Université Claude Bernard Lyon I — 2nd semestre 2019/2020
Master 1 — Introduction à la logique mathématique

Théorie des modèles

Feuille 9 : saturation

The following are equivalent for a structure $\mathcal{M}$ and infinite cardinal κ :

1. For every $A \subseteq M$, if $|A| < \kappa$ then every $p \in S_n(A)$ is realised in $\mathcal{M}$.
2. For every $A \subseteq M$, $|A| < \kappa$, and every set of formulas $\pi(\bar{x})$ with parameters in A, if every finite subset of π is realised in $\mathcal{M}$ (π is *finitely realised* in $\mathcal{M}$) then π is realised in $\mathcal{M}$.

[Idea : every complete type is in particular a partial type, and every partial type can be completed into a complete type.]

2 implies 1 : If $p \in S_n(A)$, then p is a set of formulas with parameters in A, and it is finitely realised, so it is realised.

1 implies 2 : It is enough to check this for maximal π (i.e., finitely realised set of formulas in A and maximal as such). By the compactness theorem (for sets of formulas with variables), or directly by Łoś, π is realised in some elementary extension of $\mathcal{M}$: i.e., $\mathcal{N} \succeq \mathcal{M}$ and $\bar{a} \in \mathcal{N}$ such that $\mathcal{N} \vDash \pi$. Since π is maximal : $\pi = \mathrm{tp}(\bar{a}/A) \in S_n(A)$. By 1 it is realised in $\mathcal{M}$.

Exercice 1. Parmi les structures de corps $\mathbf{Q}$, $\mathbf{Q}^a$ (clôture algébrique de $\mathbf{Q}$), $\mathbf{R}$ et $\mathbf{C}$ (dans le langage $L = \{0, 1, -, +, \cdot\}$) quelles sont celles qui sont $\aleph_0$-saturées ?

$\mathbf{Q}^a$? No.

We know that $\mathbf{C} \succeq \mathbf{Q}^a$ (since algebraically closed fields eliminate quantifiers)

Take an element $a \in \mathbf{C}$. What can $tp(a)$ be ? By QE, we only need to know the atomic formulas that a satisfies, i.e., the polynomials that a satisfies.

If a is algebraic, then it satifies some irreducible $P(x) \in \mathbf{Z}[x]$, and this determines $tp(a)$ (so $tp(a)$ is isolated by the formula $P(x) = 0$).

If a is transcendental : it doesn't satisfy any polynomial over $\mathbf{Z}$.

So $S_1(ACF) = \{\text{many algebraic types}\} \cup \{\text{unique transcendental type}\}$

Are all these types realised in $\mathbf{Q}^a$? The transcendental type is not realised.

The transcendental type is a type over a finite subset of $\mathbf{Q}^a$ (the empty set), and is not realised. So $\mathbf{Q}^a$ is not $\aleph_0$-saturated.

$\mathbf{Q}$? No. Why ?

Give a partial type over inspired by the previous argument :

$$\pi(x) = \{P(x) \neq 0 : P \in \mathbf{Z}[x] \setminus \{0\}\} = \text{"}x \text{ is transcendental"}$$

Any finite $\pi_0 \subseteq \pi$ is of the form $\{P_i(x) \neq 0 : i = 0, ..., k-1\}$ It is equivalent to $P(x) \neq 0$ for $P = \prod P_i$, and this is satisfies by at least one element (infinitely many, in fact) in $\mathbf{Q}$. So π is finitely realised in $\mathbf{Q}$, but not realised (no transcendental elements in $\mathbf{Q}$).

$\mathbf{R}$? No, because we can use both the order and the field structure. (the order is definable in the field language : $x \leq y$ if $(\exists z)(x + z^2 = y)$) Let

$$\pi(x) = \{x > 0\} \cup \{x < 1/n : n \in \mathbf{N} \setminus \{0\}\} = \text{"}x \text{ is infinitesimal"}$$

Any $\pi_0 \subseteq \pi$ finite is contained in a set of the form

$$\pi_0(x) \subseteq \{x > 0\} \cup \{x < 1/n : 0 < n < m\}$$

This is realised by $1/m \in \mathbf{Q} \subseteq \mathbf{R}$. So π is finitely realised in $\mathbf{R}$ but not realised in $\mathbf{R}$.

(remark : the same π can be used to prove that $\mathbf{Q}$ is not saturated, since it is finitely realised even in $\mathbf{Q}$)

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$\mathbf{C}$? Yes

(We cannot define the order in $\mathbf{C}$ – by QE every formula is equivalent to a Boolean combination of polynomial equalities.)

Let $A \subseteq \mathbf{C}$ finite (or even $|A| < 2^{\aleph_0}$). What is $S_1(A)$? An atomic formula in one variable x, with parameters in A is of the form $P(x, \bar{a}) = 0$ where $\bar{a} \in A$ and $P(x, \bar{y}) \in \mathbf{Z}[x, \bar{y}]$. It is the same thing as a polynomial in $K[x]$, where K is the subfield of $\mathbf{C}$ generated by A. . . [Analogous argument to what we did earlier] . $S_1(A)$ = algebraic types over K (each determined by an irreducible polynomials over K) or the unique transcendental (over K) type.

– The algebraic types are realised in $\mathbf{C}$ because $\mathbf{C}$ is algerbaically closed.

– The transcendental type : (recall that $K \supseteq \mathbf{Q}$, so $|K| \geq \aleph_0$)

$$|K| = |A| + \aleph_0$$
$$|K[x]| \leq \sum |K[x]_{\leq n}| .... \leq |K| + \aleph_0 = |K|$$
$$|K^a| \leq \aleph_0 \cdot |K[x]| = |K| + \aleph_0 = |K| = |A| + \aleph_0 < 2^{\aleph_0} = |\mathbf{C}|.$$

So there exists $a \in \mathbf{C} \setminus K^a$, and this a realises the transcendental type.

So every 1-type over A is realised for every finite A (and even for any A strictly smaller than the continuum).

Therefore, $\mathbf{C}$ is $\aleph_0$-saturated, and even $2^{\aleph_0}$-saturated.

For a structure $\mathcal{M}$ and κ infinite, TFAE :

1. $\mathcal{M}$ is κ saturated.
2. For every $A \subseteq M$, $|A| < \kappa$, every type in $S_1(A)$ is realised in $\mathcal{M}$.
3. For every $A \subseteq M$, $|A| < \kappa$, and $n \in \mathbf{N}$, every type in $S_n(A)$ is realised in $\mathcal{M}$.

Important observation : $A \subseteq M$, and let $\bar{a} \in M^n$ and $\bar{b} \in M^m$. Let $p(\bar{x}, \bar{y}) = \mathrm{tp}(\bar{a}, \bar{b}/A) \in S_{n+m}(A)$.

– Then $p(\bar{x}, \bar{b}) = \mathrm{tp}(\bar{a}/A\bar{b}) \in S_n(A, \bar{b})$ (here $A\bar{b} = A, \bar{b} = A \cup \{b_0, \ldots, b_{m-1}\}$).

– If $\bar{c} \in M^m$ and $\mathrm{tp}(\bar{c}/A) = \mathrm{tp}(\bar{b}/A)$, then $p(\bar{x}, \bar{c}) \in S_n(A, \bar{c})$ (why ? check that is it finitely satifiable because of the quality of types between $\bar{b}$ and $\bar{c}$, and it is maximal because p is maximal....)

This allows us to "replace parameters" in a type.

Now, let $\mathcal{M}$ be κ-saturated, $A \subseteq M$, $|A| < \kappa$, and let $p \in S_n(A)$. We want to show that p is realised in $\mathcal{M}$. We do know by definition that p is realised in some $\mathcal{N} \succeq \mathcal{M}$, say by $\bar{a} \in N^n$. We now apply induction on n.

$n = 1$: our assumption.

$n \to n+1$: [idea : realise an n-type, then, up to replacing parameters, we only need to realise one more 1-type] $\bar{a} \in N^{n+1}$, say $\bar{a} = a', \bar{a}''$ where $\bar{a}'' \in N^n$. By the induction hypothesis there exists $\bar{b}'' \in M^n$ such that $\mathrm{tp}(\bar{a}''/A) = \mathrm{tp}(\bar{b}''/A)$ Write $p = \mathrm{tp}(\bar{a}/A) = \mathrm{tp}(a', \bar{a}''/A)$ as $p(x, \bar{y})$. So $p(x, \bar{b}'') \in S_1(A, \bar{b}'')$ and $A, \bar{b}'' \subseteq M$ and $|A, \bar{b}''| \leq |A| + n < \kappa$. Therefore $p(x, \bar{b}'')$ is realised in M say by b'.

Then :

$$\mathcal{M} \vDash p(b', \bar{b}'')$$

And we are done.

Exercice 2. Soient $\mathcal{M}$ une structure $\aleph_0$-saturée, $\mathcal{N} \equiv \mathcal{M}$, et $(n_i)_{i \in \mathbf{N}}$ une famille d'éléments de $\mathcal{N}$. Montrer qu'il existe une famille $(m_i)_{i \in \mathbf{N}}$ d'éléments de $\mathcal{M}$ tel que pour tout $k \in \mathbf{N}$, $(m_0, ..., m_{k-1})$ et $(n_0, ..., n_{k-1})$ ont même type.
Indication : on peut considérer d'abord le cas où $\mathcal{M} \preceq \mathcal{N}$.

With the hint, we have $\mathcal{M} \preceq \mathcal{N}$, $\mathcal{M}$ is $\aleph_0$-saturated, and $(n_i : i \in \mathbf{N})$ is a sequence in $\mathcal{N}$. Want to find $(m_i : i \in \mathbf{N})$ in $\mathcal{M}$ "of the same type as (n_i)" i.e. $\mathrm{tp}(n_0, \ldots, n_{k-1}) = \mathrm{tp}(m_0, \ldots, m_{k-1})$ for all k (equivalently : the satisfy the same formulas).

We do this my the same inductive argument as earlier.

In the general case, we only have $\mathcal{M} \equiv \mathcal{N}$. We have seen that there exists an ultrapower $\mathcal{N}^{\mathcal{U}}$ and an elementary embedding $\mathcal{M} \hookrightarrow \mathcal{N}^{\mathcal{U}}$. In this case we may replace $\mathcal{N}$ with $\mathcal{N}^{\mathcal{U}}$ (recall that $\mathcal{N} \preceq \mathcal{N}^{\mathcal{U}}$) and $\mathcal{M}$ with its image in $\mathcal{N}^{\mathcal{U}}$, and we have reduced to the previous case. [What we really use here is that we can embed both $\mathcal{M}$ and $\mathcal{N}$ elementarily in some third structure, so we can replace $\mathcal{N}$ with that structure.]

Exercice 3 (Critère du va-et-vient)**.**

1. Montrer qu'une théorie T élimine les quantificateurs si et seulement si pour tous deux modèles $\aleph_0$-saturés $\mathcal{M}$ et $\mathcal{N}$ de T, dès lors qu'ils admettent au moins un isomorphisme partiel (non vide), la famille de tous les isomorphismes partiels finis entre $\mathcal{M}$ et $\mathcal{N}$ forme un va-et-vient.
2. Montrer qu'une théorie complète T élimine les quantificateurs si et seulement si pour tous deux modèles $\aleph_0$-saturés $\mathcal{M}$ et $\mathcal{N}$ de T, la famille de tous les isomorphismes partiels finis entre $\mathcal{M}$ et $\mathcal{N}$ forme un va-et-vient.

Sufficient : by what we did last time + every structure admits an $\aleph_0$-saturated elementary extension.

Necessary : Assume T has QE, $\mathcal{M}, \mathcal{N} \vDash T$ are $\aleph_0$-saturated. We need to show that any finite partial isomorphism between $\mathcal{M}$ and $\mathcal{N}$ can be extended to one more element. Let f be a finite partial isomorphism. So it is $\bar{a} \mapsto \bar{b}$ for $\bar{a} \in M^n$ and $\bar{b} \in N^n$, and $\bar{a}$ and $\bar{b}$ satisfy the same qf formulas. By QE : $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$. Let $c \in M$. We apply the same "change of parameters" idea : Let $p(x, \bar{y}) = \mathrm{tp}(c, \bar{a})$. Then $(p(x, \bar{a}) = \mathrm{tp}(c/\bar{a}) \in S_1(\bar{a})$, and) since $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$: $p(x, \bar{b}) \in S_1(\bar{b})$.

Since $\mathcal{N}$ is $\aleph_0$-saturated, $p(x, \bar{b})$ is realised in $\mathcal{N}$, say by d, and now $\mathrm{tp}(d, \bar{b}) = p = \mathrm{tp}(c, \bar{a})$. In particular $(c, \bar{a})$ and $(d, \bar{b})$ satisfy the same qf formulas so the map sending $(c, \bar{a}) \mapsto (d, \bar{b})$ is a finite partial isomorphism extending f. So we have back-and-forth.

If T is incomplete, it may be that there are no finite partial isomorphisms (of non-empty domain), so we need the extra condition. If T is complete, choose any $a \in \mathcal{M}$, and by the exercise, there exists $b \in \mathcal{N}$ such that $\mathrm{tp}(a) = \mathrm{tp}(b)$, so a non-empty finite partial isomorphism always exists (e.g., $a \mapsto b$).

Exercice 4. Soit le langage $L = \{<, c_i : i \in \mathbf{N}\}$ où $<$ est une relation binaire et les c_i sont des constantes. Soit T la théorie des ordres totaux denses sans extrémité telle que pour tout $i \in \mathbf{N}$, $c_i < c_{i+1}$.

1. Soit $\mathcal{M}$ un modèle $\aleph_0$-saturé de T. Montrer que l'ensemble A des éléments majorants tous les c_i n'a pas de plus petit élément.
2. Montrer que T élimine les quanteurs.
   Indication : on peut le faire directement, ou faire une réduction vers un résultat déjà connu.
3. En déduire que T est complète et élimine les quanteurs.
4. Construire un modèle dénombrable de T qui contient un plus petit majorant de la suite (c_i).
5. Montrer que T a exactement trois modèles dénombrables non isomorphes.
6. Montrer qu'il y a deux des modèles dénombrables non isomorphes qui se plongent l'un dans l'autre et vice versa.

Un ultrafiltre $\mathcal{U}$ sur I est dit $\aleph_1$-*incomplet* s'il existe une famille dénombrable de $(J_n)_{n \in \mathbf{N}} \subseteq \mathcal{U}$ telle que $\bigcap J_n \notin \mathcal{U}$.

— Il est facile de vérifier qu'un ultrafiltre sur $\mathbf{N}$ est soit principal, soit $\aleph_1$-incomplet.
— Plus généralement, tous les ultrafiltres que nous avons construits sont $\aleph_1$-incomplets (vérifiez !)
— Encore plus fort : si ZFC est consistent, alors ZFC + "tout ultrafiltre non principal est $\aleph_1$-incomplet" est consistent aussi (mais pour cela, il faut travailler un peu).

Exercice 5 (Les ultraproduits sont saturées)**.** Soit I un ensemble et $\mathcal{U}$ un ultrafiltre $\aleph_1$-incomplet sur I. Soit $(\mathcal{M}_i : i \in I)$ une famille de structures, et $\mathcal{N} = \prod_{\mathcal{U}} \mathcal{M}_i$ l'ultraproduit.

Le but de cet exercice est de montrer que pour toute famille dénombrables de formules $\pi(\bar{x})$, avec paramètres dans $\mathcal{N}$, si π est finiment réalisé dans $\mathcal{N}$ alors π est réalisé dans $\mathcal{N}$ (en d'autres mots, si pour tout $\pi_0 \subseteq \pi$ fini il existe $\bar{a} \in N$ tel que $\mathcal{N} \vDash \pi_0(\bar{a})$, alors il existe $\bar{a} \in N$ tel que $\mathbf{N} \vDash \pi(\bar{a})$).

(Une structure ayant cette propriété est appelée $\aleph_1$-compacte.)

1. On peu supposer que $J_0 = I$, $J_n \supseteq J_{n+1}$, et $\bigcap J_n = \emptyset$ (et toujours $J_n \in \mathcal{U}$).
2. Considérons d'abord le cas où π est sans paramètres. On énumère $\pi = \{\varphi_n(\bar{x}) : n \in \mathbf{N}\}$, et pose

   $$\psi_n(\bar{x}) = \bigwedge_{m<n} \varphi_m(\bar{x}).$$

   Posons

   $$K_n = \{i \in I : \mathcal{M}_i \vDash (\exists \bar{x})\psi_n\}.$$

   Montrer que $K_0 = I \supseteq K_1 \supseteq K_2 \supseteq \ldots$ et $K_n \in \mathcal{U}$ pour tout n.
3. On peut supposer que $J_n \subseteq K_n$ pour tout n.
4. Pour chaque $i \in I$, choisir $\bar{a}_i \in M_i$, de sorte que

   $$J_n \supseteq \{i \in I : \mathcal{M}_i \vDash \psi_n(\bar{a}_i)\}.$$

5. Posons $\bar{a} = [\bar{a}_i : i \in I] \in \mathbf{N}$ (que cela veut-il dire, lorsque $\bar{x}$ est un uplet et non un singleton ?) Montrer que $\mathcal{N} \models \pi(\bar{a})$.
6. Démontrer le cas général, par réduction au cas où π n'a pas de paramètres.
7. Montrer que si L est dénombrable, alors $\mathcal{N} = \prod_{\mathcal{U}} \mathcal{M}_i$ est $\aleph_1$-saturé.

Exercice 6. Soit κ un cardinal fortement inaccessible. Montrer que toute théorie complète sur un langage de cardinalité strictement inférieur à κ, qui a des modèles infinis, a un modèle κ-saturé de cardinal κ.