

Examples :

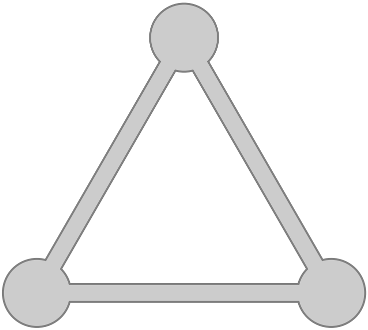




Natural Duality

aka Euler-Poincaré duality

It exchanges faces and vertices while conserving genus.



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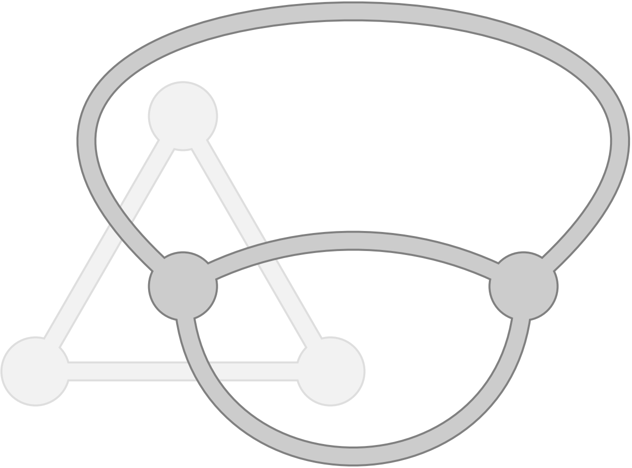
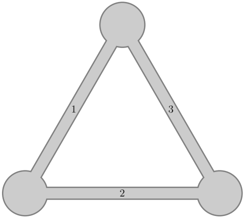


Partial Duality

A generalisation of natural duality

Let G be a ribbon graph and $A \subseteq E(G)$. Here is how to construct $\boldsymbol{G^A}$, the **partial dual of G wrt** the spanning subgraph $\boldsymbol{G_A}$:



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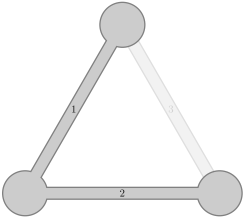

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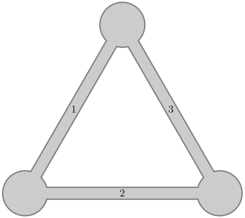

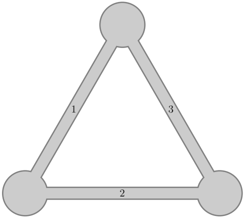

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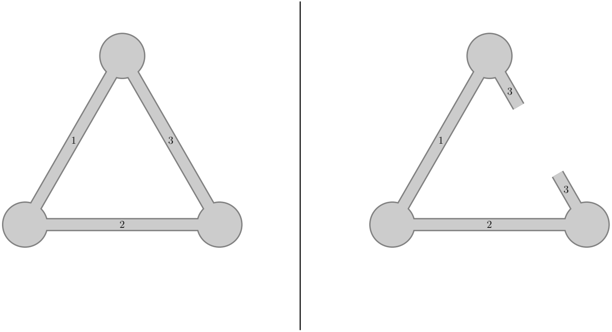

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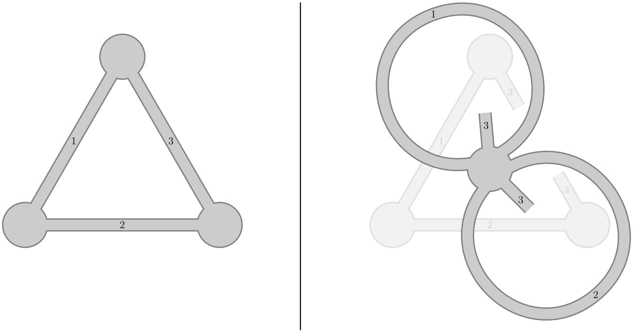

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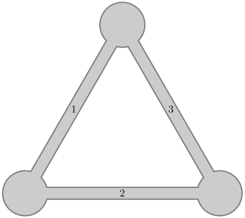

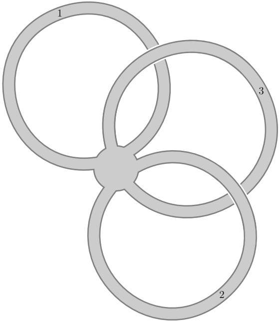

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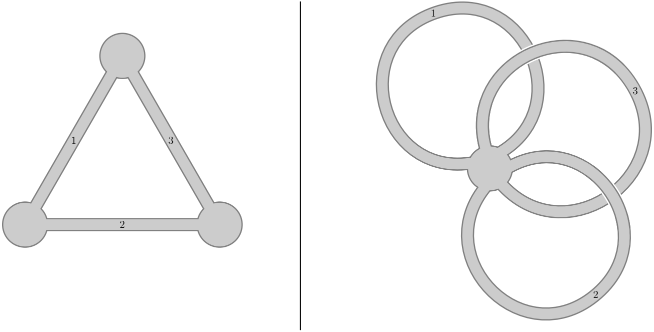


Genus is *not* conserved.

Partial Duality

Properties

Can be proven within the formalism of combinatorial maps.

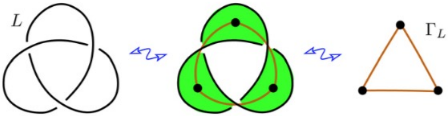
1. Partial duality generalises natural duality: $G^{E(G)} = G^\star$.
2. It's an involution: $\left(G^A\right)^A = G$.
3. It can be done edge by edge: if $e \notin A$ then $G^{A \cup \{e\}} = \left(G^A\right)^{\{e\}}$.
4. Let $A, B \subseteq E(G)$, then $\left(G^A\right)^B = G^{A \Delta B}$.

Thistlethwaite's type theorems

Classical links

- M. Thistlethwaite (1987): L an alternating link, Γ_L a planar graph

$$V_L(t) \propto T_{\Gamma_L}(-t, -t^{-1})$$



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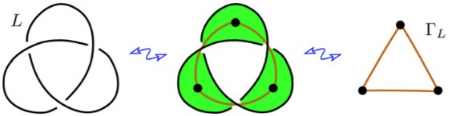
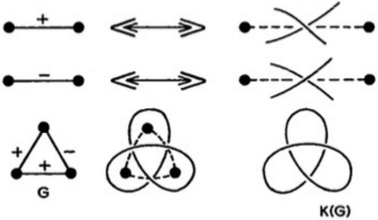


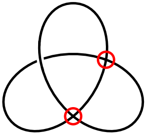
- L. Kauffman (1989): generalises to any link.



Thistlethwaite's type theorems

Virtual links

- *Virtual* knots are embeddings of the circle into $S_g \times [0,1]$.



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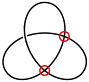

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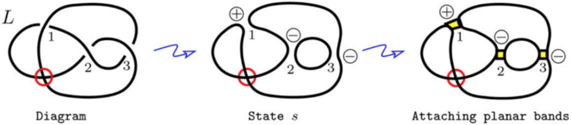

$$[L](A,B,d) = A^{e(G_s)}\left(x^k y^v z^{v+1}\ R_{G_s}(x,y,z)|_{x=Ad/B, y=Bd/A, z=1/d}\right).$$

Thistlethwaite's type theorems

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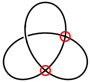

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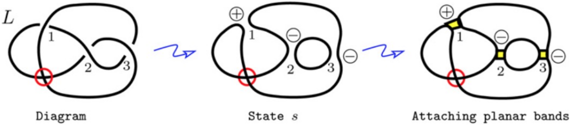

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- S. Chmutov (2007): $A \subseteq E(G)$, $G' := G^A$

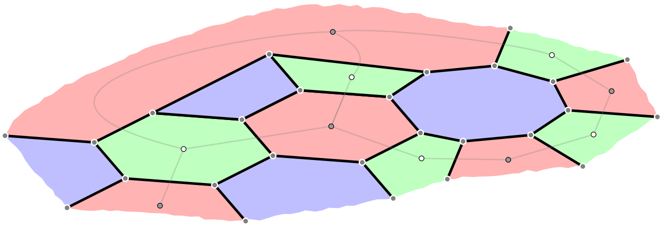
$$x^{k(G)}y^{v(G)}z^{v(G)+1}R_G(x,y,z)|_{xyz^2=1} = x^{k(G')}y^{v(G')}z^{v(G')+1}R_{G'}(x,y,z)|_{xyz^2=1}.$$

And, if s, s' are 2 states of L, G_s and $G_{s'}$ are partial duals of each other.

Hypermaps

What is a hypermap?

- Both a generalisation and a particularisation of maps.
- A cellular embedding of a bipartite graph.
- A 3-coloured polygonal tessalation of a closed compact 2-manifold.



What is a hypermap?

After Tutte (1973)

Definition (Combinatorial hypermap)

Let X be a finite set of even cardinality. A pre-hypermap on X is a 3-constellation $(\sigma_0, \sigma_1, \sigma_2)$ on X and a fixed point free involution θ_1 which obey the following axioms:

1. $\sigma_0\theta_1$ *and* $\theta_1\sigma_1$ *are involutions.*
2. $\forall x \in X$, $\mathcal{O}_{\sigma_0}(x) \cap \mathcal{O}_{\sigma_0}(\theta_1 x) = \varnothing$.
3. $\forall x \in X$, $\mathcal{O}_{\sigma_1}(x) \cap \mathcal{O}_{\sigma_1}(\theta_1 x) = \varnothing$.

A pre-hypermap such that $\langle \sigma_0, \sigma_1, \theta_1 \rangle$ acts transitively on X is called a combinatorial hypermap.

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Definition (Orientability)

A hypermap $\mathcal{H} = (\sigma_0, \sigma_1, \sigma_2; \theta_1)$ is orientable if there are two equivalence classes of $\langle \sigma_0, \sigma_1 \rangle$.

Operations on Hypermaps

cut

Notation: σ a permutation on X and $Y \subseteq X$. $\sigma_{\restriction Y} := \sigma$ on Y and id on $\overline{Y}$.

Definition (cut)

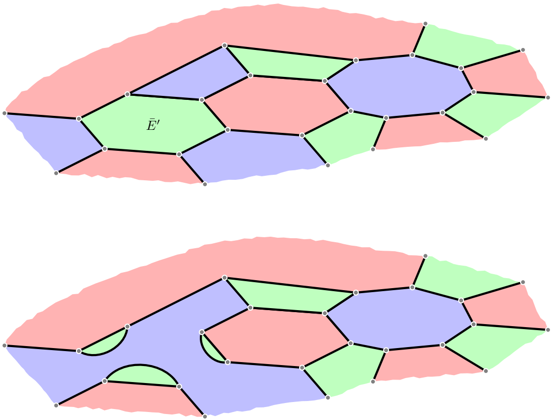
Let $\mathcal{H} = (\sigma_0, \sigma_1, \sigma_2; \theta_1)$ be a pre-hypermap on X and E' (resp. V', F') be a subset of conjugate pairs of σ_1 (resp. σ_0, σ_2). Then

$$\begin{aligned}
cut_{1,0}(\mathcal{H}, E') &:= \left(\sigma_0, \sigma_{1\restriction E'}, (\sigma_0\sigma_{1\restriction E'})^{-1}; \theta_1\right),\\
cut_{1,2}(\mathcal{H}, E') &:= \left((\sigma_{1\restriction E'}\sigma_2)^{-1}, \sigma_{1\restriction E'}, \sigma_2; \theta_1\sigma_{1\restriction \overline{E}'}\right),\\
cut_{0,1}(\mathcal{H}, V') &:= \left(\sigma_{0\restriction V'}, \sigma_1, (\sigma_{0\restriction V'}\sigma_1)^{-1}; \theta_1\right),\\
cut_{0,2}(\mathcal{H}, V') &:= \left(\sigma_{0\restriction V'}, (\sigma_2\sigma_{0\restriction V'})^{-1}, \sigma_2; \sigma_{0\restriction \overline{V}'}\theta_1\right),\\
cut_{2,0}(\mathcal{H}, F') &:= \left(\sigma_0, (\sigma_{2\restriction F'}\sigma_0)^{-1}, \sigma_{2\restriction F'}; \theta_1\right),\\
cut_{2,1}(\mathcal{H}, F') &:= \left((\sigma_1\sigma_{2\restriction F'})^{-1}, \sigma_1, \sigma_{2\restriction F'}; \theta_1\right).
\end{aligned}$$

Operations on Hypermaps

$\text{cut}_{1,0}$

An example: vertices are red, edges are green.



Definition (Colour change)

Let $\mathcal{H} = (\sigma_0, \sigma_1, \sigma_2; \theta_1)$ be a pre-hypermap on X and S_3 be the permutations on $\{0,1,2\}$. Let $\pi \in S_3$ be a transposition. Then

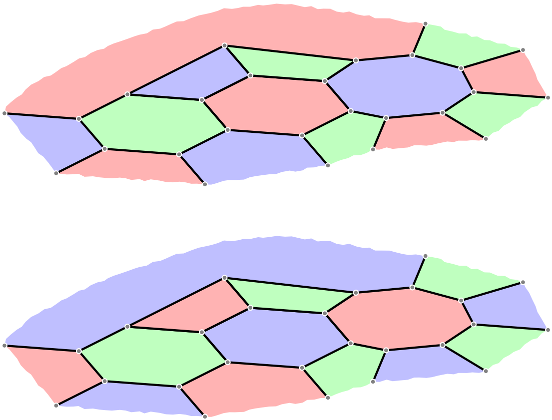
$$\Pi\mathcal{H} := (\sigma_{\pi(0)}^{-1}, \sigma_{\pi(1)}^{-1}, \sigma_{\pi(2)}^{-1}; \theta_1'), \quad \theta_1' := \begin{cases} \theta_1 & \text{if } \pi = (01) \\ \theta_1\sigma_1 & \text{if } \pi = (02) \\ \sigma_0\theta_1 & \text{if } \pi = (12) \end{cases}.$$

Any permutation (of S_3) is a product of transpositions. Its action on (pre-)hypermaps is defined as the composed actions of these transpositions.

Operations on Hypermaps

Colour change

An example: π_{02} (vertices are red, faces are blue).



Definition (glue)

Let $\mathcal{H} = (\sigma_0, \sigma_1, \sigma_2; \theta_1)$ *be a pre-hypermap on* X *and* Y *be a subset of* X*. Let* σ *be a permutation on* X *such that* $\sigma = \mathrm{id}$ *on* Y*. Assume that the set of fixed points of* σ_1 *is* $\overline{Y}$ *and* $\theta_1\sigma_1\sigma$ *is a fixed-point free involution. Then*

$$
\begin{aligned}
glue_{1,0}(\mathcal{H}, \sigma) &:= \left(\sigma_0, \sigma_1\sigma, (\sigma_0\sigma_1\sigma)^{-1}; \theta_1\right), \\
glue_{1,2}(\mathcal{H}, \sigma) &:= \left((\sigma_1\sigma\sigma_2)^{-1}, \sigma_1\sigma, \sigma_2; \sigma\theta_1\right).
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glue_{0,1}(\mathcal{H}, \sigma) &:= \left(\sigma_0\sigma, \sigma_1, (\sigma_0\sigma\sigma_1)^{-1}; \theta_1\right), \\
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\end{aligned}
$$

...

Partial Duality of Hypermaps

Operations on Hypermaps

Definition

The partial duals of $\mathcal{H}$ with respect to E' (resp. V', F') are defined as

$$\mathcal{H}^{E',V} := glue_{1,0}\big(\pi_{02} cut_{1,0}(\mathcal{H}, E'), \sigma_{1\restriction\overline{E'}}\big) = (\sigma_0\sigma_{1\restriction E'}, \sigma_{1\restriction E'}^{-1}\sigma_{1\restriction\overline{E'}}, \sigma_{1\restriction E'}\sigma_2; \theta_1\sigma_{1\restriction E'}),$$

$$\mathcal{H}^{E',F} := glue_{1,2}\big(\pi_{02} cut_{1,2}(\mathcal{H}, E'), \sigma_{1\restriction\overline{E'}}\big),$$

$$\mathcal{H}^{V',E} := glue_{0,1}\big(\pi_{12} cut_{0,1}(\mathcal{H}, V'), \sigma_{0\restriction\overline{V'}}\big) = (\sigma_{0\restriction V'}^{-1}\sigma_{0\restriction\overline{V'}}, \sigma_{0\restriction V'}\sigma_1, \sigma_2\sigma_{0\restriction V'}; \sigma_{0\restriction V'}\theta_1),$$

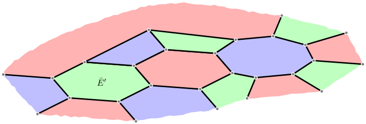
$$\mathcal{H}^{V',F} := glue_{0,2}\big(\pi_{12} cut_{0,2}(\mathcal{H}, V'), \sigma_{0\restriction\overline{V'}}\big),$$

$$\mathcal{H}^{F',V} := glue_{2,0}\big(\pi_{01} cut_{2,0}(\mathcal{H}, F'), \sigma_{2\restriction\overline{F'}}\big) = (\sigma_{2\restriction F'}\sigma_0, \sigma_1\sigma_{2\restriction F'}, \sigma_{2\restriction F'}^{-1}\sigma_{2\restriction\overline{F'}}; \theta_1),$$

$$\mathcal{H}^{F',E} := glue_{2,1}\big(\pi_{01} cut_{2,1}(\mathcal{H}, F'), \sigma_{2\restriction\overline{F'}}\big).$$

Partial Duality of Hypermaps

Example 1: $\mathcal{H}^{E',V}$ (vertices are red, edges are green)



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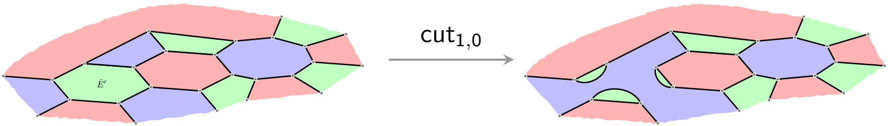

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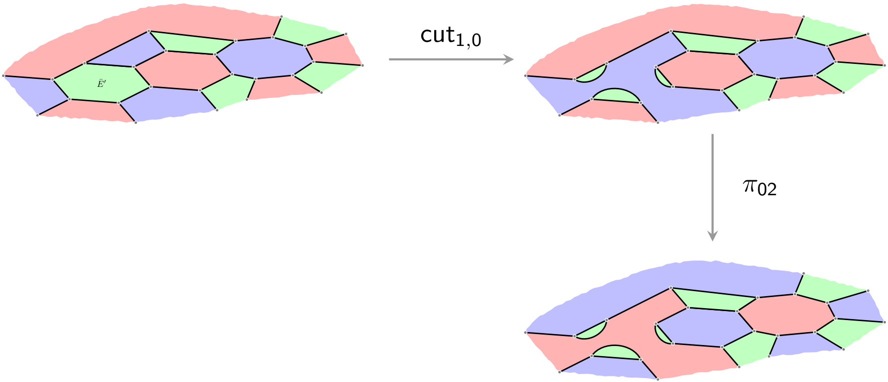

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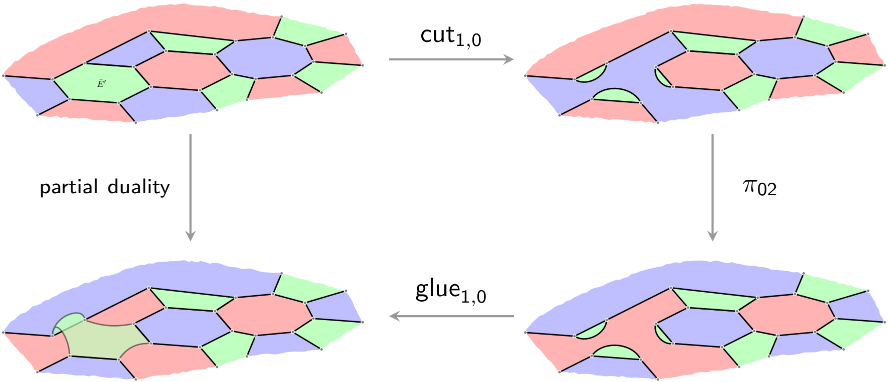
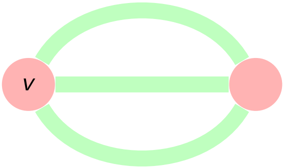


Partial Duality of Hypermaps

Example 2: $\mathcal{H}^{v,E}$ (vertices are red, edges are green)



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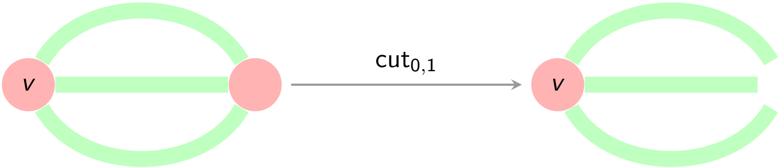

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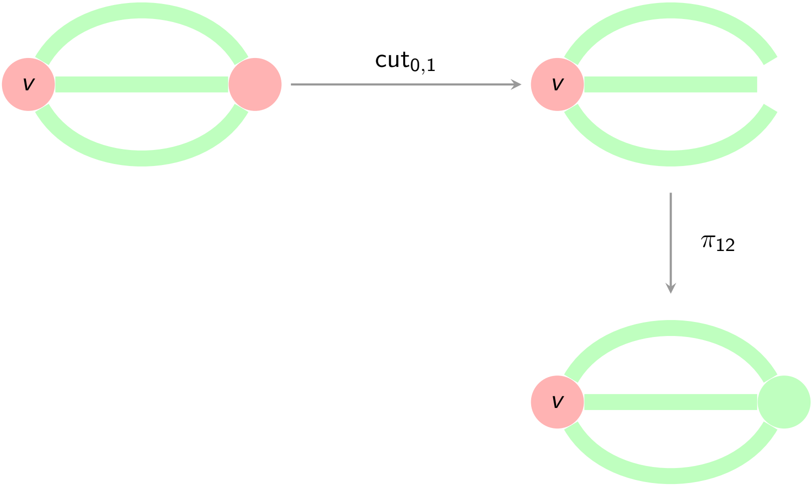

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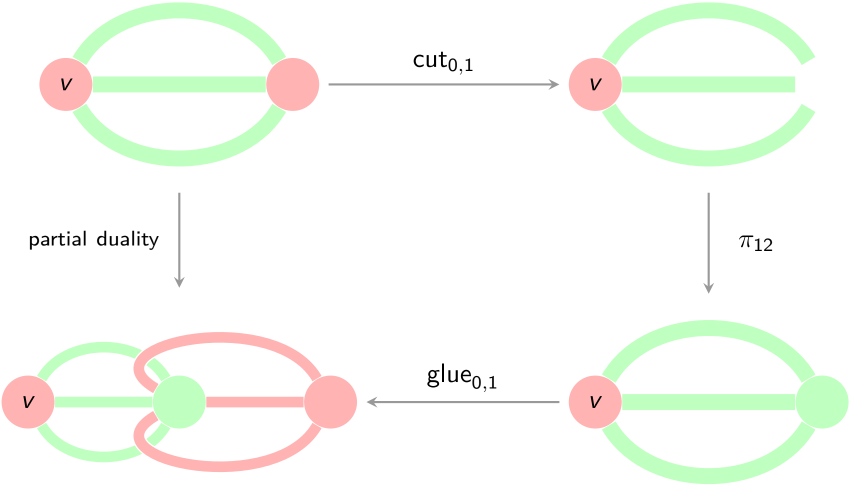

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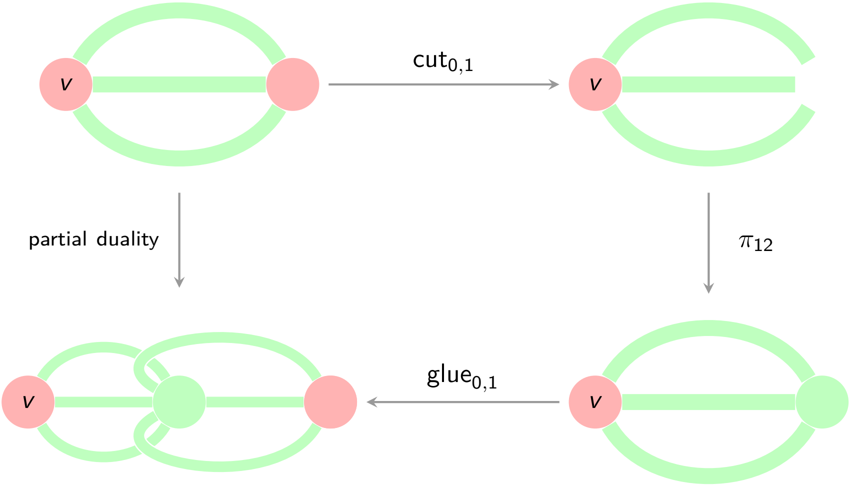

Partial Duality of Hypermaps

Properties

Lemma

Let $\mathcal{H}$ be a pre-hypermap on X and $E', E'' \subseteq X$ (resp. V', V'', resp. F', F'') be subsets of conjugate pairs of σ_1 (resp. σ_0, resp. σ_2). Then

$$(\mathcal{H}^{E'})^{E''} = \mathcal{H}^{E' \Delta E''}, \quad (\mathcal{H}^{V'})^{V''} = \mathcal{H}^{V' \Delta V''}, \quad (\mathcal{H}^{F'})^{F''} = \mathcal{H}^{F' \Delta F''}$$

where Δ denotes the symmetric difference of sets. Also

$$\mathcal{H}^{\varnothing} = \mathcal{H}, \quad \mathcal{H}^{E'=X} = \pi_{02}\mathcal{H}, \quad \mathcal{H}^{V'=X} = \pi_{12}\mathcal{H}, \quad \mathcal{H}^{F'=X} = \pi_{01}\mathcal{H}.$$

Moreover, partial duality preserves connexity and orientability.

Edge-coloured Graphs

Equivalence

Theorem (Hypermaps = regular 3-edge coloured graphs)

Let $\mathcal{H} = (\sigma_0, \sigma_1, \sigma_2)$ *be a 3-constellation on a finite set* X. *Then* $\mathcal{H}$ *is a hypermap on* X *if and only if there exists three fixed-point free involutions* τ_0, τ_1, τ_2 *on* X *such that* $\sigma_0 = \tau_0\tau_1$, $\sigma_1 = \tau_1\tau_2$, $\sigma_2 = \tau_2\tau_0$, *and* $\langle\tau_0, \tau_1, \tau_2\rangle$ *acts transitively on* X.

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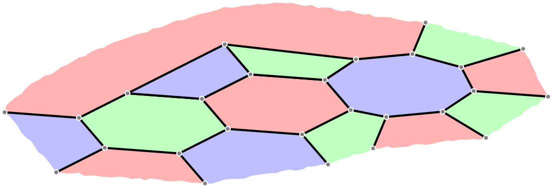

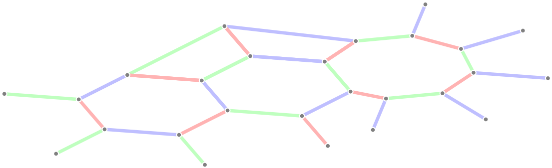

Partial Duality of Edge-coloured Graphs

Definition

Let $\Gamma = (\tau_0, \tau_1, \tau_2)$ *be a* $[2]$*-coloured graph and* E' *(resp.* V', F'*) be a subset of* 12- *(resp.* 01-, 02-*)cycles. Then*

$$\begin{aligned}
\Gamma^{E'} &:= (\tau_0, \tau_{2E'}\tau_{1\overline{E}'}, \tau_{1E'}\tau_{2\overline{E}'}),\\
\Gamma^{V'} &:= (\tau_{1V'}\tau_{0\overline{V}'}, \tau_{0V'}\tau_{1\overline{V}'}, \tau_2),\\
\Gamma^{F'} &:= (\tau_{2F}\tau_{0\overline{F}'}, \tau_1, \tau_{0F'}\tau_{2\overline{F}'}).
\end{aligned}$$

Perspectives

General study of partial duality

Partial duality can be generalized (and extended) to hypermaps.
Very few is known about it (even for maps).

My interests:

- Partial duality and edge-colouring of regular graphs
- Topology of coloured Δ-complexes:
  - manifold,
  - degree,
  - coverings
- Bijection *à la* Schaeffer between Δ-complexes and some decorated trees