Renormalization in Tensor Group Field Theory

Fabien Vignes-Tourneret

Institut Camille Jordan

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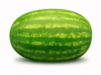

Outline

Motivations

Group Field Theory

Models of Tensor Field Theory are largely inspired by Group Field Theory (GFT).

$$S_D[\phi] = \frac{1}{2}\int_{G^D} \Big(\prod_{i=1}^{D} dg_i d\tilde{g}_i\Big)\phi(g_1,\dots,g_D)C^{-1}(g_i\tilde{g}_i^{-1})\phi(\tilde{g}_1,\dots,\tilde{g}_D)$$
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- Feynman graph amplitude $=$ spin foam model

Stranded graphs = gluing of simplices

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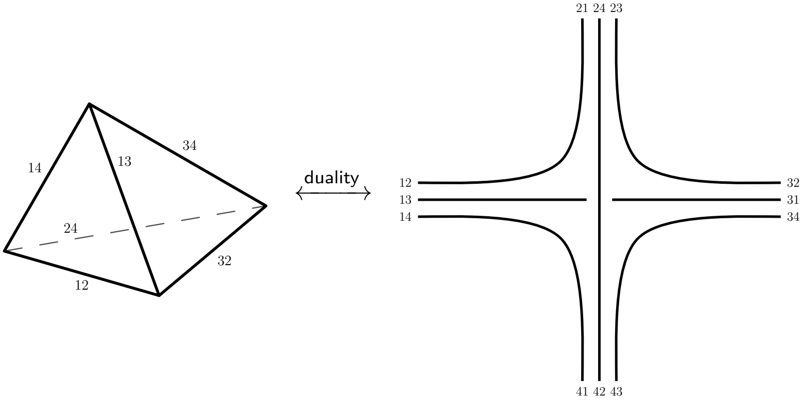

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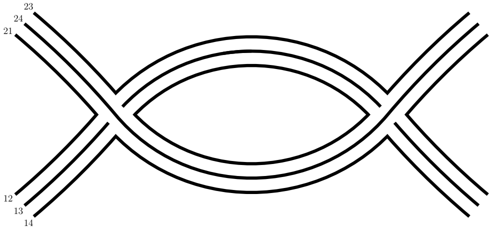

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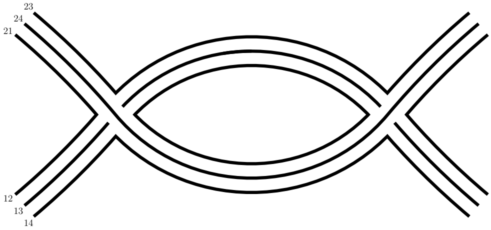

But not a D-complex!

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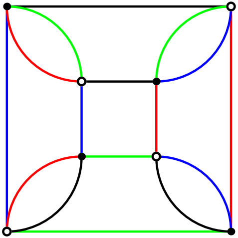
Random Tensors

Coloured Graphs

Definition

A k-coloured graph is a bipartite graph endowed with a proper edge-colouring with k colours.



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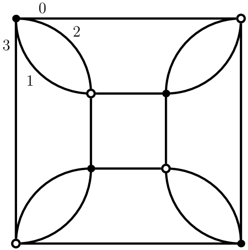

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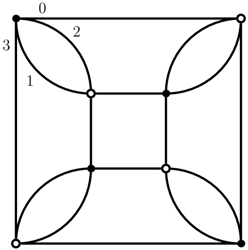

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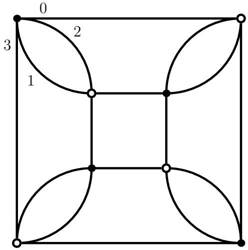

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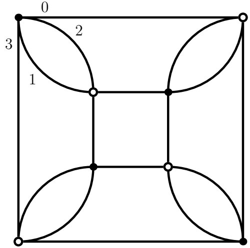

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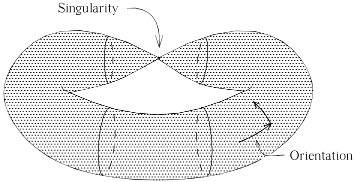
- Better, every such graph is dual to a normal pseudo-manifold [Gurau'10].

Pseudo-manifolds

These are manifolds with singularities.



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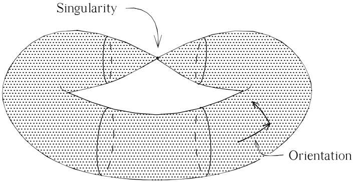

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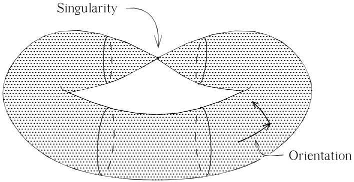

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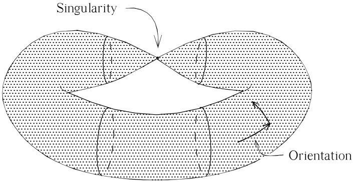

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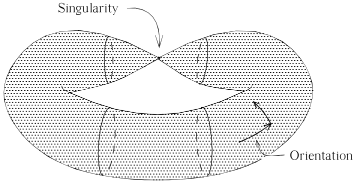

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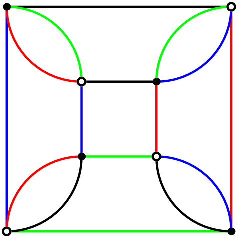
- GEMs: a combinatorial and algorithmic approach to the classification of 3-manifolds [Ferri, Gagliardi, Lins, etc '80].

Notions of coloured graph theory

Definition (Faces)

An internal face of a coloured graph $\mathcal{G}_c$ is a bicoloured cycle.



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Definition (Jacket)

Let $\mathcal{G}_c$ be a k-coloured graph. Let $\sigma = (i_0, i_1, \dots, i_{k-1})$ be a permutation on its k colours. The jacket J_σ is the ribbon graph whose vertices are those of $\mathcal{G}_c$, whose edges are those of $\mathcal{G}_c$ and whose faces are those given by σ.

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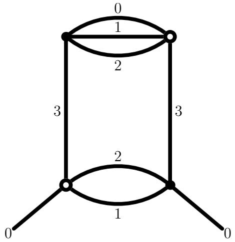

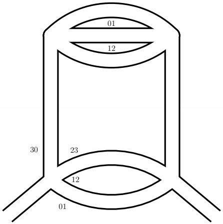

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The degree controls the $1/N$-expansion of tensor models.

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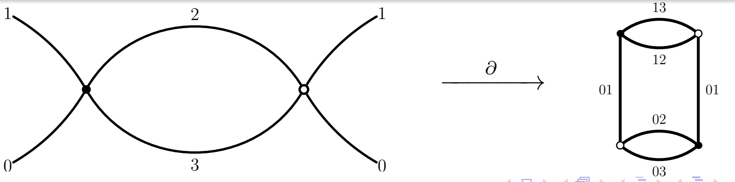

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The boundary graph $\partial\mathcal{G}_c$ triangulates the boundary of $\mathcal{G}_c$.

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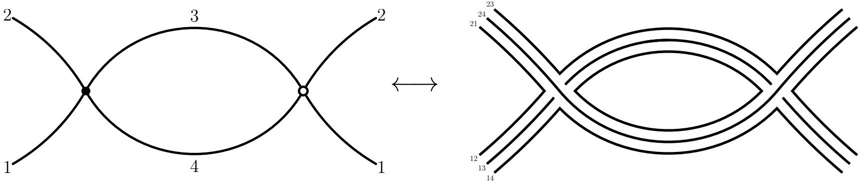

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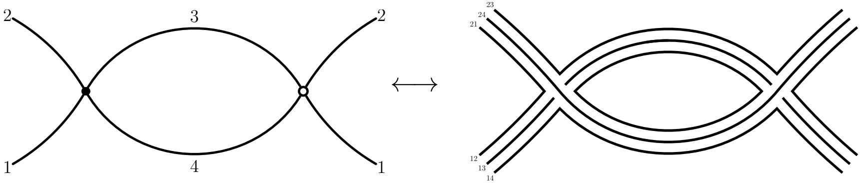

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Theorem ([Gurau, Rivasseau'11])

$$\frac{1}{N^D} \log Z = \sum_{\omega=0}^{\infty} N^{-\frac{2}{(D-1)!}\omega} \sum_{\mathcal{G}:\omega(\mathcal{G})=\omega} (\lambda\overline{\lambda})^{n(\mathcal{G})/2} / S_{\mathcal{G}}.$$

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$1/N$-expansion

- $A_{\mathcal{G}} = \left(\frac{\lambda\bar{\lambda}}{N^{D(D-1)/2}}\right)^{n(\mathcal{G})/2} N^{F(\mathcal{G})}$

Theorem ([Gurau, Rivasseau'11])

$$\frac{1}{N^D}\log Z = \sum_{\omega=0}^{\infty} N^{-\frac{2}{(D-1)!}\omega} \sum_{\mathcal{G}:\omega(\mathcal{G})=\omega} (\lambda\bar{\lambda})^{n(\mathcal{G})/2}/S_{\mathcal{G}}.$$

- $D = 2$, $\omega(\mathcal{G}) = g(\mathcal{G})$.

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- The degree ω *is not* a topological invariant!
- One can enumerate the coloured graphs of any fixed degree [Gurau, Schaeffer'13].

Renormalizable Models

Faded Effective Theories

Idea 1: integrate over D fields among $D+1$ to get an effective "colourless" theory.

$$S_{\text{eff}}[\phi^0, \overline{\phi}^0] = \sum_{p=1}^{\infty} \sum_{\mathcal{B} \in \Gamma_{2p}^{(D)}} (\lambda\overline{\lambda})^p N^{-\frac{2}{(D-2)!}\omega(\mathcal{B})} \operatorname{Tr}_{\mathcal{B}}(\phi^0, \overline{\phi}^0),$$

where $\mathcal{B}$ is a D-bubble i.e. a D-coloured graph and $\operatorname{Tr}_{\mathcal{B}}(\phi^0, \overline{\phi}^0)$ is an invariant (under the action of $U(N)^{\otimes D}$) canonically associated to $\mathcal{B}$.

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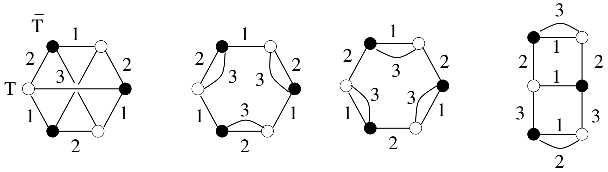
Connected Tensorial Invariants



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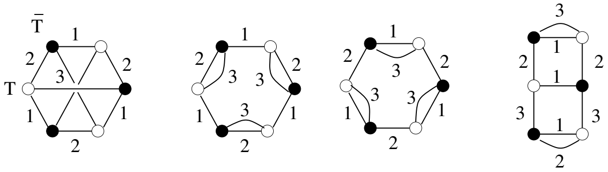


Idea 2: keep only the dominant traces **and** use a non trivial propagator.

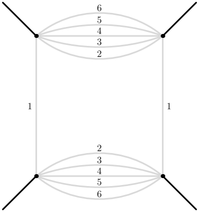
A Renormalizable ϕ^4_6

$\varphi : U(1)^6 \to \mathbb{C}$, $\varphi(g_1, \dots, g_6) = \varphi(hg_1, \dots, hg_6)$, $\forall h \in U(1)$.

$$S_4[\overline{\varphi}, \varphi] = \sum_{\mathbb{Z}^6} \overline{\varphi}_{654321}\, \delta(\sum_i p_i)(p^2 + m^2)\, \varphi_{123456} + \lambda\, V,$$

$$V = \sum_{\mathbb{Z}^{12}} \overline{\varphi}_{654321}\, \varphi_{12'3'4'5'6'}\, \overline{\varphi}_{6'5'4'3'2'1'}\, \varphi_{1'23456} + \text{ permutations.}$$

Unique connected melonic invariant of order 4



BPHZ Theorem

Theorem (D. Ousmane Samary, F. V.-T.)

The model defined by the action S_4 is renormalizable to all orders of perturbation.

One proves (using multi-scale analysis)

1. that the divergent graphs have a (uniformly) bounded number of external legs,
2. that the dangerous graphs are "tracial".

BPHZ Theorem

Sketch of the proof

We have to identify the divergent graphs and characterize their topology.

1. The divergence degree is $\omega_d = -2L + F - R$ with R the rank of the face/edge incidence matrix (+ optimized version). [BGKMR'10, BGR'11, COR'12]

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3. $\omega_d = 4 - N + \rho$, $\rho \leqslant 0$ and $\rho = 0$ iff $\mathcal{G}$ $(\mathcal{G}/\mathcal{T})$ is "fully melonic" (recursive condition, not explicit). [COR'13]
   ➥ $\mathcal{G}$ divergent iff ($N = 2$, $\rho = 0, -1, -2$) or ($N = 4$, $\rho = 0$).

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   $\rho = 0$ iff $\widetilde{\omega}(\mathcal{G}) = \omega(\partial\mathcal{G}) = (C_{\partial\mathcal{G}} - 1) = 0$.
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5. The divergent graphs are tracial (their divergent part is coded by their boundary graph). [BGR'11, OSVT'12]

Summary and perspectives

- Many renormalizable models (but who's next?).
- Most of them are asymptotically free [Ben Geloun, Ousmane Samary].
- Find phase transitions. Coupling to matter.
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- Find phase transitions. Coupling to matter.
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- Very rich combinatorics: tensor integrals, branched covers of $\mathbb{S}^2$, meanders, cellular trees...
- Random manifolds in higher dimensions (à la Le Gall, Miermont et al.).