Université Lyon 1 / ENS Lyon Semestre de printemps 2019-2020
M1A Logique et Théorie des Ensembles

Examen du 21 avril 2021, durée 3h.

This exam consists of three problems, divided into 4+1+5=10 questions. All questions have equal weight.

Problem 1. Let T be a theory, $\mathcal{M} \models T$ a model. Let $\varphi(x, \bar{y})$ be a formula, and $\bar{a} \in M$ of the same length as $\bar{y}$, so $\varphi(x, \bar{a})$ is a formula with parameters in M. Let

$$\varphi(M, \bar{a}) := \{b \in M : \mathcal{M} \models \varphi(b, \bar{a})\}.$$

denote the set of solutions of $\varphi(x, \bar{a})$ in M.
We say that $\varphi(x, \bar{a})$ is *algebraic* if the set $\varphi(M, \bar{a})$ is finite.

1. Show that if $\mathcal{N} \succeq \mathcal{M}$, then exactly one of the following holds:

   - either $\varphi(M, \bar{a}) = \varphi(N, \bar{a})$ is finite,
   - or both $\varphi(M, \bar{a})$ and $\varphi(N, \bar{a})$ are infinite.

   In particular, while the definition of an algebraic formula seems to depend on the structure $\mathcal{M}$, it remains unchanged when replacing $\mathcal{M}$ with an elementary extension.

   Answer: let us define (also for later):

   $$\psi_n(\bar{y}) = \exists x_0, \ldots, x_{n-1} \left(\bigwedge_{i<j<n} x_i \neq x_j \right) \wedge \left(\bigwedge_{i<n} \varphi(x_i, \bar{y}) \right).$$

   In particular, $|\varphi(M, \bar{a})| = n$ iff $\mathcal{M} \models \psi_n(\bar{a}) \wedge \neg\psi_{n+1}(\bar{a})$.

   Now, since $\mathcal{M} \preceq \mathcal{N}$, we have:

   - $\varphi(M, \bar{a}) \subseteq \varphi(N, \bar{a})$
   - $\mathcal{M} \models \psi_n(\bar{a}) \wedge \neg\psi_{n+1}(\bar{a})$ iff $\mathcal{M} \models \psi_n(\bar{a}) \wedge \neg\psi_{n+1}(\bar{a})$, i.e., $|\varphi(M, \bar{a})| = n$ iff $|\varphi(N, \bar{a})| = n$ (for any n).

   In particular, as soon as one of the two sets is finite, they have the same cardinal, and two comparable sets of the same finite cardinal are equal.

2. Show that if $\varphi(x, \bar{a})$ is algebraic, then there exists a formula $\psi(\bar{y})$ such that:

   - $\mathcal{M} \models \psi(\bar{a})$
   - for every model $\mathcal{N} \models T$ and all $\bar{b} \in N$, if $\mathcal{N} \models \psi(\bar{b})$, then $\varphi(x, \bar{b})$ is algebraic as well.

   (The formula ψ may depend on $\bar{a}$.)

   Answer: Assume that $\varphi(M, \bar{a})$ is algebraic, say $|\varphi(M, \bar{a})| = n$. Then ψ_n (or ψ_m for any $m \geq n$) will do.

3. Show that for a theory T and formula $\varphi(x, \bar{y})$, the following are equivalent:

- There exists a formula $\psi(\bar{y})$ such that for every $\mathcal{M} \vDash T$ and every $\bar{b} \in M$ we have $\mathcal{M} \vDash \psi(\bar{b})$ if and only if $\varphi(x, \bar{b})$ is algebraic.
- There exists $m \in \mathbf{N}$ such that for every $\mathcal{M} \vDash T$ and every $\bar{b} \in M$, if $\varphi(x, \bar{b})$ is algebraic, then $|\varphi(M, \bar{b})| \leq m$.

(When this is true, the formula $\neg\psi(\bar{y})$ is often denoted $\exists^\infty x\, \varphi(x, \bar{y})$.)

Answer: bottom-to-top: ψ_m will do.

top-to-bottom: We argue by absurd. Assume that ψ existed, but no such m does. This means that for any m, $T \cup \{\psi(\bar{y}) \wedge \neg\psi_m(\bar{y})\}$ is consistent (consider some $\bar{a}$ in a model of T that witnesses the failure of m). It follows that the set

$$T \cup \{\psi(\bar{x})\} \cup \{\neg\psi_m(\bar{x}) : m \in \mathbf{N}\}$$

is finitely consistent. By the Compactness Theorem, it is consistent. In other words, there exists a model $\mathcal{M} \vDash T$, and $\bar{a} \in M$, such that $\psi(\bar{a})$ holds, but $|\varphi(M, \bar{a})| > m$ for all m, absurd.

4. Let $T = ACF$, the theory of algebraically closed fields. Show that the equivalent conditions of the previous item hold for every formula $\varphi(x, \bar{y})$.

   Hint: Start by showing this for formulas of the form $P(x, \bar{y}) = 0$ and $P(x, \bar{y}) \neq 0$, where $P \in \mathbf{Z}[x, \bar{y}]$ is a polynomial.

   First, consider $P(x, \bar{y}) = 0$. Then for every $\bar{a}$, either $P(x, \bar{a}) = 0$, and $P(K, \bar{a}) = K$ is infinite, or $P(x, \bar{a}) \neq 0$ and $|P(K, \bar{a})| \leq \deg P$.

   Second, consider $P(x, \bar{y}) \neq 0$. By the same cases as earlier, $P(K, \bar{a})$ is either empty or infinite.

   Therefore, if $\varphi(x, \bar{y})$ is either $P(x, \bar{y}) = 0$ or its negation, there exists m such that for all $\bar{a}$, either $|\varphi(K, \bar{a})| \leq m$ or it is infinite. Moreover, as we argue below, we realise that we need, and have, a stronger property: if $\varphi(K, \bar{a})$ is infinite, then it is *co-finite* (its complement is finite).

   Let us name this stronger property: say that $\varphi(x, \bar{y})$ has $*$ if there exist m_φ such that for every ACF K, and every $\bar{a} \in K$, either $|\varphi(K, \bar{a})| \leq m_\varphi$ or $\varphi(K, \bar{a})$ is co-finite.

   Now consider two formulas $\varphi_i(x, \bar{y})$, for $i = 1, 2$ that have $*$. Then $\varphi_1 \vee \varphi_2$ has $*$ (with $m_{\varphi_1} + m_{\varphi_2}$).

   For $\varphi_1 \wedge \varphi_2$, we get stuck at first: the intersection of two infinite sets may well be finite, and we cannot control its size. This is why we need the co-finiteness: the intersection of two co-finite sets is co-finite (and therefore infinite). It follows that $\varphi_1 \wedge \varphi_2$ has $*$ as well (with m_1, or m_2, or their minimum, any one would do). Conclusion: if $\varphi(x, \bar{y})$ is a conjunction of disjunctions of polynomial equalities and inequalities, then φ satisfies $*$, and in particular the weaker property that always $|\varphi(K, \bar{a})| \leq m_\varphi$ or $\varphi(K, \bar{a})$ is infinite.

   By QE, every formula is equivalent (modulo ACF) to a conjunction of disjunctions of polynomial equalities and inequalities, and we are done.

Definition. Recall that if $\mathcal{M}$ and $\mathcal{N}$ are L-structures, an *embedding* $f\colon \mathcal{M} \hookrightarrow \mathcal{N}$ is a map $f\colon M \to N$ such that for every atomic (equivalently, quantifier-free) formula $\varphi(\bar{x})$ and every $\bar{m} \in M$ (of the right length) we have

$$\mathcal{M} \vDash \varphi(\bar{m}) \qquad \Longleftrightarrow \qquad \mathcal{N} \vDash \varphi\big(f(\bar{m})\big).$$

We say that two L-theories are *companions* if every model of one embeds in a model of the other.

Problem 2. Let $L = \{0, 1, -, +, \cdot\}$ be the language of rings. Let T_1 be the theory of fields, and let T_2 be the theory of algebraically closed fields. Show that they are companions.
Answer: Any ACF is, in particular, a field, and embeds in itself. Conversely, it is a theorem (which we admit here) that every field admits an algebraic closure, i.e., it embeds in an ACF.

Problem 3. Let T_1 and T_2 be two theories in a language L.

1. A sentence φ is called *universal* if it is of the form $\forall \bar{x} \psi(\bar{x})$, where $\psi(\bar{x})$ is quantifier-free. Show that if $\mathcal{M} \subseteq \mathcal{N}$ are structures, φ is a universal sentence, and $\mathcal{N} \vDash \varphi$, then $\mathcal{M} \vDash \varphi$.

   Answer: Let φ be as above. Let $\bar{a} \in M$. Then $\bar{a} \in N$, and since $\mathcal{N} \vDash \forall \bar{x} \psi(\bar{x})$, we have in particular $\mathcal{N} \vDash \psi(\bar{a})$. Since $\mathcal{M} \subseteq \mathcal{N}$ and ψ is q-f: $\mathcal{M} \vDash \psi(\bar{a})$. Since $\bar{a}$ was arbitrary, $\mathcal{M} \vDash \forall \bar{x} \psi(\bar{x})$.

2. Deduce that if T_1 and T_2 are companions, then they have the same universal consequences. That is to say that if φ is a universal sentence that holds in every model of one, then it also holds in every model of the other.

   Answer: Let φ be a universal consequence of, say, T_1. Let $\mathcal{M}$ be any model of T_2. By hypothesis, $\mathcal{M}$ embeds in some model $\mathcal{N} \vDash T_1$. Then $\mathcal{M}$ is isomorphic to a substructure of $\mathcal{N}$, and $\mathcal{N} \vDash \varphi$, so $\mathcal{M} \vDash \varphi$ by the previous questions. We conclude that φ is also a universal consequence of T_2. Since the argument is symmetric, every universal consequence of T_2 is also one of T_1.

3. Let $\mathcal{M}$ be any L-structure. Let $L(M)$ consist of L together with a constant symbol for every $m \in M$. Define the *quantifier-free diagram* of $\mathcal{M}$ as:

   $$D^{qf}(\mathcal{M}) = \big\{\varphi(\bar{m}) : \mathcal{M} \vDash \varphi(\bar{m}), \text{ where } \varphi(\bar{x}) \text{ is a q-f } L\text{-formula and } \bar{m} \in M\big\}.$$

   Show that if $\mathcal{N}$ is an $L(M)$-structure, and $\mathcal{N} \vDash D^{qf}(\mathcal{M})$, then there exists a natural embedding $\mathcal{M} \hookrightarrow \mathcal{N}$.

   Answer: Assume that $\mathcal{N}$ is an $L(M)$-structure, and $\mathcal{N} \vDash D^{qf}(\mathcal{M})$. In particular, $\mathcal{N}$ is an L-structure, and we may define a map $f \colon M \to N$ by $f(m) = m^{\mathcal{N}}$ (the interpretation of $m \in M$, viewed as a constant symbol of $L(M)$, in $\mathcal{N}$). Then for any qf formula $\varphi(\bar{x})$ and every $\bar{m} \in M$ for the appropriate length:

   $$\mathcal{M} \vDash \varphi(\bar{m}) \Longrightarrow \varphi(\bar{m}) \in D^{eq}(\mathcal{M}) \Longrightarrow \mathcal{N} \vDash \varphi(\bar{m}^{\mathcal{N}}) \Longrightarrow \mathcal{N} \vDash \varphi\big(f(\bar{m})\big)$$

   Considering $\neg\varphi$, we have $\Longleftrightarrow$. Therefore f is an embedding.

4. Assume that T_1 and T_2 have the same universal consequences, and let $\mathcal{M} \vDash T_1$. Show that $T_2 \cup D^{qf}(\mathcal{M})$ is consistent.

   Answer: By compactness it will suffice to show that $T_2 \cup \{\varphi^i(\bar{m}^i) : i < n\}$ is consistent for every finite subset $\{\varphi^i(\bar{m}^i) : i < n\} \subseteq D^{qf}(\mathcal{M})$. The conjunction $\varphi(\bar{m}) = \bigwedge_i \varphi^i(\bar{m}^i)$ also holds in $\mathcal{M}$. In particular, $\forall \bar{x} \neg\varphi(\bar{x})$ fails in $\mathcal{M}$, so it cannot be a universal consequence of T_1. By hypothesis, it is not a universal consequence of T_2 either. Therefore there exists a model $\mathcal{N} \vDash T_2$, and some $\bar{n} \in \mathcal{N}$, such that $\mathcal{N} \vDash \varphi(\bar{n})$. Let us make $\mathcal{N}$ into an $L(M)$-structure, by interpreting $\bar{m}$ as $\bar{n}$, and all other members of M quite arbitrarily.

   Then we have $\varphi(\bar{n}) = \varphi(\bar{m}^{\mathcal{N}}) = \bigwedge \varphi^i((\bar{m}^i)^{\mathcal{N}})$. Therefore $\mathcal{N}$, viewed as an $L(M)$-structure in this manner, is a model of $T_2 \cup \{\varphi^i(\bar{m}^i) : i < n\}$.

5. State and prove the converse of question 2 of this problem.

   Answer: if T_1 and T_2 have the same universal consequences, then they are companions (so: they are companions iff they have the same universal consequences). Indeed, by q4, if $\mathcal{M} \models T_1$, then $T_2 \cup D^{qf}(\mathcal{M})$ is consistent, and therefore admits a model $\mathcal{N}$. This $\mathcal{N}$ is an $L(M)$-structure. Viewed merely as an L-structure, it is a model of T_2, and by q3 $\mathcal{M}$ embeds in $\mathcal{N}$. Similarly in the other direction.