Ample questions and simple answers

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Plan

Introduction

Definition

ă'mple *a.*
spacious; extensive; abundant, copious; (euphem.) stout; quite enough.

(The Concise Oxford Dictionary, 1982)

Introduction

Definition

sĭ'mple *a. & n.*

1. *a.* not compound, consisting of one element, all of one kind, involving only one operation or power, not divided into parts, not analysable.

. . .

4. not complicated or elaborate or adorned or involved or highly developed.

5. absolute, unqualified, mere, neither more nor less than.

6. plain in appearance or manner, unsophisticated, ingenuous, artless.

7. foolish, ignorant, inexperienced; feeble-minded.

8. easily understood or done, presenting no difficulty.

9. of low rank, humble, insignificant, trifling.

The set-up

Throughout this talk, we shall be working in the monster model of a simple theory T. All tuples and parameters will be hyperimaginary, i.e. classes of countable tuples modulo type-definable equivalence relations over $\emptyset$. We denote the definable closure of a set A by $\mathrm{dcl}(A)$, and the bounded closure by $\mathrm{bdd}(A)$.
If you prefer, you can work in a stable theory and replace the bounded closure by the imaginary algebraic closure. This will not significantly simplify the proofs, however.

One-basedness

Definition

A simple theory T is *one-based* if for all A and B

$$A \underset{\mathrm{bdd}(A)\cap\mathrm{bdd}(B)}{\smile\!\!\!\!\!\!|} B.$$

In other words, $\mathrm{Cb}(A/B) \subseteq \mathrm{bdd}(A)$.
Hrushosvki and Pillay have shown that one-based stable groups are abelian-by-finite, and definable subsets of G^n are boolean combinations of cosets of almost $\emptyset$-definable subgroups.
In the simple case we have to allow for random predicates: A group in a simple theory is one-based iff every n-type is generic for some coset of an almost $\emptyset$-definable subgroup of G^n.

CM-triviality

Definition

A simple theory T is *CM-trivial* if for all boundedly closed $A \subset B$ and all c, whenever $\mathrm{bdd}(Ac) \cap B = A$, then $\mathrm{Cb}(c/A) \subseteq \mathrm{bdd}(\mathrm{Cb}(c/B))$.

Pillay has shown that a CM-trivial group of finite Morley rank is nilpotent-by-finite. In fact, the conclusion holds for groups in stable theories with enough regular types (where every type is non-orthogonal to a regular type).

Ampleness

Pillay has proposed a hierarchy for the complexity of forking.

Definition

T is *n-ample* if there are A and tuples $a_0, \ldots, a_n$ such that

1. $a_n \not\perp_A a_0$.
2. $a_{i+1} \perp_{Aa_i} a_0 \ldots a_{i-1}$ for $1 \leq i < n$.
3. For all $0 \leq i < n$

$$\mathrm{bdd}(Aa_0 \ldots a_{i-1} a_i) \cap \mathrm{bdd}(Aa_0 \ldots a_{i-1} a_{i+1}) = \mathrm{bdd}(Aa_0 \ldots a_{i-1}).$$

- $(n+1)$-ample implies n-ample.
- T is one-based iff it is not 1-ample.
- T is CM-trivial iff it is not 2-ample.
- An infinite field is n-ample for all $n < \omega$.
- Pillay in fact defines ampleness locally for a type.

Internality and analysability

The definitions so far use the bounded closure, which is appropriate for theories of finite rank. However, in infinite rank, or when no rank is available, other closure operators may be more relevant.

Let Σ be an $\emptyset$-invariant family of partial types.

Definition

Let π be a partial type over A. Then π is

- *(almost) Σ-internal* if for every realization a of π there is $B \mathop{\smile\!\!\!\!\!\!\mid}_A a$ and $\bar b$ realizing types in Σ based on B, such that $a \in \mathrm{dcl}(B\bar b)$ (or $a \in \mathrm{bdd}(B\bar b)$, respectively).
- *Σ-analysable* if for any realization a of π there are $(a_i : i < \alpha) \in \mathrm{dcl}(A, a)$ such that $\mathrm{tp}(a_i/A, a_j : j < i)$ is Σ-internal for all $i < \alpha$, and $a \in \mathrm{bdd}(A, a_i : i < \alpha)$.

Σ-closure

Definition

The Σ-*closure* $\Sigma\text{cl}(A)$ of a set A is the collection of all hyperimaginaries a such that $\text{tp}(a/A)$ is Σ-analysable.

We think of Σ as small. We always have $\text{bdd}(A) \subseteq \Sigma\text{cl}(A)$; equality holds if Σ is the family of all bounded types.
Other choices for Σ are the family of all types of *SU*-rank $< \omega^\alpha$ for some ordinal α, the family of all supersimple types in a properly simple theory, or the family of *p*-simple types of *p*-weight 0 for some regular type *p*, giving rise to Hrushovski's *p*-closure.
Buechler and Hoover use such a general closure operator in order to analyze types of rank ω, and prove Vaught's conjecture for a special class of superstable groups of rank ω.

Properties of Σ-closure

Theorem

The following are equivalent:

1. $tp(a/A)$ is foreign to Σ.
2. $a \underset{A}{\perp} \Sigma cl(A)$.
3. $a \underset{A}{\perp} dcl(aA) \cap \Sigma cl(A)$.
4. $dcl(aA) \cap \Sigma cl(A) \subseteq bdd(A)$.

Unless it equals bounded closure, Σ-closure has the size of the monster model and thus violates the usual conventions. The equivalence (2) $\Leftrightarrow$ (3) can be used to cut it down to some small part.

Properties of Σ-closure

Theorem

1. *Suppose $A \mathop{\smile\!\!\!\!\!\!|}_B C$. Then*

$$\Sigma cl(A) \mathop{\smile\!\!\!\!\!\!|}_{\Sigma cl(B)} \Sigma cl(C).$$

In particular,

$$\Sigma cl(AB) \cap \Sigma cl(BC) = \Sigma cl(B).$$

2. *If $\Sigma cl(C) = \Sigma cl(A) \cap \Sigma cl(B)$ and $D \mathop{\smile\!\!\!\!\!\!|}_C AB$, then*

$$\Sigma cl(AD) \cap \Sigma cl(BD) = \Sigma cl(CD).$$

Σ-ampleness

Let Φ and Σ be $\emptyset$-invariant families of partial types.

Definition

Φ is *n-Σ-ample* if there are tuples $a_0, \ldots, a_n$, with a_n a tuple of realizations of partial types in Φ over some A, such that

1. $a_n \not\perp_{\Sigma\mathrm{cl}(A)} a_0$.
2. $a_{i+1} \perp_{\Sigma\mathrm{cl}(Aa_i)} a_0 \ldots a_{i-1}$ for $1 \leq i < n$.
3. For all $0 \leq i < n$

$$\Sigma\mathrm{cl}(Aa_0 \ldots a_{i-1} a_i) \cap \Sigma\mathrm{cl}(Aa_0 \ldots a_{i-1} a_{i+1}) = \Sigma\mathrm{cl}(Aa_0 \ldots a_{i-1}).$$

One may require $a_0, \ldots, a_{n-1}$ to lie in Φ^{heq}.
If $a_0, \ldots, a_n$ witness n-Σ-ampleness over A, then $a_i, \ldots, a_n$ witness $(n-i)$-Σ-ampleness over $Aa_0 \ldots a_{i-1}$. Thus n-Σ-ample implies i-Σ-ample for all $i \leq n$.

Alternative definitions

For $n = 1$ and $n = 2$ there are alternative definitions:

Definition

1. Φ is Σ-*based* if for any tuple a of realizations of partial types in Φ over some A and any $B \supseteq A$
$$\mathrm{Cb}(a/\Sigma\mathrm{cl}(B)) \subseteq \Sigma\mathrm{cl}(aA).$$
2. Φ is Σ-*CM-trivial* if for any tuple a of realizations of partial types in Φ over some A and any $B \subseteq C$ with $\Sigma\mathrm{cl}(ABa) \cap \Sigma\mathrm{cl}(AC) = \Sigma\mathrm{cl}(AB)$
$$\mathrm{Cb}(a/\Sigma\mathrm{cl}(AB)) \subseteq \Sigma\mathrm{cl}(A, \mathrm{Cb}(a/\Sigma\mathrm{cl}(AC))).$$

1. Φ is Σ-based if and only if Φ is not 1-Σ-ample.
2. Φ is Σ-CM-trivial if and only if Φ is not 2-Σ-ample.

Closure properties of ampleness

Lemma

1. *If Φ is not n-Σ-ample, neither is $tp(b/A)$ for any $b \in \Sigma cl(aA)$, where a is a tuple of realizations of partial types in Φ over A.*
2. *If $B \underset{A}{\downarrow} a_0 \ldots a_n$ and $a_0, \ldots, a_n$ witness n-Σ-ampleness over A, they do so over B.*
3. *For $i < \alpha$ let Φ_i be an $\emptyset$-invariant family of partial types. If Φ_i is not n-Σ-ample for all $i < \alpha$, neither is $\bigcup_{i<\alpha} \Phi_i$.*
4. *If $a \downarrow A$ and $tp(a/A)$ is not n-Σ-ample, neither is $tp(a)$.*
5. *Let Ψ be an $\emptyset$-invariant family of types. If Ψ is Φ-internal and Φ is not n-Σ-ample, neither is Ψ.*

Closure properties of ampleness

Theorem (Ample Analysability)

Let Ψ be an $\emptyset$-invariant family of types. If Ψ is Φ-analysable and Φ is not n-Σ-ample, neither is Ψ.

This was shown by Pillay for superstable theories of (finite) Lascar rank (with algebraic closure).
For $n = 1$ (one-basedness), there were partial results by Buechler, Hrushovski and Chatzidakis, and a general proof by myself. The difficult part was to establish the result for analyses in two steps: If $\text{tp}(a)$ and $\text{tp}(b/a)$ are one-based, so is $\text{tp}(ab)$.
Using an appropriate theory of levels, this is in fact easy.
The main part of the proof is to show closure under unions.

Levels

In his proof of Vaught's conjecture for superstable theories of finite rank, Buechler defines the first level $\ell_1(a)$ of an element a of finite Lascar rank as the set of all $b \in \text{acl}^{eq}(a)$ internal in the family of all types of Lascar rank one; higher levels are defined inductively by $\ell_{n+1}(a) = \ell_1(a/\ell_n(a))$.
The notion has been studied by Prerna Bihani Juhlin in her thesis in connection with a reformulation of the canonical base property.
We shall generalise the notion to arbitrary simple theories.

Definition

The *first* Φ*-level of a over A* is given by

$$\ell_1^\Phi(a/A) = \{b \in \text{bdd}(aA) : \text{tp}(b/A) \text{ is } \Phi\text{-internal}\}.$$

Inductively, $\ell_{n+1}^\Phi(a/A) = \ell_1^\Phi(a/\ell_n^\Phi(a/A))$.

Domination-equivalence

Theorem

Suppose tp(a/A) is Φ-analysable. Then a and $\ell_1^\Phi(a/A)$ are domination-equivalent over A.

Proof.

Since $\ell_1^\Phi(a) \in \mathrm{bdd}(Aa)$, clearly a dominates $\ell_1^\Phi(a)$ over A. For the converse, suppose $b \not\perp_A a$. We have to show $b \not\perp_A \ell_1^\Phi(a)$.

Let $b' = \mathrm{Cb}(a/Ab)$. Then $\mathrm{tp}(b'/A)$ is $\mathrm{tp}(a/A)$-internal, and hence Φ-analysable. So there is a sequence $(b_i : i < \alpha)$ in $\mathrm{bdd}(Ab')$ such that $\mathrm{tp}(b_i/A, b_j : j < i)$ is Φ-internal for all $i < \alpha$ and $b' \in \mathrm{bdd}(A, b_i : i < \alpha)$.

Since $a \not\perp_A b'$ there is minimal $i < \alpha$ such that $a \not\perp_{A,(b_j:j<i)} b_i$.

Proof (continued).

Put $a' = \mathrm{Cb}(b_j : j \leq i/Aa)$, and let $(b_j^k : j \leq i, k < \omega)$ be a Morley sequence in $\mathrm{tp}(b_j : j \leq i/Aa)$. Then

$$a' \in \mathrm{dcl}(b_j^k : j \leq i, k < \omega).$$

As $a' \mathop{\smile\!\!\!\!\!\mid}_A (b_j : j < i)$ by minimality of i we have

$$a' \mathop{\smile\!\!\!\!\!\mid}_A (b_j^k : j < i, k < \omega).$$

Now $\mathrm{tp}(b_i^k/A, b_j^k : j < i)$ is Φ-internal by $\emptyset$-invariance of Φ, so $\mathrm{tp}(a'/A)$ is Φ-internal, and $a' \subseteq \ell_1^\Phi(a)$.
Clearly $a' \not\mathop{\smile\!\!\!\!\!\mid}_A (b_j : j \leq i)$, whence $a' \not\mathop{\smile\!\!\!\!\!\mid}_A b$ and finally $\ell_1^\Phi(a) \not\mathop{\smile\!\!\!\!\!\mid}_A b$. □

Minimal Levels

If tp(a/A) is Φ_0-analysable and Φ_1 is a subfamily of Φ_0 such that tp(a/A) remains Φ_1-analysable, then

$$\ell_1^{\Phi_1}(a) \subseteq \ell_1^{\Phi_0}(a) \subseteq \mathrm{bdd}(aA)$$

and $\ell_1^{\Phi_1}(a)$ et $\ell_1^{\Phi_0}(a)$ are both domination-equivalent to a over A. In fact it would be sufficient to have Φ_1 such that tp$(\ell_1^{\Phi_0}(a)/A)$ is Φ_1-analysable.

Question: When is there a minimal (boundedly closed) $a_0 \in \mathrm{bdd}(aA)$ domination-equivalent with a over A?

If T has finite SU-rank, one can take $a_0 \in \mathrm{bdd}(aA) \setminus \mathrm{bdd}(A)$ with $SU(a_0/A)$ minimal possible.

Flatness

Definition

The type $\mathrm{tp}(a/A)$ is Φ-*flat* if $\ell_1^\Phi(a/A) = \mathrm{bdd}(aA)$. It is *flat* if it is Φ-flat for all Φ it is analysable in. T is *flat* if all its types are.

- Generic types of simple fields or definably simple groups in a simple theory are flat.
- Minimal $a_0 \in \mathrm{bdd}(aA)$ domination-equivalent with a over A are flat.
- In a small simple theory there are many flat types over finite sets, as the lattice of boundedly closed subsets of $\mathrm{bdd}(aA)$ is scattered for finitary aA.

Question: Is every (finitary) type in such a theory non-orthogonal to a flat type?

Proof of Ample Analysability

Theorem (Ample Analysability)

If Ψ is Φ-analysable and Φ is not n-Σ-ample, neither is Ψ.

Let $a_0, \ldots, a_n$ witness n-Σ-ampleness over A, with $\mathrm{tp}(a_n/A)$ Φ-analysable. This means:

1. $a_n \not\perp_{\Sigma\mathrm{cl}(A)} a_0$.
2. $a_{i+1} \perp_{\Sigma\mathrm{cl}(Aa_i)} a_0 \ldots a_{i-1}$ for $1 \le i < n$.
3. For all $0 \le i < n$

$\Sigma\mathrm{cl}(Aa_0 \ldots a_{i-1} a_i) \cap \Sigma\mathrm{cl}(Aa_0 \ldots a_{i-1} a_{i+1}) = \Sigma\mathrm{cl}(Aa_0 \ldots a_{i-1})$.

Put $a_n' = \ell_1^{\Phi}(a/\Sigma\mathrm{cl}(A)) \subseteq \Sigma\mathrm{cl}(Aa_n)$.
Easily, (2) and (3) hold with a_n' instead of a_n.
Domination-equivalence yields $a_n' \not\perp_{\Sigma\mathrm{cl}(A)} a_0$.
As $\mathrm{tp}(a_n'/\Sigma\mathrm{cl}(A))$ is Φ-internal, we are done by the Lemma.

Strong Σ-basedness

We can define a strengthening of Σ-basedness.

Definition

Φ is *strongly Σ-based* if for any tuple a of realizations of partial types in Φ over some A and any $B \supseteq A$

$$\mathrm{Cb}(a/B) \subseteq \Sigma\mathrm{cl}(aA).$$

Similarly, one can define:

Definition

Φ is *strongly Σ-CM-trivial* if for any tuple a of realizations of partial types in Φ over some A and any $B \subseteq C$ with $\Sigma\mathrm{cl}(ABa) \cap \Sigma\mathrm{cl}(AC) = \Sigma\mathrm{cl}(AB)$

$$\mathrm{Cb}(a/AB) \subseteq \Sigma\mathrm{cl}(A, \mathrm{Cb}(a/\Sigma\mathrm{cl}(AC))).$$

Is this really stronger ?

It is easy to see that

$$\mathrm{Cb}(a/\Sigma\mathrm{cl}(B)) \subseteq \Sigma\mathrm{cl}(\mathrm{Cb}(a/B)).$$

Conjecture

$$\mathrm{Cb}(a/B) \subseteq \Sigma\mathrm{cl}(\mathrm{Cb}(a/\Sigma\mathrm{cl}(B))).$$

If this were true, strong and normal Σ-basedness and Σ-CM-triviality would obviously coincide.

Weak Ampleness

Definition

Φ is *weakly n-Σ-ample* if there are tuples $a_0, \ldots, a_n$, where a_n is a tuple of realizations of partial types in Φ over A, with

1. $a_n \not\perp_A a_0$.
2. $a_{i+1} \perp_{\Sigma\text{cl}(Aa_i)} a_0 \ldots a_{i-1}$ for $1 \leq i < n$.
3. $\text{bdd}(Aa_0) \cap \Sigma\text{cl}(Aa_1) = \text{bdd}(A)$.
4. For all $1 \leq i < n$

$\Sigma\text{cl}(Aa_0 \ldots a_{i-1} a_i) \cap \Sigma\text{cl}(Aa_0 \ldots a_{i-1} a_{i+1}) = \Sigma\text{cl}(Aa_0 \ldots a_{i-1})$.

Note that (3) implies that $\text{tp}(a_0/A)$ is foreign to Σ.

1. Φ is strongly Σ-based iff Φ is not weakly 1-Σ-ample.
2. Φ is strongly Σ-CM-trivial iff Φ is not weakly 2-Σ-ample.

Weakly Ample Aanalysability

Theorem (Weakly Ample Analysability)

Let Ψ be an $\emptyset$-invariant family of types. If Ψ is Φ-analysable and Φ is not weakly n-Σ-ample, neither is Ψ.

Let now Σ be the family of non-one-based regular types.

Corollary

Suppose every type in T is non-orthogonal to a regular type. Then T is strongly Σ-based, i.e. $tp(Cb(a/b)/a)$ is Σ-analysable for all a, b.

Proof.

A one-based type is clearly Σ-based. So all regular types are Σ-based. But every type is analysable by regular types by the non-orthogonality hypothesis. $\square$

The Canonical Base Property

The Corollary above is due to Zoé Chatzidakis for types of finite *SU*-rank in simple theories. In fact, she even obtains $\text{tp}(\text{Cb}(a/b)/\text{bdd}(a) \cap \text{bdd}(b))$ to be Σ-analysable.
However, for the applications one would like (and has) more:

Definition (Canonical Base Property)

T has the *Canonical Base Property CBP* if $\text{tp}(\text{Cb}(a/b)/a)$ is almost Σ-internal for all a, b.

It had been conjectured that all supersimple theories of finite rank have the CBP, but there is a probable counter-example due to Hrushovski.
Chatzidakis has shown that the CBP implies that even $\text{tp}(\text{Cb}(a/b)/\text{bdd}(a) \cap \text{bdd}(b))$ is almost Σ-internal.

Applications

Theorem (Kowalski, Pillay)

Let G be a hyperdefinable group in a simple theory.

1. *If $g \in G$ and $H = Stab(g)$, then $tp(gH)$ is Σ-analysable.*
2. *If $H \leq G$ is locally connected with canonical parameter c, then $tp(c)$ is Σ-analysable.*
3. *$G/Z(G)$ is Σ-analysable.*

If G has the CBP, we can replace analysable *by* almost internal.

The results are particularly useful when we have a good control of Σ, for instance when the Zilber trichotomy holds.

The CBP holds for types of finite rank in

- Differential fields (Pillay, Ziegler).
- Difference fields (Pillay, Ziegler; Chatzidakis).
- Compact complex spaces (Moosa, Pillay).

Final Remarks

We have seen that for (weak) Σ-ampleness only the first level of an element is important. However, the difference between strong Σ-basedness and the CBP is precisely the possible existence of a second (or higher) Σ-level of $\mathrm{Cb}(a/b)$ over a, i.e. its non-Σ-flatness.
A possible approach to the CBP could be to replace the Σ-closure by its first Σ-level (over the appropriate parameters) and attempt to prove a corresponding version of the Ample Analysability Theorem. However, the current proof uses the fact the Σcl is a closure operator, and so far we have not found a way around this.
Finally, it might be interesting to look for a variant of ampleness which does take all levels into account, as one might hope to obtain stronger structural consequences.

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Thank You