Die böse Farbe

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10 November 2006

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Introduction

A bad field is a field of finite Morley rank with a predicate for a proper divisible non-trivial subgroup.

The question of the existence of such fields arose naturally in the study of groups of finite Morley rank, where a Borel subgroup might have the form

$$K^+ \rtimes T$$

for some $T \leq K^\times$; if one could show that $T = K^\times$, this would imply the existence of involutions (i.e. a finite Morley rank version of the Feit-Thompson Theorem), and more generally the existence of elements of any finite order.

Positive characteristic

In characteristic $p > 0$ the existence of a bad field implies that there are only finitely many p-Mersenne primes, i.e. primes of the form

$$\frac{p^n - 1}{p - 1},$$

which is generally believed to be false.

Moreover, the absolutely algebraic numbers form an elementary substructure; it is thus impossible to construct a bad field of positive characteristic by generic (Hrushovski amalgamation style) methods, since they cannot tell us anything about $\mathrm{acl}(\emptyset)$.

It may, however, still be possible to construct a non-saturated generic structure (not of finite Morley rank), or a simple bad field of finite SU-rank.

Characteristic zero

I shall sketch the recent construction of a bad field of characteristic 0

It is obtained by collapsing Poizat's green field of characteristic zero and Morley rank $\omega \cdot 2$ with a multiplicative subgroup of Morley rank ω, following the ideas of the collapse of Poizat's red field of positive characteristic and Morley rank $\omega \cdot 2$ with an additive subgroup of Morley rank ω by Baudisch, Martín Pizarro and Ziegler.

This is joint work with Andreas Baudisch, Martin Hils and Amador Martín Pizarro.

Tori

Let K be an algebraically closed field of characteristic 0.

A variety V will be a subvariety of some $(K^\times)^n$.

A torus is a connected algebraic subgroup of $(K^\times)^n$.
It is given by equations of the form $x_1^{r_1} \cdot \ldots \cdot x_n^{r_n} = 1$.
For tori, linear dimension (of a generic point over $\mathbb{Q}$, modulo torsion) equals algebraic dimension.

Given an irreducible variety V, its minimal torus is the smallest torus T such that V lies in some coset $\bar{a} \cdot T$.
The codimension of V is then

$$\mathrm{cd}(V) := \dim(T) - \dim(V) = \mathrm{lin.dim}_{\mathbb{Q}}(V) - \dim(V).$$

A subvariety $W \subseteq V$ is cd-maximal if $\mathrm{cd}(W') > \mathrm{cd}(W)$ for every subvariety $W \subsetneq W' \subseteq V$. Clearly, irreducible components of V and tori cosets maximally contained in V are examples of cd-maximal subvarieties.

The weak CIT

Poizat used Zilber's weak CIT, a consequence of Ax' differential Schanuel conjecture:

For any uniform family $\mathcal{V}$ of varieties there is a finite set $\{T_0, \dots, T_n\}$ of associated tori, such that for any torus T, any $V \in \mathcal{V}$ and any irreducible component $W \ni \bar{a}$ of $V \cap \bar{a} \cdot T$ there is some i with $W \subseteq \bar{a} \cdot T_i$ and

$$\dim(T_i) - \dim(V \cap \bar{a} \cdot T_i) = \dim T - \dim W.$$

Moreover, the minimal torus of every cd-maximal subvariety of V belongs to the collection $\{T_0, \dots, T_r\}$.

We will assume that the above tori are all distinct, and $T_0 = (K^\times)^n$ and $T_1 = \{1\}^n$.

Consequences

1. Given T, the set of irreducible $V \in \mathcal{V}$ with minimal torus T is definable. In particular, for any irreducible $V \in \mathcal{V}$ there is a definable neighbourhood where $\mathrm{lin.dim}_{\mathbb{Q}}$ and cd remain constant.
2. Suppose $V \in \mathcal{V}$ decomposes into m irreducible components W_k with $d_k := \dim(W_k)$, $l_k := \mathrm{lin.dim}_{\mathbb{Q}}(W_k)$ and $c_k := \mathrm{cd}(W_k)$. Then this holds in some definable neighbourhood of V.

The class $\mathcal{C}$

Let $\mathcal{C}$ be the class of divisible hulls of finitely generated multiplicative subgroups of a field of characteristic 0, augmented by 0, and with a predicate $\ddot{U}$ (German: grün) for a torsion-free multiplicative subgroup, such that for all finitely generated subgroups A

$$\delta(A) = 2\,\mathrm{tr.deg}(A) - \mathrm{lin.dim}_{\mathbb{Q}}(\ddot{U}(A)) \geq 0.$$

We shall consider structures in $\mathcal{C}$ in a relational (apart from multiplication) Morleyization of ACF_0; embeddings will be with respect to this language (i.e. will extend to the fields generated).

Using the weak CIT, Poizat has axiomatized $\mathcal{C}$:

1. $\ddot{U}(M)$ is a torsion-free divisible multiplicative subgroup.
2. For every $\emptyset$-definable variety $V(\bar{x})$ of dimension n with $|\bar{x}| = 2n+1$, any $\bar{a} \in V \cap \ddot{U}(M)$ lies in some associated torus. (This uses that $\ddot{U}$ is torsion-free.)

Basic properties

Let $\bar{\mathcal{C}}$ be the class of structures whose finitely generated substructures are in $\mathcal{C}$, and for $\langle \bar a B\rangle \in \bar C$ put

$$\delta(\bar a/B) = 2\,\mathrm{tr.deg}(\bar a/B) - \mathrm{lin.dim}_{\mathbb{Q}}(\ddot U(\langle \bar a B\rangle)/\ddot U(\langle B\rangle)).$$

Properties:

1. $\delta(\bar a\bar b/C) = \delta(\bar b/C) + \delta(\bar a/\bar b C)$.
2. Submodularity: $\delta(\bar a/B) \le \delta(\bar a/B \cap \langle C\bar a\rangle)$ for $C \subseteq B$.
3. Let W be the locus of $\bar a$ over $\mathrm{acl}(B)$. Then $\delta(\bar a/\mathrm{acl}(B)) = \dim(W) - \mathrm{cd}(W)$.
4. In general, $\delta(\bar a/B) = \dim(W) - \mathrm{cd}(W) - \mathrm{lin.dim}_{\mathbb{Q}}(\langle \bar a B\rangle \cap \mathrm{acl}(B)/B)$.

Strong embeddings

Given $A \subseteq B \in \bar{C}$, we say that A is strong in B, denoted $A \leq B$, if $\delta(\bar{b}/A) \geq 0$ for every $\bar{b} \in B$.

1. If $C \leq M$ and $C' \leq M$, then $C \cap C' \leq M$.
2. For every $A \subseteq M$ there exists a unique $A \subseteq C = \langle C \rangle \leq M$ minimal such. We call such a set the (strong) closure of A (in M) and denote it by $\mathrm{cl}_M(A)$.
3. If $(A_i)_{i<\alpha}$ is an increasing sequence with $A_i \leq K$ for all $i < \alpha$, then $\bigcup_i A_i \leq M$.

The class $\mathcal{C}$ is countable up to isomorphism, and has AP and JEP with respect to strong embeddings. Let $\mathfrak{M}_\omega$ be its Fraïssé-Hrushovski limit. Using the weak CIT, Poizat has axiomatized its theory T_ω, and shown that $\mathfrak{M}_\omega$ is ω-saturated. It follows that $RM(\mathfrak{M}_\omega) = \omega \cdot 2$, and $RM(\ddot{U}(\mathfrak{M}_\omega)) = \omega$.

Pre-algebraicity

- Let $A \subseteq B \in \mathcal{C}$ with $\mathrm{lin.dim}_{\mathbb{Q}}(B/A) = n \geq 2$. The extension B/A is minimal prealgebraic of length n if $\delta(B/A) = 0$ and $\delta(B'/A) > 0$ for every $A \subsetneq B' \subsetneq B$ (or equivalently, if $\delta(B/B') < 0$).
- Let $B \subseteq \mathcal{C}$. A strong ACF_0-type $p(\bar{x}) \in S_n(B)$ is minimal prealgebraic, if the extension $\langle B\bar{a}\rangle/\langle B\rangle$ is minimal prealgebraic of length n for some $\bar{a} \models p$ with $\ddot{U}(\bar{a})$. In particular, $\bar{a}$ is multiplicatively independent over B. This is invariant under parallelism and multiplicative translation.
- An ACF_0-formula $\varphi(\bar{x})$ of Morley degree 1 is minimal prealgebraic if its generic type is minimal prealgebraic.

Note that if B/A is minimal prealgebraic and $\bar{b}$ is a multiplicative green basis of B over A, then $\mathrm{stp}(\bar{b}/A)$ is minimal prealgebraic.

Minimal extensions

If $A \leq B$ in $\mathcal{C}$ with $\mathrm{lin.dim}_{\mathbb{Q}}(B/A) < \infty$, we can find a decomposition $A = A_0 \leq A_1 \leq \ldots \leq A_{n-1} \leq A_n = B$ such that A_{i+1}/A_i is minimal strong for all $i < n$.

Let $A \leq B$ be minimal strong. There are four possibilities:

1. algebraic: $\ddot{U}(A) = \ddot{U}(B)$ and $B = \langle Ab \rangle$ for some $b \in \mathrm{acl}(A) \setminus A$. Then $\delta(B/A) = 0$.
2. white generic: $\ddot{U}(A) = \ddot{U}(B)$ and $B = \langle Ab \rangle$ for some element b transcendental over A. Then $\delta(B/A) = 2$.
3. green generic: B contains a basis consisting of a green singleton b over A. Moreover, b is transcendental over A and $\delta(B/A) = 1$.
4. minimal prealgebraic: $A \leq B$ is minimal prealgebraic, i.e. B contains a green basis $\bar{b}$ over A such that $\mathrm{stp}(\bar{b}/A)$ is minimal prealgebraic. Then $\delta(B/A) = 0$.

Codes

A code is an ACF_0-formula $\varphi(\bar{x}, \bar{y})$ with $n_\varphi = |\bar{x}|$ such that

1. For all $\bar{b}$ either $\varphi(\bar{x}, \bar{b})$ is empty, or has Morley degree 1.
2. $RM(\bar{a}/\bar{b}) = n_\varphi/2$ and $\mathrm{lin.dim}_{\mathbb{Q}}(\bar{a}/\bar{b}) = n_\varphi$ for generic $\bar{a} \models \varphi(\bar{x}, \bar{b})$.
3. Let $T_0, T_1, \ldots$ be the tori associated to the Zariski closure V of $\varphi(\bar{x}, \bar{b})$.
   For any $\bar{a} \models \varphi(\bar{x}, \bar{b})$, any $i = 2, \ldots, r$ and any irreducible component W of $V \cap \bar{a} \cdot T_i$ of maximal dimension, $\dim(T_i) > 2 \cdot \dim(W)$ if $V \cap \bar{a} \cdot T_i$ is infinite.
4. If $RM(\varphi(\bar{x}, \bar{b}) \cap \varphi(\bar{x}, \bar{b}')) = n_\varphi/2$, then $b = b'$.
5. For any invertible $\bar{m}$ and $\bar{b}$ there is $\bar{b}'$ with $\varphi(\bar{x} \cdot \bar{m}, \bar{b}) \equiv \varphi(\bar{x}, \bar{b}')$.

Properties

Let $\varphi(\bar{x}, \bar{y})$ be a code, and suppose $\varphi(\bar{x}, \bar{b})$ is non-empty.

1. If $\bar{a} \models \varphi(\bar{x}, \bar{b})$ is generic over $B \ni \bar{b}$ and green, then the extension $B \subseteq \langle B\bar{a} \rangle$ is minimal prealgebraic.
2. For all green $\bar{a} \models \varphi(\bar{x}, \bar{b})$ and $B \ni \bar{b}$
   - $\delta(\bar{a}/B) \leq 0$.
   - If $\delta(\bar{a}/B) = 0$, either $\bar{a} \in \langle B \rangle$ or $\bar{a}$ is generic in $\varphi(\bar{x}, \bar{b})$ over B.
3. $\bar{b}$ is the canonical parameter for $\varphi(\bar{x}, \bar{b})$.

Every minimal prealgebraic extension gives rise to some code.

Toric correspondences

$\mathrm{GL}_n(\mathbb{Q})$ acts on the codes. Since this group is infinite, we cannot put invariance under $\mathrm{GL}_n(\mathbb{Q})$ into the axioms, but have to deal with it externally. Using the weak CIT we obtain:

Let φ and ψ be codes. There is a finite set $G(\varphi, \psi)$ of tori such that if $\varphi(\bar{x}, \bar{b}) \neq \emptyset$ and $T \cap (\varphi(\bar{x}, \bar{b}) \times \psi(\bar{x}, \bar{b}'))$ projects generically onto $\varphi(\bar{x}, \bar{b})$ and $\psi(\bar{x}, \bar{b}')$ for some torus T with $\dim(T) = |\bar{x}|$ (a toric correspondence), then $T \in G(\varphi, \psi)$.

There exists a collection $\mathcal{S}$ of codes such that for every minimal prealgebraic definable set X there is a unique code $\varphi \in \mathcal{S}$ and finitely many tori T such that T induces a toric correspondence between X and some instance of φ.

Proof

Suppose T induces a toric correspondence between $\varphi(\bar{x}, \bar{b})$ and $\psi(\bar{x}, \bar{b}')$. Let $\mathcal{T}$ be the finite family of tori associated to the Zariski closure V of $\varphi(\bar{x}, \bar{b}) \times \psi(\bar{x}, \bar{b}')$, and put $B = \mathrm{acl}(\bar{b}\bar{b}')$. Choose some B-generic point $(\bar{a}, \bar{a}') \in V \cap T$. Let $W \subseteq V \cap T$ be the locus of $(\bar{a}, \bar{a}')$ over B. Then T is the minimal torus of W, and $\dim(W) = \mathrm{cd}(W) = n_\varphi/2$.

Suppose $W \subsetneq W' \subseteq V$ and $(\bar{g}, \bar{g}')$ is B-generic in W'.

$$\begin{aligned}
\mathrm{cd}(W') &= \mathrm{lin.dim}_{\mathbb{Q}}(\bar{g}, \bar{g}'/B) - \mathrm{tr.deg}(\bar{g}, \bar{g}'/B) \\
&= [\mathrm{lin.dim}_{\mathbb{Q}}(\bar{g}/B) - \mathrm{tr.deg}(\bar{g}/B)] \\
&\quad + [\mathrm{lin.dim}_{\mathbb{Q}}(\bar{g}'/B\bar{g}) - \mathrm{tr.deg}(\bar{g}'/B\bar{g})] \\
&= \mathrm{cd}(W) + \mathrm{lin.dim}_{\mathbb{Q}}(\bar{g}'/B\bar{g}) - \mathrm{tr.deg}(\bar{g}'/B\bar{g}) \\
&> \mathrm{cd}(W) - \delta(\bar{g}'/B\bar{g}) \geq \mathrm{cd}(W),
\end{aligned}$$

since $W \subsetneq W'$ implies $\mathrm{tr.deg}(\bar{g}'/B\bar{g}) > 0$, and $\bar{g}'$ realizes the code $\psi(\bar{x}, \bar{b}')$. So $W \subseteq V$ is cd-maximal, and $G(\varphi, \psi) \subseteq \mathcal{T}$.

The collection $\mathcal{S}$ can now be constructed recursively. List all minimal prealgebraic subsets $(X_i : i < \omega)$ up to isomorphism. Suppose that $\mathcal{S}_i$ has been already defined encoding all X_j for $j < i$. If X_i can be encoded by some element in $\mathcal{S}_i$ and some torus T, then set $\mathcal{S}_{i+1} = \mathcal{S}_i$. Otherwise X_i is equivalent to some code instance $\varphi(\bar{x}, \bar{b})$. Put

$$\rho(\bar{z}) := \forall \bar{y} \left(\bigwedge_{\psi \in \mathcal{S}_i} \quad \bigwedge_{T \in G(\psi,\varphi)} \neg \chi^T_{\psi,\varphi}(\bar{y}, \bar{z}) \right),$$

where $\chi^T_{\psi,\varphi}(\bar{b}, \bar{b}')$ expresses that T induces a toric correspondence between $\psi(\bar{x}, \bar{b})$ and $\varphi(\bar{x}, \bar{b}')$. Then $\mathcal{S}_{i+1} := \mathcal{S}_i \cup \{\varphi(\bar{x}, \bar{z}) \wedge \neg\rho(\bar{z})\}$ will do.

Difference sequences

For a code φ and some $\bar{b}$ consider a generic Morley sequence $(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_k, f)$ for $\varphi(\bar{x}, \bar{b})$, and put $\bar{e}_i = \bar{a}_i \cdot \bar{f}^{-1}$. We can then find a formula $\psi_\varphi^k \in \mathrm{tp}(\bar{e}_0, \ldots, \bar{e}_k)$ such that

1. ψ_φ^k implies $\psi_\varphi^{k'}$ for all $k' < k$.
2. ψ_φ^k is invariant under the finite group of derivations generated by $\partial_i : \bar{x}_j \mapsto \begin{cases} \bar{x}_j \cdot \bar{x}_i^{-1} & \text{if } j \neq i \\ \bar{x}_i^{-1} & \text{if } j = i \end{cases}$.
3. Any realization $(\bar{e}'_0, \ldots, \bar{e}'_k)$ of ψ_φ^k is disjoint, and $\models \varphi(\bar{e}'_i, \bar{b}')$ for some unique canonical parameter $\bar{b}'$ definable over any m_φ elements among the $\bar{e}'_i$.
4. If $\bar{e}'_i$ is generic and there is some toric correspondence T on φ with $(\bar{e}_j, \bar{e}') \in T$ for some $i \neq j$ and $\bar{e}'$, then $\bar{e}_i \not\downarrow_{\bar{b}} \bar{e}' \cdot \bar{e}_i^{-1}$.

A difference sequence for φ of length k is a realization of ψ_φ^k.

Proof

Consider the following type-definable property $\Sigma(\bar{e}_0, \ldots, \bar{e}_\lambda)$:

there exist $\bar{b}'$ and a Morley sequence $\bar{e}'_0, \ldots, \bar{e}'_\lambda, \bar{f}$ in $\varphi(\bar{x}, \bar{b}')$ with $\bar{e}_i = \bar{e}'_i \cdot \bar{f}^{-1}$.

Σ satisfies 1.–3. Now $(\bar{e}_i : i \leq \lambda)$ is a Morley sequence over $\bar{b}'\bar{f}$. If $\bar{e}_i$, $\bar{e}_j$, $\bar{e}'$ and T are as in 4., then $\bar{e}' \in \mathrm{acl}(\bar{e}_j)$, so $\bar{e}' \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{b}'\bar{f}} \bar{e}_i$. If $\bar{e}_i \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{b}'\bar{f}} \bar{e}' \cdot \bar{e}_i^{-1}$, then $\bar{e}_i^{-1}$, $\bar{e}'$ and $\bar{e}' \cdot \bar{e}_i^{-1}$ will determine a pairwise $\bar{b}'\bar{f}$-independent triple.

By a Lemma of Ziegler all three are generic types for cosets of some torus T. A contradiction, since by code property 2.

$$0 \geq \delta(\bar{e}_i/\bar{b}'\bar{f}) = \delta(T) = \dim(T).$$

Let $\psi_0 \in \Sigma$ imply properties 1.,3. and 4., and put

$$\psi^k_\varphi(\bar{x}_0, \ldots, \bar{x}_k) := \bigwedge_{\partial \text{ derivation}} \psi_0(\partial(\bar{x}_0, \ldots, \bar{x}_\lambda)).$$

A counting Lemma

Given a code φ and natural number n, there is some $\lambda = \lambda_\varphi(n)$ such that for every $M \leq N \in \mathcal{C}$ and difference sequence $(\bar{e}_0, \ldots, \bar{e}_\lambda)$ in N with canonical parameter $\bar{b}$, either

- the canonical parameter for some derived sequence lies in M, or
- the sequence $(\bar{e}_0, \ldots, \bar{e}_\lambda)$ contains a generic subsequence over $M\bar{b}$ of length n.

Suppose the first part does not hold. Put

$$\begin{aligned} X_1 &= \{i \in [m_\varphi, \lambda] : \bar{e}_i \text{ generic over } M \cup \{\bar{e}_0, \ldots, \bar{e}_{i-1}\}\}, \\ X_2 &= \{i \in [m_\varphi, \lambda] : \bar{e}_i \subseteq \langle M \cup \{\bar{e}_0, \ldots, \bar{e}_{i-1}\}\rangle\}, \\ X_3 &= [m_\varphi, \lambda] \setminus (X_1 \cup X_2). \end{aligned}$$

We may assume that $X_1 < X_3 < X_2$.

Bounding $|X_3|$

Note that

$$\begin{aligned} \delta(\bar{e}_i/M, \bar{e}_0, \ldots, \bar{e}_{i-1}) &\leq -1 \text{ for } i \in X_3 \text{ and} \\ \delta(\bar{e}_i/M, \bar{e}_0, \ldots, \bar{e}_{i-1}) &= 0 \quad \text{ for } i \in X_1 \cup X_2. \end{aligned}$$

Since $M \leq N$

$$\begin{aligned} 0 &\leq \delta(\bar{e}_0, \ldots, \bar{e}_\lambda/M) \\ &\leq \delta(\bar{e}_0, \ldots, \bar{e}_{m_\varphi - 1}/M) + \sum_{i=m_\varphi}^{\lambda} \delta(\bar{e}_i/M, \bar{e}_0, \ldots, \bar{e}_{i-1}) \\ &\leq m_\varphi n_\varphi + (-1)|X_3| , \end{aligned}$$

whence $|X_3| \leq m_\varphi n_\varphi$.

Bounding $|X_2|$

Put $r = m_\varphi + |X_1| + |X_3|$ and $s = r(n_\varphi + 1)$.

For simplicity, assume that there are varieties V, W with $\psi_\varphi = V \setminus W$, and let $\mathcal{T}$ be the family of tori associated to V.

Let $I \subset [r, \lambda]$ of cardinality $rn_\varphi + 1$; for simplicity assume $I = [r, s]$. Let W' be the locus of $(\bar{e}_0, \ldots, \bar{e}_s)$ over $\mathrm{acl}(M)$, and choose $W' \subseteq W'' \subseteq V$ maximal with $\mathrm{cd}(W'') \leq \mathrm{cd}(W')$.

By construction W'' is cd-maximal, so its minimal torus is some $T \in \mathcal{T}$. Fix some $\bar{m} \in \mathrm{acl}(M)$ with $W'' \subseteq \bar{m}T \cap V$.

Choose $(\bar{a}_0, \ldots, \bar{a}_s)$ a generic point of W'' over $\mathrm{acl}(M)$ and paint it green. It lies in $V \setminus W$, since $(\bar{e}_0, \ldots, \bar{e}_s)$ is an specialization of $(\bar{a}_0, \ldots, \bar{a}_s)$, so $\models \psi_\alpha(\bar{a}_0, \ldots, \bar{a}_s)$.

Bounding $|X_2|$

$$\begin{aligned} rn_\varphi &\geq \text{lin.dim}_{\mathbb{Q}}(\bar{e}_{<r}/M) = \text{lin.dim}_{\mathbb{Q}}(\bar{e}_{\leq s}/M) \geq \text{cd}(W') \geq \text{cd}(W'') \\ &= \sum_{i \leq s} \text{lin.dim}_{\mathbb{Q}}(\bar{a}_i/\widetilde{M}, \bar{a}_{<i}) - \text{tr.deg}(\bar{a}_i/\widetilde{M}, \bar{a}_{<i}) \\ &\geq \sum_{r \leq i \leq s} \text{lin.dim}_{\mathbb{Q}}(\bar{a}_i/\widetilde{M}, \bar{a}_{<i}) - \text{tr.deg}(\bar{a}_i/\widetilde{M}, \bar{a}_{<i}). \end{aligned}$$

By property 2. of codes $\delta(\bar{a}_i/\widetilde{M}, \bar{a}_{<i}) \leq 0$ for $i \geq r \geq m_\varphi$, so

$$2\,\text{tr.deg}(\bar{a}_i/\widetilde{M}, \bar{a}_{<i}) \leq \text{lin.dim}_{\mathbb{Q}}(\bar{a}_i/\widetilde{M}, \bar{a}_{<i}).$$

Hence, if $\bar{a}_i \notin \langle \widetilde{M}, \bar{a}_{<i} \rangle$ then

$$\text{lin.dim}_{\mathbb{Q}}(\bar{a}_i/\widetilde{M}, \bar{a}_{<i}) - \text{tr.deg}(\bar{a}_i/\widetilde{M}, \bar{a}_{<i}) \geq 1.$$

Therefore, there is some $t \in \{r, \ldots, s\}$ with $\bar{a}_t \in \langle \widetilde{M}, \bar{a}_{<t} \rangle$.

Bounding $|X_2|$

The linear dependence will be determined by the coset $\bar{m}T$. So $\bar{m}T$ also determines that $\bar{e}_t \in \langle \widetilde{M}, \bar{e}_{<t} \rangle$.

Consider now all possible pairs (t, T). This determines a $(rn_\varphi + 1)|\mathcal{T}|$-coloring of all $(rn_\varphi + 1)$-subsets of $\{r, \ldots, \lambda\}$.

By the (finite) Ramsey theorem, there is some number λ_0, such that for $\lambda \geq \lambda_0$ there is a monochromatic subset $I \subseteq \{r, \ldots, \lambda\}$ of cardinality $|I| \geq m_\alpha + rn_\alpha + 1$.

Thus for some $t \in \{r, \ldots, s\}$ and some $T \in \mathcal{T}$

$$\bar{e}_{i_t} \in \langle \widetilde{M}, \bar{e}_{<r}, \bar{e}_{i_r}, \ldots, \bar{e}_{i_{t-1}} \rangle,$$

for all $i_r < \cdots < i_s$ in I, and the linear dependence comes from some $\bar{m}T_j$ with $\bar{m} \in \widetilde{M}$.

Let γ_i be the $(t+i)^{\text{th}}$ element in I. For $i > 0$ we have that $\bar{e}_{\gamma_i}\bar{e}_{\gamma_0}^{-1} \in \widetilde{M}$, so the canonical parameter of the derived sequence lies in $\widetilde{M}$, a contradiction.

The class $\mathcal{C}_\mu$

Let μ^*, μ be finite-to-one functions from $\mathcal{S}$ to ω with

$$\mu^*(\varphi) \geq \max\{\frac{n_\varphi^2}{2}+1, \lambda_\varphi(m_\varphi+1)\} \quad \text{and} \quad \mu(\varphi) \geq \lambda_\varphi(\mu^*(\varphi)).$$

Let $\mathcal{C}_\mu$ be the class of $M \in \mathcal{C}$ such that no $\varphi \in \mathcal{S}$ has a (green) difference sequence in M of length $> \mu(\varphi)$.

The class $\mathcal{C}_\mu$ is universally axiomatizable relative to ACF_0.

Characterizing minimal extensions

Let $M \le M'$ be minimal prealgebraic, $M \in \mathcal{C}_\mu$ but $M' \in \mathcal{C} \setminus \mathcal{C}_\mu$, as witnessed by some difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu(\varphi)})$ for some code instance $\varphi(\bar{x}, \bar{b})$. If $\bar{b} \in \mathrm{acl}(M)$, there is an M-generic $\bar{e}_i$ generating M' over M, and $\bar{e}_j \in M$ for $j \neq i$.

We may assume that M is algebraically closed. As $M \le M'$, any $\bar{e}_j \notin M$ is M-generic. Since $M \in \mathcal{C}_\mu$ there must be some generic $\bar{e}_i$, and $M' = \langle M\bar{e}_i \rangle$ by minimality.

Suppose $\bar{e}_j$ is also M-generic. Since $M' = \langle M\bar{e}_i \rangle = \langle M\bar{e}_j \rangle$ there is $\bar{m} \in M$ and a toric correspondence $T \ni (\bar{e}_i \cdot \bar{m}, \bar{e}_j)$.

Let $\bar{e}_j' := \bar{e}_i \cdot \bar{m}$, so $\bar{e}_j' \cdot \bar{e}_i^{-1} \in M$. Since $\bar{e}_i$ is M-generic,

$$\bar{e}_i \underset{\bar{b}}{\downarrow} \bar{e}_j' \cdot \bar{e}_i^{-1},$$

contradicting property 4. of a difference sequence.

Characterizing minimal extensions

Let $M \leq M' \in \mathcal{C}$ be minimal, $M \in \mathcal{C}_\mu$. If $\mathrm{lin.dim}_{\mathbb{Q}}(M'/M) = 1$, then $M' \in \mathcal{C}_\mu$. Otherwise M'/M is minimal prealgebraic, and $M' \notin \mathcal{C}_\mu$ iff there is $\varphi \in \mathcal{S}$ and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu(\varphi)})$ for φ in M' with canonical parameter $\bar{b}$, s.t.

1. φ is unique, $\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)-1} \in M$ and $\langle M, \bar{e}_{\mu(\varphi)} \rangle = M'$, or
2. there is a subsequence of length $\mu^*(\varphi)$ which is a Morley sequence for $\varphi(\bar{x}, \bar{b})$ over $M\bar{b}$.

$M' \notin \mathcal{C}_\mu$ yields a generic realisation $\bar{e}$ of some code $\varphi(\bar{x}, \bar{b})$ over $M\bar{b}$, and $\mathrm{lin.dim}_{\mathbb{Q}}(M'/M) \geq \mathrm{lin.dim}_{\mathbb{Q}}(\bar{e}/M\bar{b}) \geq 2$.

1. or 2. imply $M' \notin \mathcal{C}_\mu$. Conversely, if $M' \notin \mathcal{C}_\mu$ we obtain a long difference sequence for a code $\varphi \in \mathcal{S}$; if 1. does not hold, the counting Lemma yields 2. If φ' is a second such code, $\langle M\bar{e} \rangle = M' = \langle M\bar{e}' \rangle$ yields a toric correspondence in $G(\varphi, \varphi')$, whence $\varphi = \varphi'$ by construction of $\mathcal{S}$.

Thrifty amalgamation

Suppose $B \leq A$ and $B \leq C$ are in $\mathcal{C}_\mu$ and minimal, and the free amalgam M' is not in $\mathcal{C}_\mu$. Then both extensions are prealgebraic, and there is $\varphi \in \mathcal{S}$ and a difference sequence $(\bar e_0, \ldots, \bar e_{\mu(\varphi)}) \subset M'$ for φ in M' with canonical parameter $\bar b$.

If (after derivation) $\bar b \notin \mathrm{acl}(A) \cup \mathrm{acl}(C)$ we obtain a pairwise independent triple, a contradiction.

So wlog $\bar b \in A$, $\bar e_i \in A$ for $i < \mu(\varphi)$, $\bar e_{\mu(\varphi)}$ is A-generic and $M' = \langle A \bar e_{\mu(\varphi)} \rangle$; write $\bar e_{\mu(\varphi)} = \bar a \cdot \bar c$ for some $\bar a \in A$ and $\bar c \in C$.

Suppose (after derivation) $\bar b \in C$. As $\bar e_{\mu(\varphi)} \notin C$ would imply $\bar a, \bar c, \bar e_{\mu(\varphi)}$ pairwise B-independent, we have $C = \langle B \bar e_{\mu(\varphi)} \rangle$; $C \in \mathcal{C}_\mu$ yields $\bar e_i \in A \setminus B$, and $\bar e_i \mapsto \bar e_{\mu(\varphi)}$ induces $A \cong_B C$.

Ow there is $\bar e_i \mathop{\smile\!\!\!\!\!\mid}_{B\bar b} \bar a$. Then $\bar e_{\mu(\varphi)}$ and $\bar e_i$ have the same type over $B\bar b \bar a$, as do $\bar c = \bar e_{\mu(\varphi)} \cdot \bar a^{-1}$ and $\bar e_i \cdot \bar a^{-1}$. So $\bar c \mapsto \bar e_i \cdot \bar a^{-1}$ is the required isomorphism.

Axiomatization

When we want to axiomatize richness for the Fraïssé-Hrushovski limit $\mathfrak{M}_\mu$, we have to say that for all $\bar{a} \in A \leq \mathfrak{M}$, a code instance $\varphi(\bar{x}, \bar{a})$ has an A-generic realization in $\mathfrak{M}_\mu$, unless for a generic realization $\bar{b}$ we would have $\langle A\bar{b} \rangle \notin \mathcal{C}_\mu$.

The weak CIT allows us to limit the possible $\mathbb{Q}$-linear dependencies we have to consider, with an extra twist: We may first have to extend by finitely many green generic points.

It follows that $\aleph_0$-saturated models of $T_\mu = \mathrm{Th}(\mathfrak{M}_\mu)$ are rich, $\mathfrak{M}_\mu$ has Morley rank 2, and $\ddot{U}(\mathfrak{M}_\mu)$ has Morley rank 1.

Model-completeness

We show $M \leq N$ for any two models $M \subseteq N$ of T_μ; since then $\mathrm{cl}_M = \mathrm{cl}_N$, homogeneity for closed sets yields $M \prec N$.

Claim. If $M \models T_\mu$ and $M \subseteq N \in \mathcal{C}_\mu$, then $M \leq N$.

If not, assume $\mathrm{lin.dim}_{\mathbb{Q}}(N/M) = d$ is minimal. Then $d \geq 2$, as $M = \mathrm{acl}(M)$. Choose $M \subset N' \subset N$ with $\mathrm{lin.dim}_{\mathbb{Q}}(N'/M) = d - 1$. By minimality $M \leq N'$. Now

$$-1 \geq \delta(N/M) = \delta(N/N') + \delta(N'/M),$$

and $\delta(N/N') \geq -1$ implies that $\delta(N'/M) \leq 0$. So N'/M is prealgebraic.

Hence there is $M < N'' \leq N'$ with N''/M minimal prealgebraic. But there are only finitely many extensions of this type, which must all lie already in M, a contradiction.

An alternative axiomatization

Universal axioms

- Finitely generated subfields are in $\mathcal{C}_\mu$.

Inductive axioms

- ACF_0.
- The extension of the model generated by a green generic realization of some code instance $\varphi(\bar{x}, \bar{b})$ is not in $\mathcal{C}_\mu$.

Since any complete theory of fields of finite Morley rank is $\aleph_1$-categorical, Lindström's theorem implies that $\mathrm{Th}(\mathfrak{M}_\mu)$ is model-complete.

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