Reducts and Reducibility

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9 May 2013

Plan

Introduction

Often, structures can be seen as expansions of simpler structures. For instance, both differentially closed fields and generic difference fields have a common reduct, the underlying algebraically closed field. A different example is given by the fusion of two strongly minimal sets, or more generally sets of finite Morley rank with the definable multiplicity property, via Hrushovski's amalgamation construction: The resulting structure has two reducts to the initial sets (and possibly a third one to the common sublanguage of the two).

In this talk I shall survey joint work with Thomas Blossier and Amador Martín Pizarro about type-definable groups in simple expansions of stable theories.

Preliminaries

We consider a simple $\mathcal{L}$-theory T with a stable reduct T_0 to a sub-language $\mathcal{L}_0$, or even a family $(T_i : i < n)$ of stable reducts to sublanguages $(\mathcal{L}_i : i < n)$. Model-theoretic notions refer to T, unless indicated otherwise.

(Type-)definable means with parameters and of finite arity; if we consider imaginary elements or tuples, we talk about *(type-)interpretable*. Infinite tuples are indicated by $*$-.

If E is an equivalence relation which is not i-definable, then classes modulo E make no sense in T_i. Thus we mostly work with real elements. In particular, algebraic closure acl and definable closure dcl are taken among real elements.

We assume that all T_i have geometric elimination of imaginaries, i.e. every imaginary element is interalgebraic with a real tuple. Note that if we work with a single reduct, we can simply add 0-imaginary sorts to the language.

Independence and Reducts

Lemma

If B is algebraically closed and $a \mathop{\smile\!\!\!\!\!\mid}_B c$, then $a \mathop{\smile\!\!\!\!\!\mid}^0_B c$.

Proof.

Let $\varphi(x,y)$ be an $\mathcal{L}_0(B)$-formula satisfied by (a,c), and consider a Morley sequence $(c_j : j < \omega 2)$ in $\mathrm{tp}(c/B)$. Then $\bigwedge_i \varphi(x,c_i)$ is consistent, and $(c_j : \omega \le j < \omega 2)$ is 0-Morley over $(B, c_j : j < \omega)$. Thus

$$\mathrm{Cb}_0(c_\omega / Bc_j : j < \omega) \subseteq \mathrm{acl}_0^{eq}(Bc_j : j < \omega) \cap \mathrm{acl}_0^{eq}(c_j : \omega \le j < \omega 2).$$

But $(c_j : j < \omega) \mathop{\smile\!\!\!\!\!\mid}_B (c_j : \omega \le j < \omega 2)$ and weak EI imply

$$\mathrm{acl}_0^{eq}(Bc_j : j < \omega) \cap \mathrm{acl}_0^{eq}(c_j : \omega \le j < \omega 2) \subseteq \mathrm{acl}_0^{eq}(B).$$

Hence $c_\omega \mathop{\smile\!\!\!\!\!\mid}^0_B (c_j : j < \omega)$, and $(c_j : j < \omega 2)$ is 0-independent over B. Thus $a \mathop{\smile\!\!\!\!\!\mid}^0_B c$. □

A general reduction theorem

Theorem

Let G be an emptyset-type-definable group. Then (after adjunction of parameters) there is a type-definable subgroup G_0 of bounded index in G, a T_0-$*$-interpretable group H and a definable homomorphism $\phi : G_0 \to H$ such that for independent generic elements g, g' of G_0,

$$\operatorname{acl}(g), \operatorname{acl}(g') \underset{\phi(gg')}{\downarrow^0} \operatorname{acl}(gg').$$

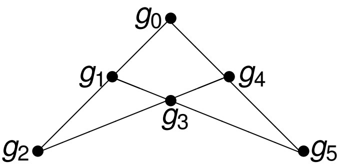
Moreover, $\ker(\phi)$ *is $\emptyset$-type-definable.*

Note that without further assumptions on the reducts, H may be trivial (for instance if there are no T_0-definable groups).

Proof

It is easy to see that up to commensurability, there is a minimal type-definable normal subgroup N such that for some G_0 of bounded index in G, the quotient G_0/N embeds into a T_0-$*$-interpretable group. Moreover, N can be taken $\emptyset$-invariant.

Let g_0, g_1 and g_4 be independent generic elements of G_0. Put $g_2 = g_0g_1$, $g_5 = g_4g_0$ and $g_3 = g_4g_0g_1$. Then the 6-tuple $(g_0, g_1, g_2, g_3, g_4, g_5)$ is an algebraic quadrangle.



We shall modify the points as to render it 0-algebraic.

Proof

There is a countable set D of independent generic elements of G_0 over g_0, g_1, g_3, such that for every collinear triplet (g_i, g_j, g_k) with $0 \leq i, j, k \leq 5$ the intersection

$$\alpha_i = \alpha_i(j,k) = \mathrm{acl}(g_i, D) \cap \mathrm{acl}_0(\mathrm{acl}(g_j, D), \mathrm{acl}(g_k, D))$$

does not depend on the choice of j, k. Moreover,

$$\mathrm{acl}(g_j, D), \mathrm{acl}(g_k, D) \mathop{\smile\!\!\!\!\!\!\,|}^{0}_{\alpha_i} \mathrm{acl}(g_i, D).$$

In fact, D is chosen inductively such that for every finite tuple $\bar{d}$ de D, any aligned couple (g_i, g_j) and non-collinear g_ℓ :

- D contains a Morley sequence in $\mathrm{tp}(g_\ell/\mathrm{acl}(g_i, g_j, \bar{d}))$.
- Both $g_i D$ and $g_i^{-1} D$ contain one in $\mathrm{tp}(g_i/\mathrm{acl}(g_j, \bar{d}))$.

If T is stable, D is just a generic Morley sequence.

Then $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ is a 0-algebraic quadrangle over $\mathrm{acl}(D)$, and we finish by the Group Configuration Theorem.$\square$

Relative One-basedness

In the sequel, we suppose that T is endowed with a finitary invariant closure operator $\langle . \rangle$ satisfying $A \subseteq \langle A \rangle \subseteq \mathrm{acl}(A)$.

Definition

T is *one-based over* $(T_i : i < n)$ *for* $\langle . \rangle$ if for all real algebraically closed $A \subseteq B$ and real tuple $\bar{c}$, whenever

$$\langle A\bar{c} \rangle \mathop{\smile\!\!\!\!\!\!\;|}^{i}_{A} B \text{ for all } i < n,$$

then $\mathrm{Cb}(\bar{c}/B)$ is bounded over A.

Every theory is one-based over itself for acl. If T is one-based over its reduct to equality for acl, then T is one-based; the converse holds if T has geometric elimination of (hyper-)imaginaries.

Properties of relative one-basedness

We shall need two conditions on the relation between the closure operator $\langle . \rangle$ and algebraic closure in the reducts:

(†) If A is algebraically closed and $b \mathop{\smile\hskip-0.9em\raise0.2ex\hbox{$|$}}_A c$, then $\langle Abc \rangle \subseteq \bigcap_{i<n} \mathrm{acl}_i(\langle Ab \rangle, \langle Ac \rangle)$.

(‡) If $\bar{a} \in \bigcup_{i<n} \mathrm{acl}_i(A)$, then $\langle \mathrm{acl}(\bar{a}), A \rangle \subseteq \bigcap_{i<n} \mathrm{acl}_i(\mathrm{acl}(\bar{a}), \langle A \rangle)$.

Note that as soon as a group is definable, algebraic closure does not satisfy (†) over the reduct to equality.

Lemma

If $\langle . \rangle$ satisfies (†), then relative one-basedness is preserved under adjunction or suppression of parameters.

Moreover, it is sufficient to verify relative one-basedness for models $A \subseteq B$.

Examples

Examples

- The theory DCF_0 of a differentially closed field of characteristic 0 is superstable, and one-based over the reduct to the theory ACF_0 of an algebraically closed field, for the model-theoretic algebraic closure acl_δ which satisfies (†) and (‡).
- The theory *ACFA* of an existentially closed difference field is supersimple, and one-based over the reduct to the theory *ACF* of an algebraically closed field, for the model-theoretic algebraic closure acl_σ which satisfies (†) and (‡).

Groups in relatively one-based expansions

Relative one-basedness allows us to show that the kernel of the homomorphism $\phi : G \to H$ is finite.

Theorem

Let T be relatively one-based over $(T_i : i < n)$ with respect to a closure operator $\langle . \rangle$ satisfying $(\dagger)$ and $(\ddagger)$.

Then a type-definable group G has a subgroup G_0 of bounded index which embeds modulo a finite kernel into a finite product $H = \prod_{i<n} H_i$ of T_i-interpretable groups H_i.

If G_0 is an intersection of definable groups (e.g. if T is stable or supersimple), we may assume G_0 has finite index in G.

Applied to DCF_0 and $ACFA$, this yields an alternative proof of the characterization of definable groups due to Kowalski and Pillay.

Proof

Consider $\phi = \prod_i \phi_i : G_0 \to H = \prod_i H_i$, and $g \mathop{\smile\!\!\!\!\mid} g'$ generic.

Then $$\mathrm{acl}(g), \mathrm{acl}(g') \mathop{\smile\!\!\!\!\mid}^{i}_{\phi_i(gg')} \mathrm{acl}(gg').$$

($\dagger$) implies $$\langle \mathrm{acl}(g), \mathrm{acl}(g') \rangle \subseteq \bigcap_{i<n} \mathrm{acl}_i(\mathrm{acl}(g), \mathrm{acl}(g')).$$

($\ddagger$) yields, as $$\phi(gg') \in \bigcup_{i<n} \mathrm{acl}_i(\mathrm{acl}(g), \mathrm{acl}(g')),$$

$$\langle \mathrm{acl}(g), \mathrm{acl}(g'), \mathrm{acl}(\phi(gg')) \rangle \subseteq \bigcap_{i<n} \mathrm{acl}_i(\mathrm{acl}(g), \mathrm{acl}(g'), \mathrm{acl}(\phi(gg'))).$$

Since $\mathrm{acl}(\phi(gg')) \subseteq \mathrm{acl}(gg')$, we get for all $i < n$
$$\langle \mathrm{acl}(g), \mathrm{acl}(g'), \mathrm{acl}(\phi(gg')) \rangle \mathop{\smile\!\!\!\!\mid}^{i}_{\mathrm{acl}(\phi(gg'))} \mathrm{acl}(gg').$$

By relative one-basedness, $$\mathrm{acl}(g), \mathrm{acl}(g') \mathop{\smile\!\!\!\!\mid}_{\mathrm{acl}(\phi(gg'))} \mathrm{acl}(gg').$$
Hence $gg' \in \mathrm{acl}(\phi(gg'))$. $\square$

Relative CM-triviality

Definition

T is *CM-trivial over* $(T_i : i < n)$ *for* $\langle . \rangle$ if for all real algebraically closed $A \subseteq B$ and real tuples $\bar{c}$, whenever

$$\langle A\bar{c}\rangle \mathop{\smile\!\!\!\!\!\!|}^{i}_{A} B \text{ for all } i < n,$$

then $\mathrm{Cb}(\bar{c}/A)$ is bounded over $\mathrm{Cb}(\bar{c}/B)$.

Every theory is CM-trivial over itself for acl. If T is CM-trivial over its reduct to equality for acl, then T is CM-trivial; the converse holds if T has geometric elimination.

Every relatively one-based theory is relatively CM-trivial.

If $\langle . \rangle$ satisfies $(\dagger)$, then relative CM-triviality is preserved under adjunction or suppression of parameters. Moreover, it is sufficient to verify relative CM-triviality for models $A \subseteq B$.

Groups in relatively CM-trivial expansions

Theorem

Let T be relatively CM-trivial over $(T_i : i < n)$ with respect to a closure operator $\langle . \rangle$ satisfying $(\dagger)$ and $(\ddagger)$. Then a type-definable group G has a subgroup G_0 of bounded index which embeds modulo an approximately central kernel into a finite product $H = \prod_{i<n} H_i$ of T_i-interpretable groups H_i.

Recall that g is approximately central in G if its centralizer $C_G(g)$ has bounded index in G.

If G_0 is an intersection of definable groups, we can assume it has finite index in G.

Corollary

A simple group in a relatively CM-trivial theory embeds into one definable in one of the reducts.

Proof

Consider $\phi = \prod_i \phi_i : G_0 \to H = \prod_i H_i$, and $g \mathop{\smile\!\!\!\!\!\!\mid}\, g'$ generic.
Put $N_i = \ker(\phi_i)$ and $N = \bigcap_{i<n} N_i = \ker(\phi)$. As before,

$$\langle \mathrm{acl}(D,g), \mathrm{acl}(D,g'), \mathrm{acl}(D,\phi(gg'))\rangle \mathop{\smile\!\!\!\!\!\!\mid}^{i}_{\mathrm{acl}(D,\phi(gg'))} \mathrm{acl}(D,gg').$$

Let $a, b, e \in D$ be distinct, and $D' = D \setminus \{e\}$. We put

$A = \mathrm{acl}(D', \phi(gg'))$, $B = \mathrm{acl}(D', gg')$ and $\bar{c} = (g, g', age, e^{-1}g'b)$.

$$\langle g, g', age, e^{-1}g'b, \mathrm{acl}(D', \phi(gg'))\rangle \mathop{\smile\!\!\!\!\!\!\mid}^{i}_{\mathrm{acl}(D',\phi(gg'),e)} \mathrm{acl}(D', gg').$$

$$e \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{acl}(D',\phi(gg'))} gg' \Rightarrow \mathrm{acl}(D', \phi(gg'), e) \mathop{\smile\!\!\!\!\!\!\mid}^{i}_{\mathrm{acl}(D',\phi(gg'))} \mathrm{acl}(D', gg').$$

$$\langle g, g', age, e^{-1}g'b, \mathrm{acl}(D', \phi(gg'))\rangle \mathop{\smile\!\!\!\!\!\!\mid}^{i}_{\mathrm{acl}(D',\phi(gg'))} \mathrm{acl}(D', gg').$$

That is, $\langle \bar{c}, A\rangle \mathop{\smile\!\!\!\!\!\!\mid}^{i}_{A} B$ for all $i < n$.
By relative CM-triviality, $\mathrm{Cb}(\bar{c}/A) \in \mathrm{bdd}(\mathrm{Cb}(\bar{c}/B))$.

$A = \mathrm{acl}(D', \phi(gg')),\ B = \mathrm{acl}(D', gg'),\ \bar{c} = (g, g', age, e^{-1}g'b).$

Let Z be the approximate centralizer of N in G. It is relatively definable, so aZ is imaginary. It suffices to show:

Lemma

If $|G : Z| = \infty$, then $aZ \in \mathrm{acl}^{eq}(\mathrm{Cb}(\bar{c}/A)) \setminus \mathrm{acl}^{eq}(\mathrm{Cb}(\bar{c}/B))$.

Proof.

Suppose $|G : Z| = \infty$. Then $aZ \notin \mathrm{bdd}(\emptyset)$. Now

$$\bar{c} \mathop{\smile\!\!\!\!|}_{gg', agg'b} D' \quad \Rightarrow \quad B \cap \mathrm{acl}(\bar{c}) = \mathrm{acl}(gg', agg'b).$$

In particular, $\mathrm{Cb}(\bar{c}/B)$ is interalgebraic with $gg', agg'b$.

$$a \mathop{\smile\!\!\!\!|} gg', agg'b \ \Rightarrow\ aZ \mathop{\smile\!\!\!\!|} \mathrm{Cb}(\bar{c}/B) \ \Rightarrow\ aZ \notin \mathrm{bdd}(\mathrm{Cb}(\bar{c}/B)).$$

Let J be a Morley sequence in $\mathrm{lstp}(\bar{c}/A)$, and $a' \models \mathrm{lstp}(a/J)$. Then $a'^{-1}a \in Z$, whence $aZ \in \mathrm{acl}^{eq}(J)$.
Finally, $J \mathop{\smile\!\!\!\!|}_{\mathrm{Cb}(\bar{c}/A)} A$ implies $aZ \in \mathrm{acl}^{eq}(\mathrm{Cb}(\bar{c}/A))$. □

Fields in relatively CM-trivial expansions

Corollary

Let T be relatively CM-trivial over $(T_i : i < n)$ with respect to a closure operator $\langle . \rangle$ satisfying $(\dagger)$ and $(\ddagger)$. Then a type-definable field K is definably isomorphic to a subfield of a field L interpretable in one of the reducts. If T has finite SU-rank, then $K = L$ is algebraically closed.

Proof.

The simple group $PSL_2(K)$ embeds into a T_i-interpretable group. Its Borel subgroup $K^+ \rtimes K^\times$ embeds naturally into a group of the form $L^+ \rtimes L^\times$, where L is a T_i-interpretable field.

If T has finite SU-rank, then $[L : K]$ is of finite degree. Since T_i is stable of finite SU-rank, L is algebraically closed. But K cannot be real closed by simplicity, so $K = L$. □

Condition (‡)

Without the condition (‡), one can show that unless G is isogenous to an abelian group, the homomorphism $\phi : G \to H$ is non-trivial. In particular, the two Corollaries remain true without the hypothesis (‡).

Proof.

If ϕ were trivial, then $\phi_i(gg') \in \mathrm{bdd}(\emptyset)$. Then

$$\mathrm{acl}(D,g), \mathrm{acl}(D,g') \underset{D,\phi_i(gg')}{\overset{i}{\downarrow}} \mathrm{acl}(D,gg')$$

and (†) would imply trivially

$$\langle \mathrm{acl}(D,g), \mathrm{acl}(D,g'), \mathrm{acl}(D,\phi(gg'))\rangle \underset{\mathrm{acl}(D,\phi(gg'))}{\overset{i}{\downarrow}} \mathrm{acl}(D,gg').$$

The remainder of the proof of the Main Theorem does not need (‡) and shows that N is approximately central in G. □

A quick overview of Hrushovski amalgamation

Coloured We expand T_0 by a colour predicate P, and consider the class $\mathcal{K}$ of coloured models of $T_0^{\forall}$.

Fusion We consider two theories T_1 and T_2 with a common reduct T_{com}, and $\mathcal{K}$ denotes the class of models of $T_1^{\forall} \cup T_2^{\forall}$.

For finitely generated B over A we define a *predimension*

- $\delta(B/A) = 2\,\mathrm{tr.deg}(B/A) - \dim_P(P(B)/P(A))$, where $\dim_P(A) = |A|$ for the black field, $\dim_P(A) = \mathrm{lin.dim}_{\mathbb{F}_p}(A)$ for the red field, and $\dim_P(A) = \mathrm{lin.dim}_{\mathbb{Q}}(A)$ for the green field.
- $\delta(B/A) = n_1 RM_1(B/A) + n_2 RM_2(B/A) - n|B \setminus A|$, where $n = n_1 RM(T_1) = n_2 RM(T_2)$, for the fusion over equality of two theories of finite and definable Morley rank, and
- $\delta(B/A) = RM_1(B/A) + RM_2(B/A) - \mathrm{lin.dim}_{\mathbb{F}_p}(B/A)$ for the fusion of two strongly minimal sets over a common $\mathbb{F}_p$-vector space.

A quick overview of Hrushovski amalgamation

Note that the predimension is *submodular*:

$$\delta(A \cup B) \leq \delta(A) + \delta(B) - \delta(A \cap B).$$

We shall suppose that all T_i and T_{com} eliminate quantifiers and that the languages are relational except possibly for a group law used for the negative part of the predimension. If this group is divisible with bounded torsion, we shall also suppose that structures in $\mathcal{K}$ are divisible groups.

The i-diagram of a structure A determines its i-type, and $\bigcup_i \mathrm{diag}_i(A)$, together with the colouring, determines $\delta(A)$.

We consider the subclass $\mathcal{K}_0$ of structures whose finitely generated substructures have non-negative predimension. For $M \in \mathcal{K}_0$ a substructure A is *self-sufficient* in M, denoted $A \leq M$, if $\delta(\bar{a}/A) \geq 0$ for all finite $\bar{a} \in M$.

A quick overview of Hrushovski amalgamation

By submodularity, the intersection of two self-sufficient subsets is again self-sufficient. Hence every subset $A \subseteq M$ is contained in a unique smallest self-sufficient superset, its *self-sufficient closure* $\langle A \rangle_M$. Then $\langle . \rangle$ satisfies (†) and (‡).

Given a class $\mathcal{K}' \subseteq \mathcal{K}_0$ of finitely generated structures with the amalgamation property for self-sufficient embeddings, one constructs by the Fraïssé method a countable structure universal and homogeneous for self-sufficient substructures.

For self-sufficient $A \cap B = C$, independence is characterized as follows:

$$A \downarrow_C B \quad \Leftrightarrow \begin{cases} A \downarrow^i_C B \text{ for all } i\text{, and } AB \text{ is self-sufficient (and} \\ P(AB) = P(A)P(B) \text{ in the coloured case).} \end{cases}$$

Note that the last condition is obvious unless we are in the group case.

Relative CM-triviality of Hrushovski amalgams

Theorem

The coloured fields are CM-trivial for $\langle . \rangle$ over their reduct to the pure algebraically closed field.

The fusions of T_1 and T_2 (over equality or over a common vector space) are CM-trivial for $\langle . \rangle$ over (T_1, T_2).

In Ziegler's fusion of two theories of finite and definable Morley rank, the base theories do not necessarily geometrically eliminate imaginaries. However, the characterization of independence in Hrushovski amalgams implies that nevertheless independence implies *i*-independence.

The theorem applies to the collapsed and non-collapsed constructions.

Proof

Take a model $\bar{a}$, algebraically closed $\bar{b}$ and a tuple $\bar{c}$ with

1. $\mathrm{bdd}(\bar{a}) \cap \mathrm{bdd}(\bar{b}) = \mathrm{dcl}(\emptyset)$ is a model,
2. $\mathrm{acl}(\bar{a}, \bar{b}) \mathop{\smile\!\!\!\!\!\!|}^{i}_{\bar{a}} \langle \bar{c}, \bar{a} \rangle$ for all i, and
3. $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{\bar{b}} \bar{a}\bar{b}$.

Put $D = \langle \bar{c}, \bar{a} \rangle \cap \mathrm{acl}(\bar{c}, \bar{b})$. We have to show that $D \mathop{\smile\!\!\!\!\!\!|} \bar{a}$.

Conditions 2 and 3 imply that

$$\mathrm{Cb}_i(D/\mathrm{acl}(\bar{a}, \bar{b})) \subseteq \bar{a} \cap \bar{b} = \mathrm{acl}(\emptyset) \quad \text{for all } i,$$

whence $D \mathop{\smile\!\!\!\!\!\!|}^{i} \bar{a}$.

We must show that $D\bar{a}$ is self-sufficient, and $P(D\bar{a}) = P(D)P(\bar{a})$.

Condition 2 implies that $\langle \bar{c}, \bar{a} \rangle \cap \mathrm{acl}(\bar{a}, \bar{b}) = \bar{a}$. Hence

$$\langle \bar{c}, \bar{a} \rangle \cap \bar{b} \subseteq \bar{a} \cap \bar{b} = \mathrm{acl}(\emptyset).$$

The non-group case

As the language is relational, $D\bar{a}$ is a substructure. Condition 3 and the characterization of independence yield

$$\langle \mathrm{acl}(\bar{c}, \bar{b}), \mathrm{acl}(\bar{a}, \bar{b})\rangle = \mathrm{acl}(\bar{c}, \bar{b}) \cup \mathrm{acl}(\bar{a}, \bar{b}).$$

Therefore

$$\begin{aligned}\langle \bar{c}, \bar{a}\rangle \cap \langle \mathrm{acl}(\bar{c}, \bar{b}), \mathrm{acl}(\bar{a}, \bar{b})\rangle \\ &= \langle \bar{c}, \bar{a}\rangle \cap (\mathrm{acl}(\bar{c}, \bar{b}) \cup \mathrm{acl}(\bar{a}, \bar{b})) \\ &= (\langle \bar{c}, \bar{a}\rangle \cap \mathrm{acl}(\bar{c}, \bar{b})) \cup (\langle \bar{c}, \bar{a}\rangle \cap \mathrm{acl}(\bar{a}, \bar{b})) \\ &= D \cup \bar{a}.\end{aligned}$$

Since the intersection of two self-sufficient sets is again self-sufficient, we are done.

The group case

Consider, for a contradiction, a finite (coloured) tuple $\bar\gamma \in \langle D, \bar a\rangle$ of minimal length such that

1. $\delta(\bar\gamma / D \cdot \bar a) < 0$, or
2. $\bar\gamma$ is a coloured point of $(D \cdot \bar a) \setminus (P(D) \cdot P(\bar a))$.

As $\bar a \mathop{\smile\!\!\!\!\!\!|}_{\bar b} \bar c$, the group $\mathrm{acl}(\bar a, \bar b) \cdot \mathrm{acl}(\bar c, \bar b)$ is self-sufficient and

$$P(\mathrm{acl}(\bar a, \bar b) \cdot \mathrm{acl}(\bar c, \bar b)) = P(\mathrm{acl}(\bar a, \bar b)) \cdot P(\mathrm{acl}(\bar c, \bar b)).$$

Hence $\langle D, \bar a\rangle \subseteq \mathrm{acl}(\bar a, \bar b) \cdot \mathrm{acl}(\bar c, \bar b)$ and

$$\bar\gamma = \bar\gamma_1 \bar\gamma_2 \text{ with (coloured) } \bar\gamma_1 \in \mathrm{acl}(\bar a, \bar b) \text{ and } \bar\gamma_2 \in \mathrm{acl}(\bar c, \bar b).$$

We shall show that we can choose $\bar\gamma_1 \in \bar a$ and $\bar\gamma_2 \in D$, thus contradicting our assumption and finishing the proof.

The group case

$D \cup \bar\gamma \subseteq \langle D, \bar a\rangle \subseteq \langle \bar c, \bar a\rangle$ implies $D, \bar\gamma \mathop{\smile\!\!\!\!\!\!\,|}^i_{\bar a} \operatorname{acl}(\bar a, \bar b)$ for all i.

$D \cup \bar\gamma_2 \subseteq \operatorname{acl}(\bar c, \bar b)$ implies $D, \bar\gamma_2 \mathop{\smile\!\!\!\!\!\!\,|}^i_{\bar b} \operatorname{acl}(\bar a, \bar b)$ for all i.

Hence $\operatorname{tp}_i(D, \bar\gamma_2/\bar b) \parallel \bar\gamma_1^{-1} \operatorname{tp}_i(D, \bar\gamma/\bar a)$. Define

$$\bar a' E \bar a'' \quad \Leftrightarrow \quad \exists\, \bar\gamma' \bigwedge_i \bar\gamma' \cdot p_i(X, \bar x, \bar a') \parallel p_i(X, \bar x, \bar a'')$$

where $p_i(X, \bar x, \bar a) = \operatorname{tp}_i(D, \bar\gamma/\bar a)$, and γ' (coloured) acts on $\bar x$.

Then E is a type-definable equivalence relation, and $\bar a_E$ is bounded over $\bar b$. Thus $\bar a_E \in \operatorname{bdd}(\bar a) \cap \operatorname{bdd}(\bar b) = \operatorname{dcl}(\emptyset)$.

Let $\operatorname{tp}(\bar a_0/\bar a\bar b\bar c)$ be finitely satisfiable in $\operatorname{dcl}(\emptyset)$, with $\bar a_0 E \bar a$, as witnessed by $\bar\gamma_0$. Predimension calculations yield

$$\bar\gamma_0 \in \operatorname{acl}(\bar a, \bar a_0) \quad \text{and} \quad \bar\gamma_0^{-1}\bar\gamma_1 \in \operatorname{acl}(\bar b, \bar a_0).$$

There is (coloured) $\bar\gamma_0' \in \operatorname{acl}(\bar a) = \bar a$ with $\bar\gamma_0'^{-1}\bar\gamma_1 \in \operatorname{acl}(\bar b) = \bar b$,

$\bar\gamma_0'^{-1}\bar\gamma = \bar\gamma_0'^{-1}\bar\gamma_1\bar\gamma_2 \in \langle D, \bar a\rangle \cap \operatorname{acl}(\bar b, \bar c) \subseteq \langle \bar c, \bar a\rangle \cap \operatorname{acl}(\bar b, \bar c) = D.$ □

Subgroups of algebraic groups

We are thus led to consider subgroups of algebraic groups.

Theorem

In a coloured field a type-definable subgroup of an algebraic group is an extension of the coloured points of an algebraic additive or multiplicative group by an algebraic group. In particular, every type-definable simple group is algebraic.

An algebraic multiplicative group is a torus; an algebraic additive group is given by p-polynomials.

For the last sentence, a type-definable simple group embeds into an algebraic group by relative CM-triviality. But then it must be itself algebraic.

A field of finite Morley rank eliminates imaginaries. Thus an interpretable simple group in a collapsed red field is algebraic.

Proof

Let G be a connected subgroup of an algebraic group, and g, g' two independent generics. Then $\langle a\rangle$ and $\langle b\rangle$ are freeely amalgamated. Hence

$$\langle gg'\rangle \subseteq \langle \mathrm{acl}_0(g,g')\rangle \subseteq \mathrm{acl}_0(\langle g,g'\rangle) = \mathrm{acl}_0(\langle g\rangle\langle g'\rangle) \quad \text{and}$$
$$P(\langle gg'\rangle) \leq P(\mathrm{acl}_0(\langle g,g'\rangle)) = P(\langle g,g'\rangle) = P(\langle g\rangle)P(\langle g'\rangle)).$$

If $\bar r$ is a basis for $P(\langle g\rangle)$ and $\bar s$ a basis for $P(\langle g'\rangle)$, then a basis $\bar t$ for $P(\langle gg'\rangle)$ is a linear combination of $\bar r$ and $\bar s$. Since $\bar r, \bar s, \bar t$ are pairwise independent, we may assume $\bar r + \bar s = \bar t$ (red) or $\bar r \cdot \bar s = \bar t$ (green). In the black case, necessarily $\bar r = \bar s = \bar t = \emptyset$.

It follows that the stabilizer $\mathrm{Stab}(a,r)$ defines an endogeny $\phi : G \to P^{|r|}$; composing by an algebraic map, we may assume it is a homomorphism. The image is a relatively algebraic subgroup, and the kernel is algebraic. □

Groups definable in the green field

Our general results say nothing about abelian groups interpretable in Hrushovski amalgams. In order to characterize all groups interpretable in the bad field, i.e. the collapsed green field, a more detailed analysis is necessary.

Theorem

A group interpretable in the collapsed green field is isogenous to the quotient of a subgroup of an algebraic group by the green points of a central torus.

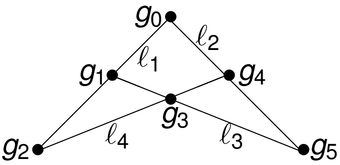
This in particular deals with groups such as $K^\times/P(K^\times)$.

Our method does not apply to the red case, as repeatedly we use the fact that connected algebraic multiplicative groups are tori, and hence parameter-free definable.

In the red field, one would have to deal with additive subgroups defined by p-polynomials.

Proof

Let g_0, g_1 and g_4 be independent generic elements of our group, and put $g_2 = g_0 g_1$, $g_5 = g_4 g_0$ and $g_3 = g_4 g_0 g_1$.

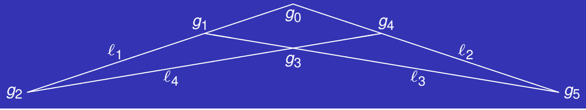


If $\phi : G \to H$ is the maximal homomorphism to an algebraic group, we may replace each g_i by some self-sufficient $\bar{g}_i$ of finite transcendence degree over $\phi(g_i)$, such that:

$g_i, g_i^{-1} \in \mathrm{acl}_0(\bar{g}_i)$ and $\bar{g}_i \in \mathrm{acl}(g_i)$.

$\ell_1 \mathop{\downarrow}^0_{\bar{g}_0} \ell_2$ and $\ell_1 \mathop{\downarrow}^0_{\bar{g}_0, \bar{g}_1} \ell_2, \ell_3$, where $\ell_1 = \mathrm{acl}_0(\langle \bar{g}_0, \bar{g}_1, \bar{g}_2 \rangle)$.

$\ell_1 \cdot \ell_2 \cdot \ell_3$ is self-sufficient, $P(\ell_1 \cdot \ell_2 \cdot \ell_3) = P(\ell_1) \cdot P(\ell_2) \cdot P(\ell_3)$, and this product is direct modulo $P(\bar{g}_0) \cdot P(\bar{g}_1) \cdot P(\bar{g}_5)$.

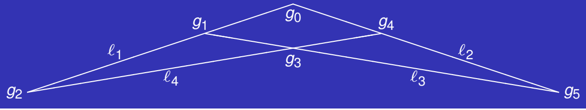


There is a green basis $\bar{t}_1$ of $P(\ell_1)$ over $P(\bar{g}_0)\cdot P(\bar{g}_1)\cdot P(\bar{g}_2)$ such that

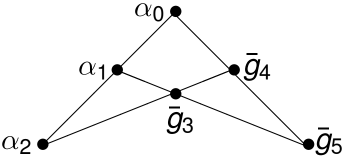
- $\bar{t}_1 \cdot \bar{t}_4 = \bar{t}_2 \cdot \bar{t}_3$,
- $\bar{t}_1 \in \mathrm{acl}_0(\bar{g}_0, \bar{g}_1, \bar{g}_2)$,
- $\bar{t}_1$ is 0-transcendent over $\bar{g}_0, \bar{g}_1$ and over $\bar{g}_0, \bar{g}_2$,
- $\bar{t}_1$ is green generic over $\bar{g}_0$,

and similarly for the other lines. Put

$$\begin{aligned}
\alpha_0 &= \mathrm{acl}_0(\bar{g}_4, \bar{g}_5) \cap \mathrm{acl}_0(\bar{g}_0, \bar{t}_2) \\
\alpha_1 &= \mathrm{acl}_0(\bar{g}_3, \bar{g}_5) \cap \mathrm{acl}_0(\bar{g}_1, \bar{t}_3) \\
\alpha_2 &= \mathrm{acl}_0(\bar{g}_3, \bar{g}_4) \cap \mathrm{acl}_0(\bar{g}_2, \bar{t}_4).
\end{aligned}$$



Then



is a 0-algebraic quadrangle.

By the Group Configuration Theorem there is a 0-algebraic group K and generic $k \in K$ which is 0-interalgebraic with α_0.

Put $S' = \mathrm{Stab}_0(k, \bar{t}_2/\bar{g}_0)$. Then S' is a torus and does not depend on $\bar{g}_0$. Moreover, k and $\bar{t}_2$ are 0-interalgebraic over $\bar{g}_0$, and $(\bar{g}_0, \bar{t}_2)$ is algebraic over k.

Hence S' is an isogeny between $\pi_1(S') \leq K$ and $(K^\times)^{|\bar{t}_2|}$.

Put $\Gamma = S'^{-1}(P(K)^{|\bar{t}_2|})$ and $H = N_K(S'^{-1}((K^\times)^{|\bar{t}_2|}))$.

Then G is isogenous to a subgroup of H/Γ. $\square$

References

Andreas Baudisch, Martin Hils, Amador Martín Pizarro, and Frank O. Wagner, *Die böse Farbe*, J. Inst. Math. Jussieu, 8(3):415–443, 2009.

Thomas Blossier, Amador Martín Pizarro and Frank O. Wagner, *Géométries relatives*, preprint (2012), HAL-00514393.

Thomas Blossier, Amador Martín Pizarro and Frank O. Wagner, *À la recherche du tore perdu*, preprint (2013), HAL-00758982.

Ehud Hrushovski, *Strongly minimal expansions of algebraically closed fields*, Isr. J. Math. 79:129–151, 1992.

Ehud Hrushovski, *A new strongly minimal set*, Ann. Pure Appl. Logic 62:147–166, 1993.

Ehud Hrushovski and Anand Pillay, *Groups definable in local fields and pseudofinite fields*, Isr. J. Math. 85:203–262, 1994.

Ehud Hrushovski and Anand Pillay, *Affine Nash roups over real closed fields*, Confl. Math. 3(4):577–585, 2011.

Piotr Kowalski and Anand Pillay, *A note on groups definable in difference fields*, Proc. Amer. Math. Soc. 130:205–212, 2002.

Anand Pillay, *Some foundational questions concerning differential algebraic groups*, Pac. J. Math. 179:179–200, 1997.

Bruno Poizat, *Le carré de l'égalité*, J. Symb. Logic 64:1338–1355, 1999.

Bruno Poizat, *L'égalité au cube*, J. Symb. Logic 66:1647–1676, 2001.

Thank You