Jonsson hybrids and their similarities

Nazerke Mussina

Karaganda Buketov University
Karagandy, Kazakhstan

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Outline

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References

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Definition 1. [1]

A theory T is Jonsson if:
1) Theory T has infinite models;
2) Theory T is inductive;
3) Theory T admits the joint embedding property (JEP);
4) Theory T admits the property of amalgam (AP).

Examples of Jonsson theories are:
1) Group Theory,
2) Theory of Abelian groups,
3) Theory of fields of fixed characteristics,
4) Theory of Boolean algebras,
5) Theory of polygons over a fixed monoid,
6) Theory of modules over a fixed ring,
7) Theory of linear order.

Homogeneous-universal model

Definition 2. [1]

Let $\kappa \geq \omega$. Model M of theory T is called

- κ-universal for T, if each model T with the power strictly less κ isomorphically imbedded in M;
- κ-homogeneous for T, if for any two models A and A_1 of theory T, which are submodels of M with the power strictly less then κ and for isomorphism $f : A \to A_1$ for each extension B of model A, wich is a submodel of M and is model of T with the power strictly less then κ there is exist the extension B_1 of model A_1, which is a submodel of M and an isomorphism $g : B \to B_1$ which extends f.

Semantic model

Definition 3. [1]

Model C of Jonsson theory T is called semantic model, if it is ω^+-homogeneous-universal.

Definition 4. [1]

The center of Jonsson theory T is called an elementary theory of the its semantic model. And denoted through T^*, i.e. $T^* = Th(C)$.

Existence of semantic model

Fact 1. [1]

Each Jonsson theory T has k^+-homogeneous-universal model of power 2^k. Conversely, if a theory T is inductive and has infinite model and ω^+-homogeneous-universal model then the theory T is a Jonsson theory.

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Fact 2. [1]

Let T is a Jonsson theory. Two k-homogeneous-universal models M and M_1 of T are elementary equivalent.

Perfectness

Definition 5. [1]

Jonsson theory T is called a perfect theory, if each a semantic model of theory T is saturated model of T^*.

Theorem 1. [1]

Let T is a Jonsson theory. Then the following conditions are equivalent:

1. theory T is perfect;
2. theory T^* is a model companion of theory T.

Let E_T be a class of all existentially closed models of theory T.

Theorem 2. [1]

If T is a perfect Jonsson theory then $E_T = ModT^*$.

Definition 6. [2]

Let $\mathcal{A}$ be an arbitrary model of signature σ. Let us call the Jonsson spectrum of model $\mathcal{A}$ a set:

$$JSp(\mathcal{A}) = \{T \mid T \text{ is the Jonsson theory in a language } \sigma$$

$$\text{and } \mathcal{A} \in Mod\ T\}.$$

Definition 7. (Mustafin T.)

We say that Jonsson theory T_1 is cosemantic to the Jonsson theory $T_2\,(T_1 \bowtie T_2)$ if $C_{T_1} = C_{T_2}$, where C_{T_i} is a semantic model T_i, $i = 1, 2$.

The relation of cosemanticness on a set of theories is an equivalence relation. Then $JSp(\mathcal{A})/_{\bowtie}$ is the factor set of the Jonsson spectrum of the model $\mathcal{A}$ with respect to $\bowtie$.

Notation for features of the Jonsson spectrum

Let $[T] \in JSp(\mathcal{A})/_{\bowtie}$. Since for each theory $\Delta \in [T]$ is $\mathcal{C}_\Delta = \mathcal{C}_T$, then the semantic model of the class $[T]$ will be called the semantic model of the theory T: $\mathcal{C}_{[T]} = \mathcal{C}_T$.

The center of Jonsson class $[T]$ will be called the elementary theory $[T]^*$ of its semantic model $\mathcal{C}_{[T]}$, i.e. $[T]^* = \mathrm{Th}(\mathcal{C}_{[T]})$ and $[T]^* = \mathrm{Th}(\mathcal{C}_\Delta)$ for each $\Delta \in [T]$.

Denote by $E_{[T]} = \bigcup_{\Delta \in [T]} E_\Delta$ the class of all existentially closed models of class $[T] \in JSp(\mathcal{A})/_{\bowtie}$.

The syntactic similarity of complete theories

Definition 8 [1] (Mustafin)

Let T_1 and T_2 are complete theories. We will speak, as T_1 and T_2 are syntactically similar ($T_1 \overset{S}{\approx} T_2$), if $f : F(T_1) \longrightarrow F(T_1)$ exists bijection such that

1. restriction f to $F_n(T_1)$ is isomorphism of Boolean algebras $F_n(T_1)$ and $F_n(T_2)$, $n < \omega$;
2. $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in F_{n+1}(T)$, $n < \omega$;
3. $f(v_1 = v_2) = (v_1 = v_2)$.

Definition 9 [1] (Yeshkeyev)

Let T_1 and T_2 are an arbitrary Jonsson theories. We say that T_1 and T_2 are Jonsson syntactically similar ($T_1 \overset{S}{\sim} T_2$) if exists a bijection $f : E(T_1) \longrightarrow E(T_2)$ such that:

1. restriction f to $E_n(T_1)$ is isomorphism of lattices $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;
2. $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in E_{n+1}(T)$, $n < \omega$;
3. $f(v_1 = v_2) = (v_1 = v_2)$.

The semantic similarity of Jonsson theories

Definition 10 [1] (Yeshkeyev)

The pure triple $\langle C, Aut(\mathcal{C}), Sub(C)\rangle$ is called the Jonsson semantic triple, where C is universe of semantic model $\mathcal{C}$ of theory T, $Aut(\mathcal{C})$ is the automorphism group of $\mathcal{C}$, $Sub(C)$ is a class of all subsets of C which are universe of the corresponding existentially closed submodels of $\mathcal{C}$.

Definition 11 [1] (Yeshkeyev)

Two Jonsson theories T_1 and T_2 are called the Jonsson semantically similar if their Jonsson semantic triples are isomorphic as pure triples.

Definition 12 [3]

1) Let T_1 and T_2 be some Jonsson theories of the countable language L of the same signature σ; C_1 and C_2 are their semantic models, respectively. In the case of common signature of Jonsson theories T_1, T_2, let us call a hybrid of Jonsson theories T_1 and T_2 of the first type the following theory $Th_{\forall\exists}(C_1 \diamond C_2)$ if that theory is Jonsson in the language of signature σ and denote it by $H(T_1, T_2)$, where the operation $\diamond \in \{\times, +, \oplus, \prod_F, \prod_U\}$ and $C_1 \diamond C_2 \in Mod\,\sigma$.

Here $\times$ means cartesian product, $+$ means sum, $\oplus$ means direct sum, $\prod_F$ means reduced product and $\prod_U$ means ultraproduct of models. Herewith, the algebraic construction $(C_1 \diamond C_2)$ is called a semantic hybrid of the theories T_1, T_2.

Definition 12 [3]

2) If T_1 and T_2 are Jonsson theories of different signatures σ_1 and σ_2, then $H(T_1, T_2) = Th_{\forall\exists}(C_1 \diamond C_2)$ will be called a hybrid of the second type, if that theory is Jonsson in the language of signature $\sigma = \sigma_1 \cup \sigma_2$ where $C_1 \diamond C_2 \in Mod\, \sigma$.
Obviously that 1) is the particular case of 2).

Examples of hybrids of Jonsson theories

1. Let T be some Jonsson theory, C be a semantic model of the theory T, $A, B \subseteq C$ and A, B be Jonsson subsets, where $dcl(A) = M_1$, $dcl(B) = M_2$ and $M_1, M_2 \in E_T$. Then, theories $T_1 = \mathrm{Th}_{\forall\exists}(M_1)$ and $T_2 = \mathrm{Th}_{\forall\exists}(M_2)$ are Jonsson theories. Let C_1 and C_2 be semantic models of these theories, respectively. Then $\mathrm{Th}_{\forall\exists}(C_1 \times C_2) = H(T_1, T_2)$ is a hybrid of the first type of Jonsson theories T_1 and T_2.
2. Let T_1, T_2 be the theories of groups. These theories are Jonsson theories. Let G_1 be the semantic model of the theory T_1, G_2 be the semantic model of the theory T_2. The particular case of this example of the hybrid is the example of acting of the group on itself.
3. Let T_1 be the theory of Abelian groups, T_2 is the theory of fields of fixed characteristic. Both T_1, T_2 are Jonsson theories. The particular case $H(T_1, T_2)$ is well-known notion of vector space.

The center of the hybrid

Since the hybrid of two Jonsson theories is a Jonsson theory, in the case when this theory is perfect, we will say for brevity – a perfect hybrid of two Jonsson theories. As the center of the hybrid $H(T_1, T_2)$, we will mean the center of the Jonsson theory $Th_{\forall\exists}(C_1 \diamond C_2)$ and denote it by $H^*(T_1, T_2)$.

Theorem 3.

Let K be an axiomatizable class of models of a countable language L of signature σ.
Let $[T_1], [T_2], [T_3], [T_4] \in JSp(K)/_{\bowtie}$, $H_1 = H([T_1], [T_2])$ and $H_2 = H([T_3], [T_4])$ are complete for existential sentences perfect hybrids, then following conditions are equivalent:

1. $H_1 \overset{S}{\sim} H_2$;
2. $H_1^* \overset{S}{\approx} H_2^*$.

Through $T_1 \overset{S}{\sim} T_2$ will be denote the Jonsson syntactic similarity of theories T_1 and T_2. The syntactic similarities of the complete theories T_1 and T_2 will be denoted $T_1 \overset{S}{\approx} T_2$.

Theorem 4.

Let K be an axiomatizable class of models of a countable language L of signature σ, $[T_1], [T_2] \in JSp(K)/_{\bowtie}$. For any perfect complete for $\exists$-sentences hybrid $H([T_1], [T_2])$ there is a Jonsson $\exists$-complete theory of the polygon T'_{Π} such that $H([T_1], [T_2]) \overset{S}{\sim} T'_{\Pi}$.

Thank you for your attention!