Quasi-Frobenius pairs in model theory

Adrien Deloro

Sorbonne Université

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In this talk

1. Model theory: dimension(s)
   - Dimensions on definable sets
   - Finite Morley rank
   - o-minimality
2. Group theory: quasi-Frobenius pairs
   - Some geometric algebra
   - The definition
   - Examples
3. Interaction: dimensional quasi-Frobenius pairs
   - Results
   - Questions

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Let's go!

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→ For simplicity I say *definable* also for interpretable sets.
So I write $\mathrm{Def}(G)$ for the class of all *interpretable* sets with parameters.
(I should be writing $\mathrm{Def}(G^{\mathrm{eq}})$.)

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Dimensions abound in model theory!

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We move to another context.

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- As a result:
  G def. in an *o*-minimal theory $\implies$ G dimensional, but much stronger.

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Nothing similar for groups of finite Morley rank!

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Time for groups!

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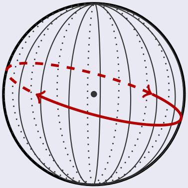

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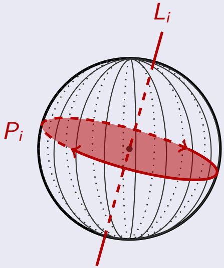

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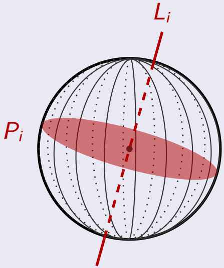

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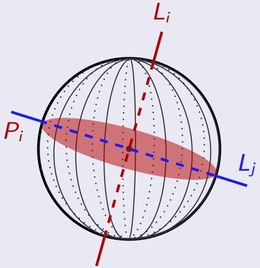

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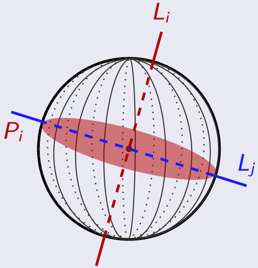

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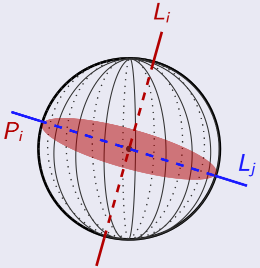

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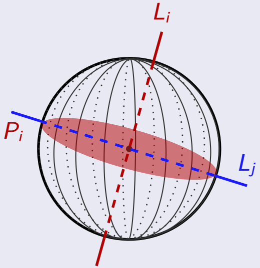
- Consider (definable) relation $i\varepsilon j$: $([i, j] = 1) \wedge (i \neq j)$.
  Thus <u>$i\varepsilon j$ iff $L_i \leq P_j$</u>.
- So $(I, \varepsilon) \simeq (\mathbb{P}^2(\mathbb{R}), \in) := (\{\text{lines}\}, \{\text{planes}\}, \leq)$
  (projective plane over the reals)...



The geometry of $\mathrm{SO}_3(\mathbb{R})$

$\mathrm{SO}_3(\mathbb{R})$ (actually, $\mathrm{SO}(\mathbb{R}^3)$) is the group of direct isometries/rotations of Euclidean $\mathbb{R}^3$.

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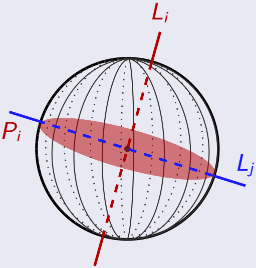

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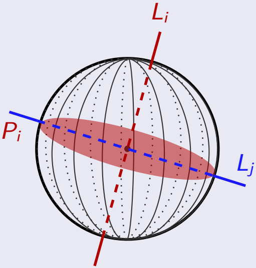

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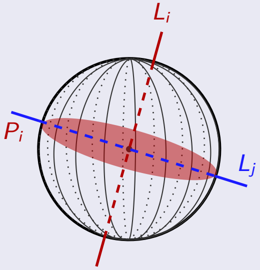

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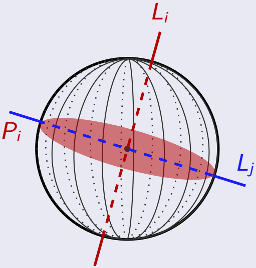

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- Then $N_G(C) = C_G(i) = C \rtimes \langle w \rangle$ and w inverts the abelian group C.

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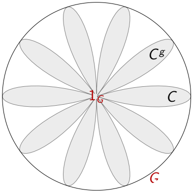

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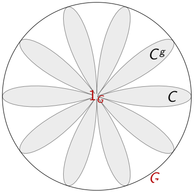

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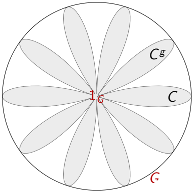
- Note: one often writes $C \cap C^g = 1$ instead of $C \cap C^g = \{1\}$.



Main example 1: affine transformations of the line

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- Bonus: $N_G(C) \setminus C$ is a single coset of C, consisting of involutions inverting C.

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- Now if $g \in G$ *does* normalise C, then g stabilises $\langle e_1 \rangle$ and also $e_1^\perp = \langle e_2, e_3 \rangle \dots$ so either $g \in C$, or g acts a reflection inside $e_1^\perp$.
- Bonus: $N_G(C) \setminus C$ is a single coset of C, consisting of involutions inverting C.

Quasi-Frobenius, but *not* Frobenius: here $[N_G(C) : C] = 2$.

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Time to add some model theory.

In this section

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I *do not* believe in the conjecture at this level of generality. Let us see what it becomes in special cases.

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Proof.

The proof uses Lie theory. I call it cheating! □

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In finite Morley rank, we must make assumptions on torsion.
G 'has odd type' if:

- G has elements of order 2;
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Remark

As a consequence it makes sense to define 'quasi-Frobenius' even in finite group theory:
C is TI in G + $[N_G(C) : C] = 2$. To my knowledge, the notion has not been studied.

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Theorem (Zamour, generalised by Corredor-D.)

Let $(C < G)$ *be a quasi-Frobenius pair fMR with* G *of odd type.*

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Proof.

Nesin-Cherlin-Jaligot computations (the ideas go back to Bender). □

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