Claude Bernard Lyon 1 University

VII FRENCH-KAZAKH COLLOQUIUM ON MODEL THEORY

PROPERTIES OF COMPANIONS OF JONSSON AP-THEORIES

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Lyon, France
November 14-18, 2022

Introduction

The talk consists of 2 parts:

1. The results on the model-theoretic properties of the theory of differentially closed fields (zero and positive characteristics) in the framework of the study of the Jonsson theories will be presented.
2. The forcing companions of the Jonsson AP-theories in the enriched signature in more general situation are considered.

Outline

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Necessary information about differential fields

Let us introduce the theory DF of differential fields.

Definition 1 ([1])

The theory of differential fields is set by axioms of field theory and two added axioms:

$$\forall x \forall y D(x+y) = D(x) + D(y),$$
$$\forall x \forall y D(x,y) = xDy + yDx,$$

where $x, y \in K$ and $D : K \to K$ is a differentiation operator.

The models of DF are called differential fields.

The language used to study differential fields is the language $L = \{+, -, \cdot, D, 0, 1\}$. Here the differentiation operator D plays the role of a single functional symbol.

Necessary information about differential fields

Let us introduce the theory *DCF* of differentially closed fields.

Definition 2 ([2])

The theory of differential closed fields is set by axioms of the theory of differential fields augmented by two axioms:

- *Every nonconstant polynomial in one variable has a solution.*
- *If $f(x)$ and $g(x)$ are differential equations such that the order of $f(x)$ is greater than the order of $g(x)$ then $f(x)$ has a solution not a solution of $g(x)$.*

The models of *DCF* are called differentially closed fields.

Necessary information about differential fields

Definition 3 ([3])

A differential field F is said to be differentially perfect if every its extension is separable.

Theorem 1 ([3])

It is sufficient and necessary for a differential field of characteristic p to be differentially perfect that either $p = 0$ or $p \neq 0$ and $C = F^p$, where C is a constant subfield, F^p is all p-th powers of the elements of F.

In this manner, to set the theory *DPF* of differentially perfect fields of characteristic p we use the axioms of *DF* and add the following axiom [4]:

$$\forall x \exists y (D(x) = 0 \rightarrow y^p = x).$$

Necessary information about differential fields

Notation:

DF_0 – the theory of differential fields of characteristic 0,
DCF_0 – the theory of differentially closed fields of characteristic 0,
DF_p – the theory of differential fields of characteristic p,
DCF_p – the theory of differentially closed fields of characteristic p,
DPF_p – the theory of differentially perfect fields of characteristic p.

Necessary information about differential fields

Here are some important facts on the theories mentioned:

Theorem 2 ([2])

DF_0 has the joint embedding and the amalgam properties.

Theorem 3 ([2])

The DCF_0 theory is a model completion of the DF_0 theory.

Necessary information about differential fields

Theorem 4 ([4])

The theory DF_p of differential fields of characteristic p does not admit the amalgam property.

Theorem 5 ([4])

The DF_p theory has no model completion.

Theorem 6 ([4])

DPF_p admits the amalgam property.

Theorem 7 ([4])

DCF_p is a model companion for DF_p and a model completion for DPF.

Jonsson theories and related concepts

Definition 4 ([5])

A theory T is called a Jonsson theory if:

1. *T has infinite models,*
2. *T is an inductive theory,*
3. *T has the amalgam property (AP),*
4. *T has the joint embedding property (JEP).*

Jonsson theories and related concepts

Examples of Jonsson theories:

- group theory;
- abelian groups theory;
- boolean algebras theory;
- linear order theory;
- the theory of fields of characteristic p, where p is zero or a prime number;
- ordered fields theory;
- modules theory.

Jonsson theories and related concepts

Definition 5 ([6])

Let T be a Jonsson theory. A model C_T of power $2^{|T|}$ is said to be a semantic model of T if C_T is an ω^+-homogeneous ω^+-universal model of the theory T.

Theorem 8 ([6])

A theory T is Jonsson if and only if it has a semantic model C_T.

Definition 6 ([6])

A Jonsson theory T is called perfect if its semantic model C_T is saturated.

Definition 7 ([7])

A Jonsson theory is said to be hereditary if, in any of its permissible enrichment, it preserves the Jonssonness.

Special subclasses of Jonsson theories

Definition 8 ([8])

A theory T is called to be

1. *AP-theory if in theory T amalgam property implies joint embedding property;*
2. *JEP-theory if in theory T joint embedding property implies amalgam property;*
3. *AJ-theory if in theory T both properties are equivalent.*
4. *Otherwise, we say that for the theory T, the properties of AP and JEP are independent of each other.*

The listed classes of theories form the corresponding subclasses in the class of Jonsson theories.

Examples [9]: Different classes of unars define all 4 classes of the described theories.

The results on differential fields

It can be proved that theories DF_0, DCF_0, DPF_p, DCF_p are AP-theories. Using this fact we obtained the following results:

Theorem 9 ([8])

DF_0 is a perfect Jonsson theory.

Theorem 10 ([8])

DCF_0 is the center of DF_0.

The results obtained

Theorem 11 ([8])

DPF_p is a perfect Jonsson theory.

Theorem 12 ([8])

DCF_p is the center of DPF_p.

The results on differential fields

Remark 1 ([8])

Every perfect (in Galois sense) field is differentially perfect. The converse does not hold.

Remark 2 ([8])

DF_p is not a Jonsson theory, however it has the model companion DCF_p which is a perfect Jonsson theory. At the same time, the perfectness of DPF_p models in the differential sense is the sufficient condition of perfectness in Jonsson sense.

Some necessary information on model companions and forcing companions

Definition 9 ([5])

Let T and T_{MC} be some L-theories. The theory T_{MC} is called a model completion of the theory T if:
1) T and T_{MC} are mutually model consistent, i.e. any model of the theory T is embedded in the model of the theory T_{MC} and vice versa;
2) T_{MC} is a model complete theory;
3) if $A \models T$, then $T_{MC} \cup D(A)$ is a complete theory. The theory T_{MC} is called a model companion if conditions 1) and 2) hold.

Definition 10 ([10])

Let T be a theory of the language L. The forcing companion of the theory T is a theory T^f that satisfies the following condition:

$$T^f = \{\phi \mid T \Vdash \neg\neg\phi\}.$$

Some necessary information on model companions and forcing companions

The following results were proved by J. Barwise and A. Robinson:

Theorem 13 ([10])

Let T_1 and T_2 be the theories of the language L. Then T_1 and T_2 are mutually model consistent if and only if $T_1^f = T_2^f$.

Theorem 14 ([10])

Let T be mutually model consistent with some inductive theory T'. Then $T' \subseteq T^f$. Therefore, if T is an inductive theory then $T \subseteq T^f$.

Some necessary information on model companions and forcing companions

The following theorem is of particular importance for this study:

Theorem 15 ([6])

Let T be a perfect Jonsson theory. Then the following statements are equivalent:

1) T^* is the model companion of T;
2) $ModT^* = E_T$;
3) $T^* = T^f$, where T^f is a forcing companion of the theory T.

A general case for studying the forcing companion in the enriched signature

We move to the setting of our problem. We consider the theories

$$\Delta_1,\ \Delta_2,\ \Delta_3$$

that satisfy the following conditions:

1) Δ_1 is an inductive theory that is not a Jonsson theory, but has a model companion which is the theory Δ_3,

2) Δ_2 is a hereditary Jonsson AP-theory that has a model companion, which is also Δ_3.

A general case for studying the forcing companion in the enriched signature

Based on the conditions above, we can make the following conclusions:

- All three theories are mutually model consistent.
- $\Delta_1^f = \Delta_2^f$.
- Δ_2 is a perfect Jonsson theory, $\Delta_2^* = Th(C) = \Delta_3$, C is a semantic model of Δ_2.
- Δ_3 is also a forcing companion of Δ_2, i.e. $\Delta_3 = \Delta_2^f$. So we get $\Delta_1^f = \Delta_2^f = \Delta_3$.

A general case for studying the forcing companion

Let $\overline{\Delta_1}$ be a theory that extends Δ_1 by enriching the language L with the predicate symbol P as follows:

$$\overline{\Delta_1} = \Delta_1 \cup \Delta_1^f \cup \{P, \subseteq\},$$

where $\{P, \subseteq\}$ is an infinite list of $\exists$-sentences and interpretation of P is an existentially closed submodel in a model of Δ_1.

Let $\overline{\Delta_2}$ be a theory that extends Δ_2 when a new constant symbol c is added to the language L and defined as follows:

$$\overline{\Delta_2} = \Delta_2 \cup \Delta_2^f \cup Th_{\forall\exists}(C_2, c),$$

where C_2 is a semantic model of Jonsson theory Δ_2. Since Δ_2 is a hereditary Jonsson theory, $\overline{\Delta_2}$ is also a Jonsson theory.

Here we pose two questions:

1) How will the addition of new symbols P and c to the language L and the subsequent expansion of Δ_1 and Δ_2 affect the forcing companion of the received theories?

2) When combining the theories $\overline{\Delta_1}$ and $\overline{\Delta_2}$, can we obtain a consistent theory and what will be its forcing companion?

The answers are the following theorems.

The results on the forcing companions

Theorem 16

$$\overline{\Delta_1}^f = \Delta_1^f.$$

Thus, we can conclude that the forcing companion of the inductive theory Δ_1 does not change when enriching the language of this theory with a new predicate symbol P.

Theorem 17

$$\overline{\Delta_2}^f = \Delta_2^f.$$

This means that the addition of the new constant c to language L does not affect the forcing companion when expanding theory Δ_2 to $\overline{\Delta_2}$.

The results on the forcing companions

The following theorem (as known as Robinson's Consistency Theorem) is necessary for the following result.

Theorem 18 ([5])

Let T be a complete theory of language L, languages L_1 and L_2 are extensions of language L such that $L_1 \cap L_2 = L$, and theories T_1 and T_2 are consistent extensions of theory T in languages L_1 and L_2 respectively. Then $T_3 = T_1 \cup T_2$ is a consistent theory.

The results on the forcing companions

Now we can formulate the following result. Let us introduce the following notation:

$$\overline{\Delta_3} = \overline{\Delta_1} \cup \overline{\Delta_2} \cup P(c),$$

where the sentence $P(c)$ means that the constant symbol c added to the language belongs to $M = P(C_3)$, i.e. this axiom refines the interpretation of P in semantic model C_3 of the theory $\Delta_3 = (\Delta_1)^f = (\Delta_2)^f$ in accordance with the position of c in C_3.

Theorem 19

i) $\overline{\Delta_3}$ *is consistent.*
ii) $(\overline{\Delta_3})^f = \Delta_1^f = \Delta_2^f$

Merci de votre attention!