Model theory of finite and pseudofinite graphs

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LOGIC OF GRAPHS

- [1]It is undecidable whether a given first-order sentence can be realized by a finite undirected graph. This means that no algorithm can correctly answer this question for all sentences.
- Some first-order sentences are modeled by infinite graphs but not by any finite graph. For instance, the property of having exactly one vertex of degree one, with all other vertices having degree exactly two, can be expressed by a first-order sentence. It is modeled by an infinite ray, but violates Euler's handshaking lemma for finite graphs.
- [2] It is also undecidable to distinguish between the first-order sentences that are true for all graphs and the ones that are true of finite graphs but false for some infinite graphs.

[1]Parys (2014) writes that this undecidability result is well known, and attributes it to Trahtenbrot (1950) on the undecidability of first-order satisfiability for more general classes of finite structures.

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Background on graph theory

- **Definition**. A graph $\Gamma = \langle G, R\rangle$ that contains no cycles is called an *acyclic graph*. A connected acyclic graph is called a *tree*. Any graph without cycles is also called a *forest* so that the components of a forest are trees.
- **Definition** [3]. A *regular graph* is a graph where each vertex has the same number of neighbors. A regular graph with vertices of degree k is called a *k-regular graph* or *regular graph of degree k*.
- **Definition** [4] A graph $\Gamma = \langle G, R\rangle$ is said to be *homogeneous* if, for $U, V \subseteq G$, the statement that $\langle U, R \upharpoonright U^2\rangle \equiv \langle V, R \upharpoonright V^2\rangle$ implies the existence of an automorphism of Γ mapping U to V.

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1. Erdős, P., Rényi, A. Asymmetric graphs. Acta Mathematica Academiae Scientiarum Hungaricae 14, 295-315 (1963). https://doi.org/10.1007/BF01895716
2. Jónsson, B. "Homogeneous universal relational systems." Mathematica Scandinavica 8, no. 1 (1960): 137–42. http://www.jstor.org/stable/24489128
3. R. Rado, Universal graphs and universal functions, Acta Arith., 9 (1964), 331-340.
4. Sheehan, J. (1974). Smoothly Embeddable Subgraphs. Journal of the London Mathematical Society, s2-9(2), 212–218. https://doi.org/10.1112/jlms/s2-9.2.212
5. C. W. Henson, "A family of countable homogeneous graphs Pacific J. of Math., 38 (1971), 69-83
6. Hodges W. A., Hodkinson I. M., Lascar D. and Shelah S. (1993), The small index property for ω-stable, ω-categorical structures and for the random graph, J. London Math. Soc. (2) 48, P. 204-218.

- [5]**Definition.** Let L be (relational) language. An infinite L-structure $\mathcal{M}$ is *pseudofinite* if for all L-sentences φ, $\mathcal{M} \models \varphi$ implies that there is a finite structure $\mathcal{M}_0$ such that $\mathcal{M}_0 \models \varphi$. The theory $T = \mathrm{Th}(\mathcal{M})$ of the pseudofinite structure $\mathcal{M}$ is called *pseudofinite*.
  A pseudofinite graph is an infinite graph that satisfies every first order sentence L that is true for some finite graphs.
- Recall that the ω-categorical structure $\mathcal{M}$ is said to be *smoothly approximated* if it is the union of an ω-chain of finite homogeneous substructures; or equivalently, if any sentence $\varphi \in Th(\mathcal{M})$ is true of some finite homogeneous substructure $\mathcal{N}$ of $\mathcal{M}$.

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Who investigated pseudofinite graphs?

• T. Tao. Expanding polynomials over finite fields of large characteristic, and a regularity lemma for definable sets. Contributions to Discrete Mathematics. Volume 10, Number 1, Pages 22–98, ISSN 1715-0868. 2014

• E. Hrushovski, Pseudo-finite fields and related structures, in: Model Theory and Applications, Bélair et al. ed., Quaderni di Matematica Vol. 11, Aracne, Rome 2005, 151 – 212

• Beyarslan, O'. (2010). Random hypergraphs in pseudofinite fields. Journal of the Institute of Mathematics of Jussieu, 9(1), 29-47. doi:10.1017/S1474748009000073

•Valizadeh, A.N., Pourmahdian, M. Pseudofiniteness in Hrushovski Constructions// *Notre Dame J. Formal Log.* 2020, vol. 61, pp. 1–10. https://doi.org/10.1215/00294527-2019-0038

• Valizadeh, A.N., Pourmahdian, M. Strict Superstablity and Decidability of Certain Generic Graphs // *Bull. Iran. Math. Soc.* 2019, vol. 45, pp. 1839–1854. https://doi.org/10.1007/s41980-019-00234-2

• Myasnikov, A.G., Remeslennikov, V.N. Generic Theories as a Method for Approximating Elementary Theories. Algebra Logic 53, 512–519 (2015). https://doi.org/10.1007/s10469-015-9314-0

- We know several methods to show that the theory T is a pseudofinite.
- The first method is via a probabilistic argument. Let $\mathcal{K}$ be finite L-structures which is closed under isomorphisms, that has only finitely many non-isomorphic models of size n for every $n \in \omega$. Let μ_n be a probability measure on the set $\mathcal{K}_n(L) = \{\mathcal{M} : \mathcal{M}$ is an L-structure with universe $\{1, \ldots, n\}\}$ and define, for any L-sentence φ,

$$\mu(\varphi) = \lim_{n\to\infty} \mu_n(\{\mathcal{A} \in \mathcal{K}_n(L) : \mathcal{A} \models \varphi\}).$$

- Given μ_n and $\mathcal{K}_n(L)$ as above, we define *the almost sure theory*[6] of $\mathcal{K}$ as $T_{as}(\mathcal{K}) = \{\varphi : \varphi$ is a first-order L-sentence and $\mu(\varphi) = 1\}$. A surprising corollary is that each finite subset of T has a finite model.

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Background on model theory

- **Erdős-Rényi model** In the $G(n,p)$ model, a graph is constructed by connecting labeled nodes randomly. Each edge is included in the graph with probability p, independently from every other edge. Equivalently, the probability for generating each graph that has n nodes and M edges is $p^M(1-p)^{\binom{n}{2}-M}$.
  Example 1. The theory of the random graph is pseudofinite.
- Its theory can be axiomatized in the language $L = \{R\}$ by the sentences

$$P_{k,l} = \forall x_1, \ldots, x_k \forall y_1, \ldots, y_l (\bigwedge_{i,j} x_i \neq y_j \to \exists z (\bigwedge_{i,j} zRx_i \wedge \neg zRy_j))$$

- **Fact**. For any $k, l \geq 1$ and every constant $p \in (0,1)$, almost every graph $G \in G(n,p)$ satisfies the property $P_{k,l}$. That is, $lim_{n\to\infty}\mu_n(P_{k,l}(G(n,p)) = 1$.

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- When $lim_{n\to\infty}\mu_n(\phi) = 1$ for some sentence ϕ, we say that the sentence ϕ occurs almost surely, i.e. almost all graphs satisfies ϕ. When $lim_{n\to\infty}\mu_n(\phi) = 0$, we say that ϕ holds almost never, i.e. almost no graphs satisfies ϕ.
- **Fact.**[7] **(Zero-One Law for Graphs)** For any L-sentence ϕ either $lim_{n\to\infty}\mu_n(\phi) = 0$ or $lim_{n\to\infty}\mu_n(\phi) = 1$. Moreover, T axiomatizes $\{\phi : lim_{n\to\infty}\mu_n(\phi) = 1\}$, the almost sure theory of graphs. The almost sure theory of graphs is decidable and complete.

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- The second method is to construct finite structures that satisfy an arbitrary finite subset T. In fact, pseudofinite fields were first introduced by J. Ax and S. Kochen in the form of non-principal ultraproducts of finite fields. J. Ax connect the notion of pseudofiniteness and an ultraproduct construction.
- **Proposition 1.** *Fix a language L and an L-structure $\mathcal{M}$. Then the following conditions are equivalent:*
  (1) $\mathcal{M}$ is pseudofinite;
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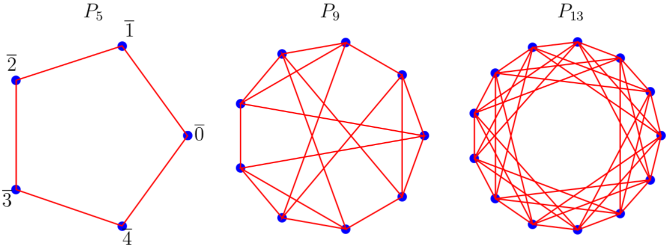
- **Paley graph** Let $q = p^n$ be a prime power with $q \equiv 1(mod4)$. We define the Paley graph P_q to be the graph with set of vertices $V = F_q$ and the edge relation defined by xRy if and only if $x \neq y$ and $(x - y)$ is a square.



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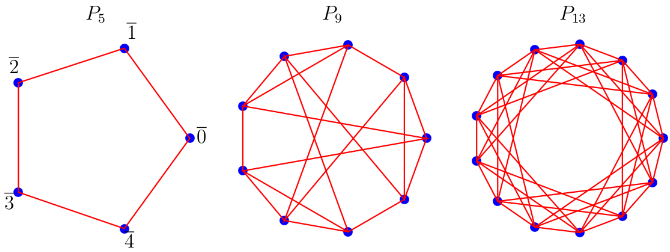

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- A similar approximation approach was developed by S.V.Sudoplatov.

  Definition.[8] Let $\mathcal{T}$ be a family of theories and T be a theory such that $T \notin \mathcal{T}$. The theory T is said to be $\mathcal{T}$*-approximated*, or *approximated by the family* $\mathcal{T}$, or a *pseudo-*$\mathcal{T}$*-theory*, if for any formula $\varphi \in T$ there exists $T' \in \mathcal{T}$ such that $\varphi \in T'$.

- If a theory T is $\mathcal{T}$-approximated, then $\mathcal{T}$ is said to be an *approximating family* for T, and theories $T' \in \mathcal{T}$ are said to be *approximations* for T. We put $\mathcal{T}_\varphi = \{T \in \mathcal{T} \mid \varphi \in T\}$. Any set $\mathcal{T}_\varphi$ is called the φ*-neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$. A family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in T$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite.

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- Recall that the *E-closure*[9] for a family $\mathcal{T}$ of complete theories is characterized by the following proposition.
  Proposition 2. Let $\mathcal{T}$ be a family of complete theories of the language L. Then $Cl_E(\mathcal{T}) = \mathcal{T}$ for a finite $\mathcal{T}$, and for an infinite $\mathcal{T}$, a theory T belongs to $Cl_E(\mathcal{T})$ if and only if T is a complete theory of the language L and $T \in \mathcal{T}$, or $T \notin \mathcal{T}$ and for any formula φ the set $\mathcal{T}_\varphi$ is infinite.
- We denote by $\overline{\mathcal{T}}$ the class of all complete theories of relational languages, by $\overline{\mathcal{T}}_{fin}$ the subclass of $\overline{\mathcal{T}}$ consisting of all theories with finite models, and by $\overline{\mathcal{T}}_{inf}$ the class $\overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{fin}$.
- **Proposition 3.**[10] For any theory T the following conditions are equivalent:
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Groundworks

[11]We denote the following classes of acyclic graphs and consider various approximations for these classes.

Let $\mathcal{G}_{fin}(\lambda)$, for arbitrary cardinality λ, be the family of all infinite acyclic graphs consisting of λ connected components of totally bounded diameters.

Let $\mathcal{G}_{inf}(\lambda)$, for arbitrary cardinality λ, be the family of all infinite acyclic graphs consisting of λ connected components of infinite diameters.

[11] Markhabatov N. D. Approximations of Acyclic Graphs. *The Bulletin of Irkutsk State University. Series Mathematics*, 2022, vol. 40, pp. 104–111. https://doi.org/10.26516/1997-7670.2022.40.104

Groundworks

- **Theorem 1**[12] Theory T of any infinite acyclic graph $\Gamma \in \mathcal{G}_{fin}(\lambda)$, for arbitrary cardinality λ, is pseudofinite.
- **Theorem 2** Let Γ be an acyclic graph from class $\mathcal{G}_{fin}(\lambda)$, for a cardinality λ. The theory $T = Th(\Gamma)$ is approximated by theories of finite trees if and only if $\lambda = 1$.

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- **Theorem 3.** [13] *Let Γ be an arbitrary countable graph in which each component contains a finite number of cycles. Then Γ is countably categorical if and only if Γ is bounded and finitely many 1-types are realized in it.*
- Let us distinguish a subclass $\mathcal{G}_{cc}(\lambda)$ of the class $\mathcal{G}_{fin}(\lambda)$ as the class of all countably categorical acyclic graphs.
- **Theorem 4**[14] *Any* $\Gamma \in \mathcal{G}_{cc}(\lambda)$ *is smoothly approximable.*

[13] Ovchinnikova E.V., Shishmarev Yu.E. Countably categorical graphs. Ninth All-Union Conference on Mathematical Logic. Leningrad, September 27-29, 1988: dedicated to the 85th anniversary of Corresponding Member of the USSR Academy of Sciences A.A. Markov: abstracts, p.120

[14] N. D. Markhabatov, "On Smoothly Approximable Acyclic Graphs", Proceedings of the International Scientific Conference «Actual Problems of Mathematics, Mechanics and Informatics» dedicated to the 80th anniversary of professor T.G. Mustafin (8-9 September, Karaganda), Karaganda Buketov University, 2022, 36–37

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Groundworks

- **Theorem 5.**[15] Let T be the theory of an acyclic graph Γ from the class $\mathcal{G}_{inf}(\lambda)$, for finite cardinality λ, with finitely many rays. If the number of rays in Γ is even then T is pseudofinite theory.
- The first version of this theorem was like the following statement
- **Theorem***[16] An acyclic graph $\Gamma \in \mathcal{G}_{inf}(\lambda)$, for finite cardinality λ, with finitely many rays is pseudofinite if and only if the number of rays in Γ is even.
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[16] N. D. Markhabatov, "On pseudofiniteness of acyclic graphs", Traditional international April mathematical conference in honor of the Day of Science Workers of the Republic of Kazakhstan, dedicated to the 1150th anniversary of Abu Nasyr al-Farabi and the 75th anniversary of the Institute of Mathematics and Mathematical Modeling : abstract. report (Republic of Kazakhstan, Almaty), Institute of Mathematics and Mathematical Modeling, 2020, 14-15

[17] Markhabatov N. D. Approximations of Acyclic Graphs II. *In preparation*, 2022

Groundworks

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[17] Markhabatov N. D. Approximations of Acyclic Graphs II. *In preparation,* 2022

Groundworks

- **Theorem 7.**[18] Any theory T of a regular graph with an infinite model is pseudofinite.
  This theorem immediately implies
- **Theorem 8.**[19] A theory T_r is the theory of an infinite tree, each vertex of which has degree r. The theory T_∞ (also known as the everywhere infinite forest theory) is the theory of an infinite tree in which each vertex has infinite degree. Both theories are pseudofinite.

[18] Markhabatov N. D., Sudoplatov S. V., Approximations of Regular Graphs, *Herald of the Kazakh-British Technical University*, 2022, vol. 19, no. 1, pp. 44–49 https://doi.org/10.55452/1998-6688-2022-19-1-44-49

[19] Garcia D., Robles M. Pseudofiniteness and measurability of the everywhere infinite forest. In preparation (2020)

Groundworks

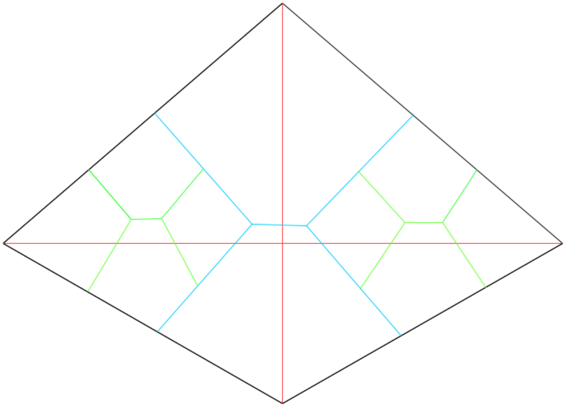

Рис.: Approximation of a 3-regular graph.

Groundworks

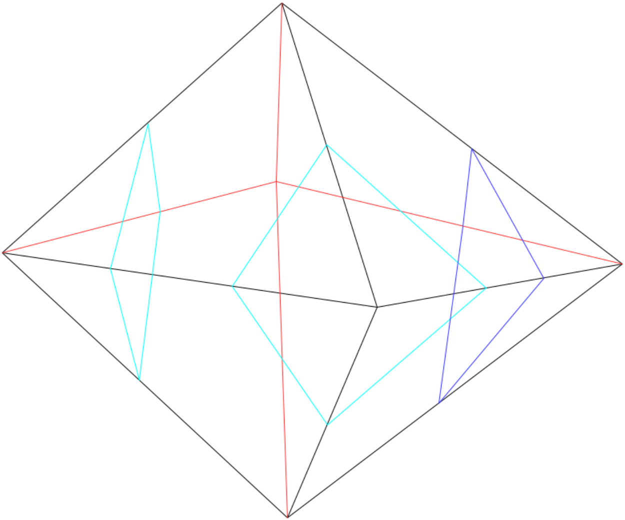

Рис.: Approximation of a 4-regular graph.

Tools

- • Probabilistic argument
- • Ultraproduct construction
- • Approximation
- • **Theorem 9.** (Hrushovski E.[20]) *Let X be a finite graph. Then there exists a finite graph Z containing X as an induced subgraph, such that every isomorphism between induced subgraphs of X extends to an automorphism of Z.*
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[20] Hrushovski E. Extending partial isomorphisms of graphs, Combinatorica. 1992. Vol. 12, No. 4. P. 204–218.

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Questions

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- **Question:** Which pseudofinite graphs are Fraïssé limits?

[21] Markhabatov N. D., Sudoplatov S. V., Approximations of Regular Graphs, *Herald of the Kazakh-British Technical University*, 2022, vol. 19, no. 1, pp. 44–49 https://doi.org/10.55452/1998-6688-2022-19-1-44-49

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- [22] We call a graph G *k-existentially complete* if every partial embedding of the graph on k + 1 vertices extends to a complete embedding. We call a graph *k-existentially triangle-free (hereinafter k-ECTF)* if G has no triangles and every partial embedding of the (k + 1)-vertex triangle-free graph extends to a complete embedding.
- Cherlin[23] asked whether there are finite k-ECTF graphs for every fixed $k \in \mathbb{N}$. To date, this problem remains poorly understood[24], and the state of the art can be summarized as follows.

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[22] C. Even-Zohar and N. Linial. Triply existentially complete triangle-free graphs. J. Graph Theory, 78:26–35, 2015

[23] G. L. Cherlin, Combinatorial problems connected with finite homogeneity, in Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989) Contemporary Mathematics, Vol. 131, American Mathematical Society, providence, RI, 1992, pp. 3–30.

[24] G.L. Cherlin. Two problems on homogeneous structures, revisited. Contemporary Mathematics, 558:319–416, 2011.

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- The case $k = 1$ is trivial; a graph is 2-ECTF if and only if it is maximally free of triangles, twins or a single edge; there are various (nontrivial) constructions of 3-ECTF graphs[25]; and the case k = 4 was opened. But in 2020 it was proved in [26] that there is no graph with this property.
- Martin Smoli'k, Pseudofinite structures, Bachelor thesis, 2016
- Jan Mycielski. Sur le coloriage des graphes // Colloq. Math.. — 1955. — T. 3. — C. 161—162.

[25] C. Even-Zohar and N. Linial. Triply existentially complete triangle-free graphs. J. Graph Theory, 78:26–35, 2015

[26] Letzter, S., Sahasrabudhe, J. On existentially complete triangle-free graphs. Isr. J. Math. 236, 591–601 (2020). https://doi.org/10.1007/s11856-020-1982-3

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Following [27] we define the *rank* $\mathrm{RS}(\cdot)$ for the families of theories, similar to Morley rank [28], and a hierarchy with respect to these ranks in the following way.
For the empty family $\mathcal{T}$ we put the rank $\mathrm{RS}(\mathcal{T}) = -1$, for finite nonempty families $\mathcal{T}$ we put $\mathrm{RS}(\mathcal{T}) = 0$, and for infinite families $\mathcal{T}$ — $\mathrm{RS}(\mathcal{T}) \geq 1$.
For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent sentences φ_n, $n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.
If α is a limit ordinal then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.
We set $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.
If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we put $\mathrm{RS}(\mathcal{T}) = \infty$.

[27] Sudoplatov S. V. Ranks for families of theories and their spectra / S. V. Sudoplatov// Lobachevskii J. Math 42, 2959–2968 (2021). https://doi.org/10.1134/S1995080221120313.

[28] *Morley M.* Categoricity in Power / M. Morley // Transactions of the American Mathematical Society. — 1965. — Vol. 114, No. 2. — P. 514–538.

Preliminaries from ranks for families of theories

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.

If $\mathcal{T}$ is e-totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

Definition[29] [30]. A *n-dimensional cube*, or a *n-cube* (where $n \in \omega$) is a graph isomorphic to the graph $\mathcal{Q}_n$ with the universe $\{0;\ 1\}^n$ and such that any two vertices $(\delta_1, \ldots,\ \delta_n)$ and $(\delta'_1, \ldots,\ \delta'_n)$ are adjacent if and only if these vertices differ exactly in one coordinate.

Let λ be an infinite cardinal. A λ-*dimensional cube*, or a λ-*cube*, is a graph isomorphic to a graph $\Gamma = \langle X;\ R\rangle$ satisfying the following conditions:

(1) the universe $X \subseteq \{0;\ 1\}^\lambda$ is generated from an arbitrary function $f \in X$ by the operator $\langle f\rangle$ attaching, to the set $\{f\}$, all results of substitutions for any finite tuples $(f(i_1), \ldots,\ f(i_m))$ by tuples $(1 - f(i_1), \ldots, 1 - f(i_m))$;

(2) the relation R consists of edges connecting functions differing exactly in one coordinate.

[29] Sudoplatov S.V., *Group polygonometries.* – Novosibirsk: Publishing House of Novosibirsk State Technical University, 2013. – 302 p.

[30] Sudoplatov S.V. *Models of Cubic Theories* // Bulletin of the Section of Logic. 2014. Vol. 43, No. 1-2. P. 19-34.

Preliminaries from cubic theories

The described graph $\mathcal{Q} \rightleftarrows \mathcal{Q}_f$ with the universe $\langle f \rangle$ is a canonical representative for the class of λ-cubes.

Note that the canonical representative of the class of n-cubes (as well as the canonical representatives of the class of λ-cubes) are generated by any its function: $\{0, 1\}^n = \langle f \rangle$, where $f \in \{0, 1\}^n$. Therefore the universes of canonical representatives $\mathcal{Q}_f$ of n-cubes like λ-cubes, will be denoted by $\langle f \rangle$.

Any graph $\Gamma = \langle X; R \rangle$, where any connected component is a cube, is called a *cubic structure*. A theory T of graph language $\{R^{(2)}\}$ is *cubic* if $T = Th(\mathcal{M})$ for some cubic structure $\mathcal{M}$. In this case, the structure $\mathcal{M}$ is called a *cubic model* of T.

Preliminaries from cubic theories

The *invariant* of theory T is the function

$$Inv_T : \omega \cup \{\infty\} \to \omega \cup \{\infty\},$$

satisfying the following conditions:
(1) for any natural n; $Inv_T(n)$ is the number of connected components in any model of T, being n-cubes, if that number is finite, and $Inv_T(n) = \infty$ if that number is infinite;
(2) $Inv_T(\infty) = 0$ if models of T do not contain infinite-dimensional cubes (i. e., dimensions of cubes are totally bounded), otherwise we set $Inv_T(\infty) = 1$.

The support (accordingly the ∞-*support*) $Supp(T)(Supp_{\infty}(T))$ of theory T is the set $\{n \in \omega | Inv_T(n) \neq 0\}(\{n \in \omega | Inv_T(n) = \infty\})$.

Proposition 4 *If T is a theory of model, whose connected components are finite cubes, then any countable connected component of a model of T is an ω-cube.*

Theorem 10.[31] *If the diameter d(T) is finite and the ∞-support is a singleton then T is a strongly minimal totally categorical theory.*

[31] Sudoplatov S.V. *Models of Cubic Theories* // Bulletin of the Section of Logic. 2014. Vol. 43, No. 1-2. P. 19-34.

Theorem 11.[32] *Any cubic theory T with an infinite model is pseudofinite.*

Let a language L consist of $R^{(2)}$. Denote by $\mathcal{T}_L$ family of all cubic theories of language L.

Theorem 12. *For any countable ordinal α and natural $n \geq 1$ there exists a family $\mathcal{T} \subseteq \mathcal{T}_L$, such that $RS(\mathcal{T}) = \alpha$ and $ds(\mathcal{T}) = n$.*

[32] Markhabatov N.D., Ranks and approximations for families of cubic theories, Siberian Electronic Mathematical Reports, 2022, submitted

Approximations and ranks for families of cubic theories

Recall [33] [34] that a countable model $\mathcal{Q}$ of a theory T is called a *limit model* if $\mathcal{Q}$ is represented as the union of a countable elementary chain of models of the theory T that are prime over tuples, and the model $\mathcal{Q}$ itself is not prime over any tuple. A theory T is called *l-categorical* if T has a unique (up to isomorphism) limit model.

Theorem 13.[35] Any model $\mathcal{Q}$ of the l-categorical cubic theory T is smoothly approximable by finite cubic structures.

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[33] Sudoplatov, S.V. Complete Theories with Finitely Many Countable Models. I. Algebra and Logic 43, 62–69 (2004). https://doi.org/10.1023/B:ALLO.0000015131.41218.f4

[34] Sudoplatov, S.V. Complete theories with finitely many countable models. II. Algebra and Logic 45, 180–200 (2006). https://doi.org/10.1007/s10469-006-0016-5

[35] Markhabatov N.D., Ranks and approximations for families of cubic theories, Siberian Electronic Mathematical Reports, 2022, submitted

Main result

Definition [36] A family $\mathcal{T}$, with infinitely many accumulation points, is called *a-minimal* if for any sentence $\varphi \in L$, $\mathcal{T}_\varphi$ or $\mathcal{T}_{\neg\varphi}$ has finitely many accumulation points.
Denote by $\mathcal{T}_{reg}$ the family of all regular graph theories. For any theory $T \in \mathcal{T}_{reg}$, consider the pair (k, γ_k), where $k \in \omega$, γ_k is the number of k-regular graphs.

Proposition 5. [37] (1) A family $\mathcal{T}$ is *e*-minimal if and only if $\mathcal{T}$ is a family of regular graph theories with one arbitrary value γ_k, $\gamma_m = 0$ for $m \neq k$.
(2) A family $\mathcal{T}$ is *a*-minimal if and only if $\mathcal{T}$ is a family of regular graph theories with two arbitrary values $\gamma_{k_1}, \gamma_{k_2}$, $\gamma_{k_0} = 0$ for $k_0 \neq k_1 \neq k_2$.

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[36] Sudoplatov S. V. Ranks for families of theories and their spectra / S. V. Sudoplatov// Lobachevskii J. Math 42, 2959–2968 (2021). https://doi.org/10.1134/S1995080221120313.

[37] N. D. Markhabatov, S. V. Sudoplatov, “Ranks for families of regular graph theories”, Herald of the Kazakh-British Technical University, 19:3 (2022), 54–59 https://doi.org/10.55452/1998-6688-2022-19-3-54-59

Proposition 6. For any countable ordinal α and natural $n \geq 1$ there exists a family $\mathcal{T} \subseteq \mathcal{T}_{reg}$, such that $RS(\mathcal{T}) = \alpha$ and $ds(\mathcal{T}) = n$.

The next result shows that the family $\mathcal{T}_{reg}$ of all regular graph theories is not e-totally transcendental.

Theorem 14. $RS(\mathcal{T}_{reg}) = \infty$.

Thanks for your attention!