Université Lyon 1 / ENS Lyon Semestre de printemps 2019-2020
M1A Logique et Théorie des Ensembles

Examen du 21 avril 2021, durée 3h.

This exam consists of three problems, divided into 4+1+5=10 questions. All questions have equal weight.

Problem 1. Let T be a theory, $\mathcal{M} \vDash T$ a model. Let $\varphi(x, \bar{y})$ be a formula, and $\bar{a} \in M$ of the same length as $\bar{y}$, so $\varphi(x, \bar{a})$ is a formula with parameters in M. Let

$$\varphi(M, \bar{a}) := \{b \in M : \mathcal{M} \vDash \varphi(b, \bar{a})\}.$$

denote the set of solutions of $\varphi(x, \bar{a})$ in M.
We say that $\varphi(x, \bar{a})$ is *algebraic* if the set $\varphi(M, \bar{a})$ is finite.

1. Show that if $\mathcal{N} \succeq \mathcal{M}$, then exactly one of the following holds:
   - either $\varphi(M, \bar{a}) = \varphi(N, \bar{a})$ is finite,
   - or both $\varphi(M, \bar{a})$ and $\varphi(N, \bar{a})$ are infinite.

   In particular, while the definition of an algebraic formula seems to depend on the structure $\mathcal{M}$, it remains unchanged when replacing $\mathcal{M}$ with an elementary extension.

2. Show that if $\varphi(x, \bar{a})$ is algebraic, then there exists a formula $\psi(\bar{y})$ such that:
   - $\mathcal{M} \vDash \psi(\bar{a})$
   - for every model $\mathcal{N} \vDash T$ and all $\bar{b} \in M$, if $\mathcal{M} \vDash \psi(\bar{b})$, then $\varphi(x, \bar{b})$ is algebraic as well.

   (The formula ψ may depend on $\bar{a}$.)

3. Show that for a theory T and formula $\varphi(x, \bar{y})$, the following are equivalent:
   - There exists a formula $\psi(\bar{y})$ such that for every $\mathcal{M} \vDash T$ and every $\bar{b} \in M$ we have $\mathcal{M} \vDash \psi(\bar{b})$ if and only if $\varphi(x, \bar{b})$ is algebraic.
   - There exists $m \in \mathbf{N}$ such that for every $\mathcal{M} \vDash T$ and every $\bar{b} \in M$, if $\varphi(x, \bar{b})$ is algebraic, then $|\varphi(M, \bar{b})| \leq m$.

   (When this is true, the formula $\neg\psi(\bar{y})$ is often denoted $\exists^\infty x\, \varphi(x, \bar{y})$.)

4. Let $T = ACF$, the theory of algebraically closed fields. Show that the equivalent conditions of the previous item hold for every formula $\varphi(x, \bar{y})$.
   Hint: Start by showing this for formulas of the form $P(x, \bar{y}) = 0$ and $P(x, \bar{y}) \neq 0$, where $P \in \mathbf{Z}[x, \bar{y}]$ is a polynomial.

Definition. Recall that if $\mathcal{M}$ and $\mathcal{N}$ are L-structures, an *embedding* $f\colon \mathcal{M} \hookrightarrow \mathcal{N}$ is a map $f\colon M \to N$ such that for every atomic (equivalently, quantifier-free) formula $\varphi(\bar{x})$ and every $\bar{m} \in M$ (of the right length) we have

$$\mathcal{M} \vDash \varphi(\bar{m}) \qquad \Longleftrightarrow \qquad \mathcal{N} \vDash \varphi\big(f(\bar{m})\big).$$

We say that two L-theories are *companions* if every model of one embeds in a model of the other.

Problem 2. Let $L = \{0, 1, -, +, \cdot\}$ be the language of rings. Let T_1 be the theory of fields, and let T_2 be the theory of algebraically closed fields. Show that they are companions.

Problem 3. Let T_1 and T_2 be two theories in a language L.

1. A sentence φ is called *universal* if it is of the form $\forall \bar{x} \psi(\bar{x})$, where $\psi(\bar{x})$ is quantifier-free. Show that if $\mathcal{M} \subseteq \mathcal{N}$ are structures, φ is a universal sentence, and $\mathcal{N} \vDash \varphi$, then $\mathcal{M} \vDash \varphi$.

2. Deduce that if T_1 and T_2 are companions, then they have the same universal consequences. That is to say that if φ is a universal sentence that holds in every model of one, then it also holds in every model of the other.

3. Let $\mathcal{M}$ be any L-structure. Let $L(M)$ consist of L together with a constant symbol for every $m \in M$. Define the *quantifier-free diagram* of $\mathcal{M}$ as:

$$D^{qf}(\mathcal{M}) = \big\{\varphi(\bar{m}) : \mathcal{M} \vDash \varphi(\bar{m}), \text{ where } \varphi(\bar{x}) \text{ is a q-f } L\text{-formula and } \bar{m} \in M\big\}.$$

   Show that if $\mathcal{N}$ is an $L(M)$-structure, and $\mathcal{N} \vDash D^{qf}(\mathcal{M})$, then there exists a natural embedding $\mathcal{M} \hookrightarrow \mathcal{N}$.

4. Assume that T_1 and T_2 have the same universal consequences, and let $\mathcal{M} \vDash T_1$. Show that $T_2 \cup D^{qf}(\mathcal{M})$ is consistent.

5. State and prove the converse of question 2 of this problem.