Cours du 14 avril 2020

Theorem 4.9 (General Omitting Types Theorem). It T is countable, I can omit a meagre set of n-types for all n.

Recall the topology on the set of n-types : basic clopen sets are $[\varphi(\bar{x})] = \{p \in S_n(T) : \varphi \in p\}$. Types are the points; they are closed. This topological space is compact (compactness theorem) and totally discontinuous.

Meagre = countable union of closed sets of empty interior

co-meagre = compement of meagre = countable intersection of dense open sets.

For a single type p, the set $\{p\}$ is meagre iff p is not isolated, as $\varphi(\bar{x})$ isolates p iff $\{p\} = [\varphi(\bar{x})]$.

So the OTT => non-principal types can be omitted (in a countable language).

But the general OTT allows us to omit several types (even infinitely many) at once.

Recall Baire category theorem : A (countable intersection of) co-meagre sets is dense. True in any locally compact topological space.

Proof : Start with a countable theory T and its Henkinisation T_H using countably many constants C. Consider a closed set p of empty interior in S_n. We want to show that $T_H \setminus [p(\bar{c})]$ is open dense in T_H.

Then $\bigcap_{\bar{c} \in C}(T_H \setminus [p(\bar{c})])$ is still dense, and in particular non-empty. So it contains a completion T' of T_H. If $\mathfrak{N}_C \models T'$, then $C^{\mathfrak{N}_C} \preccurlyeq \mathfrak{N}_C$ omitting p.

To show that $T_H \setminus [p(\bar{c})]$ is open dense in T_H, consider a basic open set $[\varphi'(\bar{c}')]$. We have to show that $(T_H \setminus [p(\bar{c})]) \cap [\varphi'(\bar{c}')]$ is non-empty. In other words,

$$T_H \cup \{\varphi'(\bar{c}')\} \not\models p(\bar{c}).$$

We know that $p(\bar{x})$ is mearge in S_n. We want that $[p(\bar{c})]$ is meagre in T_H.

Let $(c_i : i < \ell)$ contain $\bar{c}, \bar{c}'$. Among $(x_i : i < \ell)$ let $\bar{x}$ correspond to $\bar{c}$, and $\bar{x}'$ to $\bar{c}'$, and put $\bar{x}'' = (x_i : i < \ell) \setminus \bar{x}'$. Put

$$\psi(\bar{x}, \bar{x}'') = \varphi(\bar{x}') \wedge \bigwedge_{i<\ell}((\exists x \varphi_i(x)) \to \varphi_i(x_i)).$$

Then $[\exists \bar{x}'' \psi(\bar{x}, \bar{x}'')]$ is a non-empty open, as $[\varphi(\bar{c}')]$ is non-empty open in T_H, and hence $[\psi(\bar{c}, \bar{c}'')]$ is non-empty in T_H, so $\psi(\bar{x}, \bar{x}'')$ is consistent. Now $p(\bar{x})$ has non-empty interior, and does not contain $[\exists \bar{x}'' \psi(\bar{x}, \bar{x}'')]$, and there is a model $\mathfrak{M}$ and $\bar{m}$ in M such that $\mathfrak{M} \models [\exists \bar{x}'' \psi(\bar{m}, \bar{x}'')$ and $\mathfrak{M} \not\models p(\bar{m})$. Find $\bar{m}''$ in M such that $\psi(\bar{m}, \bar{m}'')$, and interpret c_i by m_i for $i < \ell$. Then I can interpre recursively c_i for $i \geq \ell$ for get a model of T_H where $p(\bar{c})$ is false and $\varphi(\bar{c}')$ is true.

Thus $[p(\bar{c})]$ has empty interior.

Section 2. (Countable) models.

Definition 4.10.

- A prime model need not be unique, nor minimal. If $\mathfrak{M}$ and $\mathfrak{M}'$ are prime over A, either embeds into the other other over A. In particular, if the embedding is proper, neither $\mathfrak{M}$ nor $\mathfrak{M}'$ are minimal.

- If $\mathfrak{M}$ is minimal and $\mathfrak{M}'$ is prime over A, then $\mathfrak{M} = \mathfrak{M}'$ is the unique minimal model : $\mathfrak{M}'$ embeds into $\mathfrak{M}$ as it is prime, and must be equal since $\mathfrak{M}$ is minimal.
- Clearly strongly λ-homogeneous implies λ-homogeneous.
Atomic = poor (only realizes types which must be realized)
Saturated = rich (realizes all types possible)

4.11.1 : Enumerate $B \setminus A = (a_i : i < \omega)$. Then $\mathrm{tp}(a_i : i \leq n/A)$ is isolated => $\mathrm{tp}(a_n/A, a_i : i < n)$ is isolated. So ANY enumeration will give a construction. This is false in hogher cardinality : If SOME enumeration of B is a construction over A, some other might not.
4.11.2. $\mathrm{tp}(a_k/A, a_i : i < k) = \mathrm{tp}(a_k/A \cup \{a_i : i < k\})$ and
$\mathrm{tp}(a_i : i < k/A) = \mathrm{tp}((a_i : i < k)/A)$.
We have $\mathrm{tp}(a_k/A, a_i : i < k)$ isolated by $\varphi(x, \bar{a}'')$ for some $\bar{a}''$ among $(a_i : i < k)$ (which may have more parameters from A). By hypothesis, for any $\bar{a}'$ among $(a_i : i < k)$ the type $\mathrm{tp}(\bar{a}'\bar{a}''/A)$ is isolated. Now $\varphi(x, \bar{a}'')$ also isolates $\mathrm{tp}(a_k/A, \bar{a}', \bar{a}'')$. By transitivity, $\mathrm{tp}(a_k, \bar{a}', \bar{a}''/A)$ is isolated. Hence $\mathrm{tp}(a_k\bar{a}'/A)$ is isolated.
4.11.2. As isolated types are realized, find in any model of $\mathrm{Th}(\mathfrak{M}, A)$ a sequence $(b_i : i < \alpha)$ such that if $\mathrm{tp}(a_i/A, a_j : j < i) = p(x, \bar{a})$, then $b_i \models \mathrm{tp}(x, \bar{b})$; if $\varphi(x, \bar{a})$ isolates $p(x, \bar{a})$, then $\varphi(x, \bar{b})$ isolates $p(x, \bar{b})$.

When do we have saturated and atomic models.
Proposition 4.14 and 4.15 : We have constructed models iff the isolated types are dense.
4.14 : If $[\varphi(x)]$ is non-empty, $\varphi(\bar{x})$ is realized in our atomic model by some $\bar{m}$, and $\mathrm{tp}(\bar{m})$ is isolated in $[\varphi(\bar{x})]$.
Conversely, in a Henkinisation, if $\exists x \varphi_i(x)$ is true, then by transitivity (φ_i contains finitely many constants $(c_j : j < i)$) $[\varphi_i(x)]$ contains an isolated type over $(c_j : j < i)$. We let c_i realize that type. Then $\mathrm{tp}(c_j : j \leq i)$ is isolated.
4.15 similar.
4.16. Any two countable atomic models are isomorphic. Any two saturated models of the same cardinal are isomorphic.
Just a back-and-forth argument. Take enumerations $(m_i : i < \lambda)$ of $\mathfrak{M}$ and $(n_i : i < \lambda)$ of $\mathfrak{N}$. Given a partial isomorphism σ_k with domain $\bar{m}$ of cadinality $< \lambda$ containing $(m_i : i < k)$, consider $\mathrm{tp}(m_k/\bar{m}) = p(x, \bar{m})$. Then $p(x, \sigma(\bar{m}))$ is realized (either because it is isolated, or by saturation) by some n_j. So we can prolong σ. By symmetry, we also have the "back" direction.
We start with σ_0 the empty partial isomorphism, and we take unions at limit stages.
For the "back" direction, if the image $\bar{n} = \sigma_k(\bar{m})$ of σ_k) contains $(n_i : i < k)$, consider $\mathrm{tp}(n/n) = p(x, \bar{n}) = p(x, \sigma(\bar{m}))$ and its preimage $p(x/\bar{m})$. It is realized in M because it is atomic, or by saturation.
I get a partial elementary map whose domain is M and whose image s N, i.e. an isomorphism.
Example 4.18.2 : $\langle \mathbb{Z}, 0, 1, + \rangle$ is atomic.
Proposition 4.22. Union of chains.

Theorem 4.25 (Ryll-Nardzewski)
If S_n is finite, say $S_n = \{p_1, \ldots, p_k\}$. For every pair of types (p_i, p_j) with $j \neq i$ choose a formule $\varphi_{ij} \in p_i \setminus p_j$. Then $\bigwedge_{j \neq i} \varphi_{ij}$ isolates p_i. Thus all n-types are isolated, and any formula $\psi(\bar{x}) = \bigvee_{p \in [\psi]} p$. There are only 2^k many possibilities.
Conversely, if there are only k inequivalent formulas in n variables, then $|S_n| \leq 2^k$.
Thus (2) $\Leftrightarrow$ (3).
(2) implies that all types in $S(T)$ are isolated, so all models are atomic. But two countable atomic models are isomorphic. Hence (1).
Conversely, if there is a non-isolated type, then there is a countable model realizing it, and another one omitting it, so T is not $\aleph_0$-categorical. Thus if T is $\aleph_0$-categorical, every type $p \in S_n$ is isolated by a formula φ_p. But if $S_n(T)$ is infinite, $\{\neg\varphi_p : p \in S_n(T)\}$ is finitely consistent, a contradiction. Hence $S_n(T)$ is finite.

Section 3 : Small theories.
Definition 4.27 : T is *small* if $S(T)$ is countable.
Lemma 4.28 : Smallness is preserved under adding finitely many parameters.
Proposition 2.29 : T has a countable saturated model iff T is small.
Remark : Countable saturated model $\neq$ countably saturated model
Countable atomic models : If T is small, isolated types are dense over finite sets (lemma 4.30), and there is a countable atomic model.
$I(T, \aleph_0)$ = the number of non-isomorphic countable models of T.
$= 1$ iff T is $\aleph_0$-categorical.
$\leq 2^{\aleph_0}$.
if $< 2^{\aleph_0}$, then T is small (Lemma 4.34).
can take values $3, 4, 5, \ldots, \aleph_0, 2^{\aleph_0}$.
$\neq 2$ (Vaught, Theorem 4.37).
Vaught's conjecture : Only interesting without CH. The interest is not the number of models, but to solve it, one has to analyse in detail the structure of countable models