Quantum Gravity, Trees, and Polynomials

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Perimeter Institute. April 3, 2014

Outline

Coloured GFT

Group Field Theory

$\phi : G^{\times D} \to \mathbb{C}$, G (compact) Lie group

$$S_D[\phi] = \frac{1}{2} \int_{G^D} \Big(\prod_{i=1}^{D} dg_i d\tilde{g}_i \Big) \phi(g_1, \ldots, g_D) C^{-1}(g_i \tilde{g}_i^{-1}) \phi(\tilde{g}_1, \ldots, \tilde{g}_D)$$
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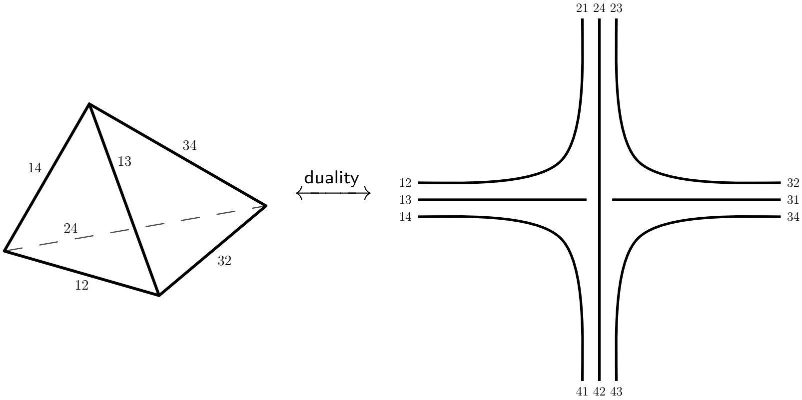
- Feynman graph amplitude = spin foam model

Stranded graphs = gluing of simplices

$D = 3$



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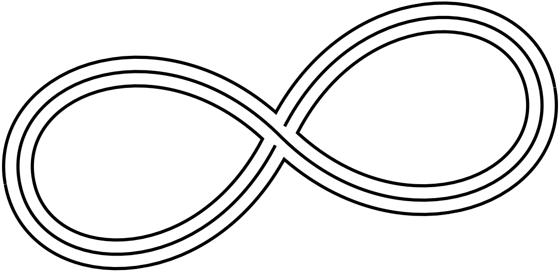

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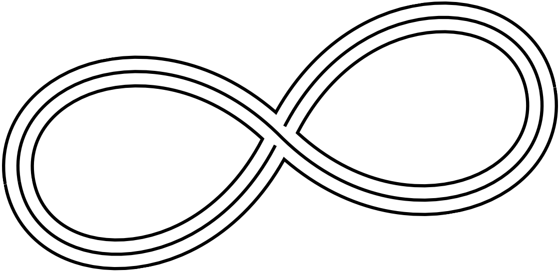
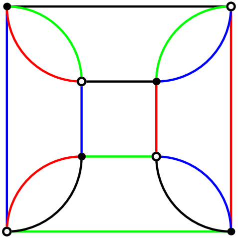


But not a *D*-complex!

Coloured Graphs

Definition

A k-coloured graph is a bipartite graph endowed with a proper edge-colouring with k colours.



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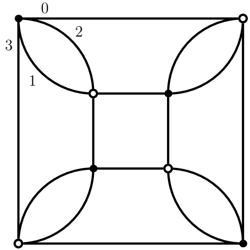

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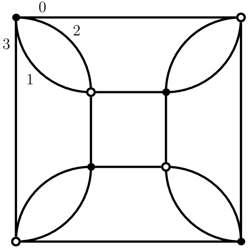

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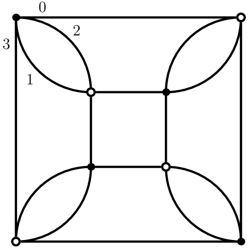

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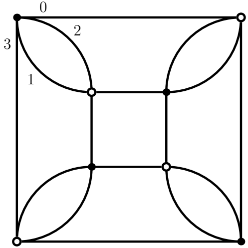

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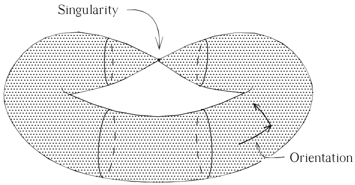
- Every $(D+1)$-coloured graph is dual to a D-dimensional triangulated space (*trisp* or Δ-complex).
- Better, every such graph is dual to a normal pseudo-manifold [Gurau'10].

Pseudo-manifolds

These are manifolds with singularities.



A pseudomanifold (the pinched torus)

A normal pseudo-manifold is such that the boundary of the neighbourhood of each of its points is a pseudo-manifold.

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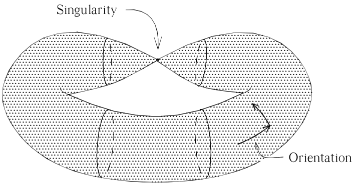

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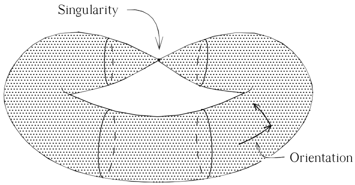

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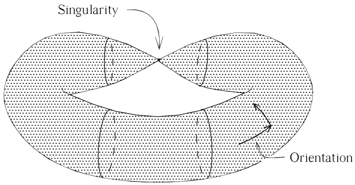

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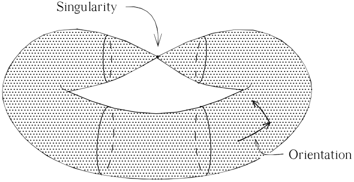

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- GEMs: a combinatorial and algorithmic approach to the classification of 3-manifolds [Ferri, Gagliardi, Lins etc '80].

Coloured Cellular Complex

Definition (Bubbles)

Let Γ *be a* $(D+1)$*-coloured graph and* $0 \leqslant k \leqslant D$*. A* k*-bubble of colours* $\{i_1, \dots, i_k\}$ *is a connected component of the subgraph of* Γ *induced by the edges of colours* $\{i_1, \dots, i_k\}$*.* 0*-bubbles are vertices.*

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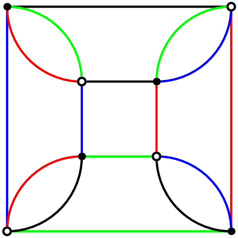

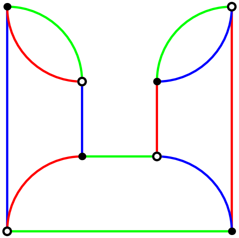

The 3-bubble of colors {red, green, blue}

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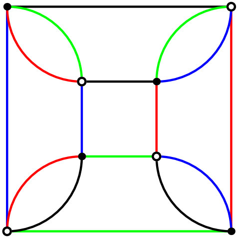

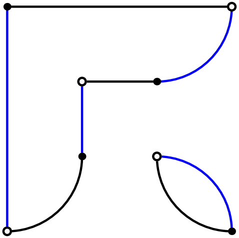

The 2-bubbles (or *faces*) of colors {blue, black}

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Definition (Cellular Complex)

For all $0 \leqslant k \leqslant D$, $C_k(\Gamma) :=$ *free Abelian group generated by the k-bubbles.*
$C_{D+1}(\Gamma) := \{0\}$ *and* $C_{-1}(\Gamma) := \mathbb{Z}$.
$\partial_k : C_k \to C_{k-1}$, $\partial_k b_{i_0,\ldots,i_{k-1}} := \sum_{j=0}^{k-1} (-1)^j b'_{i_0,\ldots,\hat{i}_j,\ldots,i_{k-1}}$.
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Definition (Cellular Homology)

For all $0 \leqslant k \leqslant D$*,* $\widetilde{H}_k(\Gamma) \coloneqq \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}} \eqqcolon \frac{Z_k(\Gamma)}{B_k(\Gamma)}$*.*

The colored Boulatov-Ooguri GFT

[BCORS'13]

Mimicking 3-dimensional gravity

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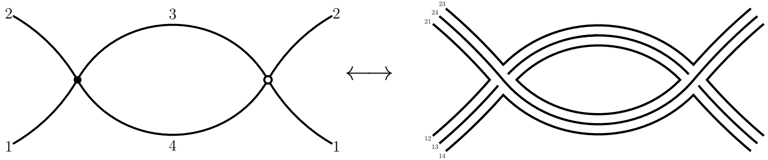
In bijection with $(D+1)$-coloured graphs.



The colored Boulatov-Ooguri GFT

Feynman Amplitudes

- Closure constraint: $P(g_1, \dots, g_D; g'_1, \dots, g'_D) = \int_G dh \prod_i \delta(hg_i {g'}_i^{-1})$.
- Regularization: $\delta(g) = \lim_{\tau \to 0} K_\tau(g)$ (heat kernel).
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- $N_\tau := \tau^{-1/2}$.

$$A_\tau(\mathcal{G}) = N_\tau^k \int \prod_{e \in \mathcal{E}} dh_e \prod_{f \in \mathcal{F}} K_{m_f \tau}\Big(\overrightarrow{\prod_{e \in \partial f}} h_e^{\epsilon_{fe}} \Big)$$

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- $G^{|\mathcal{V}|}$-symmetry: $h_e\to k_{s(e)}h_ek_{t(e)}^{-1}$. Fix it $(h_{e\in\mathcal{T}}=\mathbb{1})$.
- Small τ expansion of K_τ.

$$F_{\tau,\lambda\overline{\lambda}}\underset{\tau\to 0}{\sim}N_\tau^{(\dim G)(D-1)}F^{(0)}_{\lambda\overline{\lambda}},\qquad F^{(0)}_{\lambda\overline{\lambda}}=\sum_{p\in\mathbb{N}}\frac{(\lambda\overline{\lambda})^p}{p}\sum_{\mathcal{G}\in\boldsymbol{M_p}}a(\mathcal{G}),$$

$$a(\mathcal{G})=\Big(\mathbf{det}(\tilde{\boldsymbol{L}})\prod_f m_f\Big)^{-\frac{\dim G}{2}},\qquad L_{ee'}=\sum_f\frac{1}{m_f}\epsilon_{ef}\epsilon^T_{fe'}.$$

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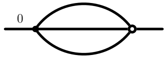
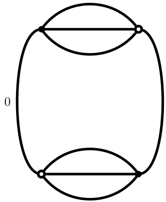
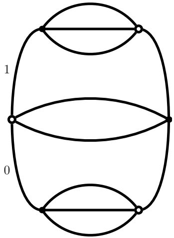


A basic 0-melon



A melonic graph in M_2



Another one in M_3

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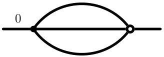

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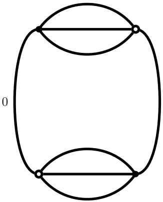

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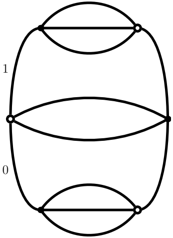
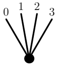
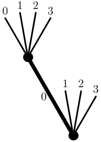
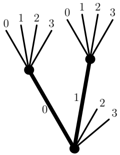


Another one in M_3

- In bijection with colored $(D+1)$-ary trees.







Cellular Trees

From Graph Theory to Algebraic Topology

Homological Definition of (1-)trees

Definition

Let G be a graph. A spanning (1-)tree of G is a subgraph g of G such that:

1. $V(g) = V(G)$ *(spanning),*
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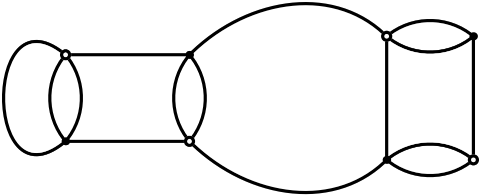

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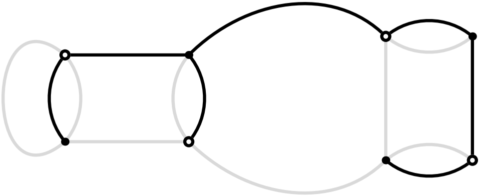

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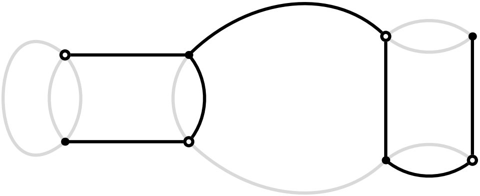

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- Examples: a triangulation of $\mathbb{S}^2$ - one 2-cell is a 2-tree. $\mathbb{R}\mathrm{P}^2$ is a 2-tree.

Matrix-Tree Theorem

Theorem ($D = 1$)

Let G be a graph and $\partial_1 : C_1(G) \to C_0(G)$ be its boundary operator. Then, for all $v \in V(G)$,

$$\det(\partial_1 \partial_1^T)_v = \# \{\textit{spanning trees in } G\}.$$

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Let Δ be a cell complex of dimension n and $k \leqslant n$. If $\widetilde{\beta}_{k-1}(\Delta) = \widetilde{\beta}_{k-2}(\Delta) = 0$, then

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$$L_{ee'} = \sum_f \frac{1}{m_f} \epsilon_{ef} \epsilon^T_{fe'}, \quad \det \widetilde{L} = \sum_{\delta \in \mathcal{T}_2(\mathcal{G})} |\widetilde{H}_1(\delta)|^2 \prod_{f \in \delta_2} m_f^{-1}.$$

An Enumeration Strategy

The Tutte Polynomial

- Remark: $D = 3$, 2-trees in Δ in bijection with 1-trees in Δ^*.
- **Aim**: Counting spanning 1-trees in the dual complex of 3 dim. melons.

Definition ((Multivariate) Tutte Polynomial)

Let G be a graph, $c(G)$ its number of connected components.

$$Z_G(q; \boldsymbol{v}) := \sum_{\text{spann. subgr. } g} q^{c(g)} \prod_{e \in E(g)} v_e.$$

$$X_G(\boldsymbol{v}) := \lim_{\lambda \to 0} \lambda^{c(G) - v(G)} \lim_{q \to 0} q^{-c(G)} Z_G(q; \lambda \boldsymbol{v})$$

Proposition

If G is connected, $X_G(\mathbf{1}) = \#\{\text{spann. trees of } G\}$.

How to Count Trees in Melons?

Let $\mathcal{G}$ be a 3-dim. melon, Δ its associated cell complex.
We need to compute $X_{\Delta^\star_{(1)}}(\mathbf{1})$.

Use the recursive structure of melons!

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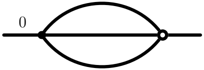
Proposition

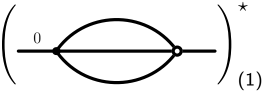
X_G obeys the following reduction relations

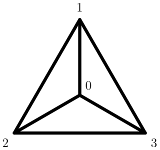
$$X_G = \begin{cases} v_e X_{G/e} & \text{if } e \text{ is a bridge,} \\ X_{G-e} & \text{if } e \text{ is a loop,} \\ v_e X_{G/e} + X_{G-e} & \text{if } e \text{ is ordinary,} \\ X_{G'}(v_e + v_{e'}) & \text{if } e \parallel e'. \end{cases}$$

Contracting Melons

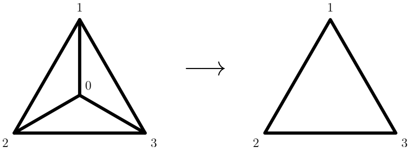
- 

$=$ 3-ball made of the gluing of 2 tetrahedra along 3 facets.

- 

$=$ 

- Contracting a melon in $\mathcal{G}$ amounts to, in $\Delta^\star$:



A Recursion

Proposition

$$X_{\Delta^\star_{(1)}}(v_{01}, v_{02}, v_{03}, v_{12}, v_{13}, v_{23}, \dots) = v_{01} X_{(\Delta/M_0)^\star_{(1)}}(v_{12}+v_{02}, v_{13}+v_{03}, v_{23}, \dots)$$
$$+ v_{02} X_{(\Delta/M_0)^\star_{(1)}}(v_{12}, v_{13}, v_{23}+v_{03}, \dots) + v_{03} X_{(\Delta/M_0)^\star_{(1)}}(v_{12}, v_{13}, v_{23}, \dots)$$

This is *not* a recursion for the number of spanning trees!

A Recursion

which converges sometimes

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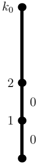
The number of 2-trees in 

is: $3^{k_0}(k_0 + 4)^2$.

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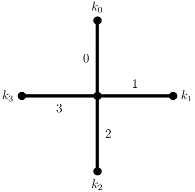
The number of 2-trees in 

is:

$$3^{\sigma_1 - 3}(6\sigma_1\sigma_2 - 4\sigma_3 + 27\sigma_1^2 + 24\sigma_2 + 216\sigma_1 + 432),$$

with $\sigma_1 = \sum_{i=0}^3 k_i$, $\sigma_2 = \sum_{i<j} k_i k_j$, $\sigma_3 = \sum_{i<j<l} k_i k_j k_l$.

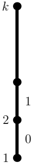
Summary and perspectives

- k-trees are k-dim. generalizations of trees.
- Such trees appear in the Feynman amplitudes of GFT.
- Their enumeration would help to characterize the critical UV behaviour of the model.

A lot remains to be done:

- Complete enumeration,
- Counting k-trees in dimension D (via the Tutte-Krushkal polynomial?),
- What about torsionful trees? Enumeration, characterization.
- After the melons, the cherry trees?

The Bicoloured Branch

The number of 2-trees in 

is: $\frac{k+3}{2\sqrt{3}}\left((2+\sqrt{3})^{k+1}-(2-\sqrt{3})^{k+1}\right)$.