Multinomial convolution polynomials

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Abstract

In [9] Knuth shows how to derive the convolution formulas of Hagen, Rothe and Abel from Vandermonde's convolution or binomial theorem for integer exponents. In the present paper, we shall first present a short and elementary proof of the multi-extension of the above convolution formulas, due to Raney and Mohanty. In the second part we shall present a multi-version of Knuth's approach to convolution polynomials and derive another short proof of the above formulas.

1 Introduction

Recall that a family of polynomials $\{F_n(x)\}_{n\geq 0}$ is said to be of convolution type if $F_n(x)$ has degree $\leq n$ and satisfies the convolution condition :

$$F_n(x+y)=\sum_{k=0}^{n} F_k(x)F_{n-k}(y).$$

The two typical families of convolution polynomials are the binomial theorem for integers

$$\frac{(x+y)^n}{n!}=\sum_{k=0}^{n}\frac{x^k}{k!}\frac{y^{n-k}}{(n-k)!}$$

and Vandermonde's convolution

$$\binom{x+y}{n}=\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}.$$

Some important extensions of the above identities have been given by Abel, Rothe and Hagen in the last centry. More precisely, for $a, b \in \mathbf{C}$ and $n \in \mathbf{N}$ let

$$A_n(a,b) = \frac{a}{a+bn}\binom{a+bn}{n}.$$

In 1891 Hagen [7, 4, 6] proved the following identity :

$$\sum_{k=0}^{n}(p+qk)A_k(a,b)A_{n-k}(c,b) = \frac{p(a+c)+aqn}{a+c}A_n(a+c,b).$$

In the case $q = 0$, this formula reduces to Rothe's convolution formula [17, 5, 16] dated 1793:

$$\sum_{k=0}^{n} A_k(a,b)A_{n-k}(c,b) = A_n(a+c,b).$$

Note that Hagen's formula implies also Abel's identity [1] :

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x(x-kz)^{k-1}(y+kz)^{n-k},$$

where $x,\ y,\ z \in \mathbf{C}$ and $n \in \mathbf{N}$.

In a *refreshing* paper about *convolution polynomials*, Knuth [9] shows how to derive these seemingly non trivial formulas from the basic binomial theorem for integers and Vandermonde's convolution.

Since the multi-extensions of the above identities are also known, it is natural to give a multi-extension of Knuth's results in [9]. To express them more concisely, we shall use the vector notation as follows. Throughout this paper m will be a fixed natural number. For $\mathbf{a} = (a_1, \cdots, a_m) \in \mathbf{N^m}$ and $\mathbf{b} = (b_1, \cdots, b_m) \in \mathbf{N^m}$ set $|\mathbf{a}| = \sum_{i=1}^{m} a_i$, $\mathbf{a}! = a_1! \cdots a_m!$, $\mathbf{a}+\mathbf{b} = (a_1+b_1, \cdots, a_m+b_m)$ and $\mathbf{a}\cdot\mathbf{b} = \sum_{i=1}^{m} a_i b_i$. We order the elements of $\mathbf{N^m}$ by *lexicographic order*, *i.e.*, $\mathbf{a} < \mathbf{a}$ if $a_1 < b_1$ or there is a $i > 1$ such that $a_1 = b_1$, …, $a_{i-1} = b_{i-1}$ but $a_i < b_i$. Also for $1 \le i \le m$ let $\mathbf{e}_i = (\delta_{i1}, \delta_{i2}, \cdots, \delta_{im})$. Finally, for a complex number x and $\mathbf{n} \in \mathbf{Z}^m$, we define

$$\binom{x}{\mathbf{n}} = \begin{cases} x(x-1)\cdots(x-|\mathbf{n}|+\mathbf{1})/\mathbf{n}!, & \text{if } \mathbf{n} = (n_1, \cdots, n_m) \in \mathbf{N^m}; \\ 0, & \text{if some } n_i \text{ is negative.} \end{cases}$$

We now define the multi-analogue of $A_n(a,b)$ by

$$A_{\mathbf{n}}(x,\mathbf{b}) = \frac{x}{x+\mathbf{b}\cdot\mathbf{n}}\binom{x+\mathbf{b}\cdot\mathbf{n}}{\mathbf{n}}.$$

The multi-extension of Rothe's convolution formula can then be stated as follows :

$$\sum_{\mathbf{k}} A_{\mathbf{k}}(a,\mathbf{b}) A_{\mathbf{n}-\mathbf{k}}(c,\mathbf{b}) = A_{\mathbf{n}}(a+c,\mathbf{b}). \tag{1.1}$$

Remark: The author first learned this identity in July 1988 at Oberwolfach, where Louck [11] communicated it as a conjecture. It was shortly proved independently by Strehl, Paule and the author by using different methods and reported at the 20th session of le sminaire Lotharingien de Combinatoire held at Alghero, Italie, in September 1988. Shortly later, we noted that formula (1.1) was already established by Raney [15, Th. 2.2, Th. 2.3] in 1960. We refer the reader to the recent paper of Strehl [18] for a good account of the various aspects of this formula.

Curiously enough Raney's derivation of (1.1) was entirely different from the three formentioned ones. Indeed, Strehl's proof is of combinatorial nature [18], Paule's proof is based on the one variable Lagrange inversion formula [14] and the author's proof is inductive and will be produced below. Note that Chu [2] quoted Raney's identity explicitly in his paper to derive a combinatorial interpretation of the generalized Catalan numbers.

A multi-extension of Hagen's identity has also been given by Mohanty [12], by applying the *multivariable Lagrange inversion formula*, in the following form :

$$\sum_{\mathbf{k}} (p+\mathbf{q}\cdot\mathbf{k}) A_{\mathbf{k}}(a,\mathbf{b}) A_{\mathbf{n}-\mathbf{k}}(c,\mathbf{b}) = \frac{p(a+c)+a(\mathbf{q}\cdot\mathbf{n})}{a+c} A_{\mathbf{n}}(a+c,\mathbf{b}), \tag{1.2}$$

where $\mathbf{q}=(q_1,\cdots,q_m)\in\mathbf{C}^m$ and $a,\ c,\ p\in\mathbf{C}$.

The aim of this paper is two folds. First we present a very short and elementary proof of (1.1) and (1.2) from scratch, which is similar in spirit to Good's short proof of Dyson's conjecture [4]. In the the second part of this paper we shall present a natural multi-extension of Knuth's approach [9] to ordinary convolution polynomials. As a consequence of this generalization we see that (1.2) and (1.1) are respectively equivalent to the following trivial convolution identities :

$$\frac{(x+y)^{|\mathbf{n}|}}{\mathbf{n}!} = \sum_{0\le\mathbf{k}\le\mathbf{n}} \frac{x^{|\mathbf{k}|}}{\mathbf{k}!}\frac{y^{|\mathbf{n}-\mathbf{k}|}}{(\mathbf{n}-\mathbf{k})!} \tag{1.3}$$

and

$$\binom{x+y}{\mathbf{n}} = \sum_{0\le\mathbf{k}\le\mathbf{n}} \binom{x}{\mathbf{k}}\binom{y}{\mathbf{n}-\mathbf{k}}. \tag{1.4}$$

As remarked by Knuth [9], in the special case that each polynomial $F_n(x)$ has degree exactly n, the polynomials $n!F_n(x)$ are said to be of *binomial type* [13]. We note that in the later case Joni [10] has given another generalization of binomial polynomials.

2 A short proof of formula (1.2)

Notice that formula (1.2) is eqivalent to (1.1) and the following identity :

$$\sum_{\mathbf{k}}(\mathbf{q}\cdot\mathbf{k})A_{\mathbf{k}}(a,\mathbf{b})A_{\mathbf{n}-\mathbf{k}}(c,\mathbf{b}) = \frac{a(\mathbf{q}\cdot\mathbf{n})}{a+c}A_{\mathbf{n}}(a+c,\mathbf{b}) \tag{2.1}$$

A further look shows that formula (1.3) is actually a consequence of (2.1). Indeed, exchanging a with c, and $\mathbf{k}$ with $\mathbf{n}-\mathbf{k}$ in (2.1) yields

$$\sum_{\mathbf{k}}(\mathbf{q}\cdot(\mathbf{n}-\mathbf{k}))A_{\mathbf{k}}(a,\mathbf{b})A_{\mathbf{n}-\mathbf{k}}(c,\mathbf{b}) = \frac{c(\mathbf{q}\cdot\mathbf{n})}{a+c}A_{\mathbf{n}}(a+c,\mathbf{b}) \tag{2.2}$$

Summing up the two above identities side by side and taking $\mathbf{q}=\mathbf{n}$, we get (1.1). Therefore it suffices to prove (2.1). However it seems useful to first give an independent proof of (1.1) to illustrate the method. In the sequel we assume a to belong to $\mathbf{N}$. Let

$$S_{\mathbf{n}}(a,c,\mathbf{b}) = \sum_{\mathbf{k}} A_{\mathbf{k}}(a,\mathbf{b})A_{\mathbf{n}-\mathbf{k}}(c,\mathbf{b}).$$

Then, first

$$S_{\mathbf{n}}(0,c,\mathbf{b}) = A_{\mathbf{n}}(c,\mathbf{b}), \qquad S_{\mathbf{0}}(a,c,\mathbf{b}) = 1. \tag{2.3}$$

Next, by the definition of $A_{\mathbf{n}}(a,\mathbf{b})$ we have

$$\begin{aligned} A_{\mathbf{n}}(a,\mathbf{b}) &= \frac{1}{a-1+\mathbf{b}\cdot\mathbf{n}}\left(a-1+\sum_{i=1}^{m}\frac{b_i n_i}{a+\mathbf{b}\cdot\mathbf{n}}\right)\binom{a+\mathbf{b}\cdot\mathbf{n}}{\mathbf{n}} \qquad (2.4)\\ &= \frac{1}{a-1+\mathbf{b}\cdot\mathbf{n}}\left[(a-1)\binom{a+\mathbf{b}\cdot\mathbf{n}}{\mathbf{n}}+\sum_{i=1}^{m} b_i\binom{a-1+\mathbf{b}\cdot\mathbf{n}}{\mathbf{n}-\mathbf{e}_i}\right]\\ &= A_{\mathbf{n}}(a-1,\mathbf{b})+\sum_{i=1}^{m}A_{\mathbf{n}-\mathbf{e_i}}(a+b_i-1,\mathbf{b}). \end{aligned}$$

The last line is due to $\binom{x}{\mathbf{n}} = \binom{x-1}{\mathbf{n}} + \sum_{i=1}^{m} \binom{x-1}{\mathbf{n}-\mathbf{e}_i}$. So that

$$\begin{aligned} S_{\mathbf{n}}(a,c,\mathbf{b}) &= \sum_{\mathbf{k}} \left(A_{\mathbf{k}}(a-1,\mathbf{b}) + \sum_{i=1}^{m} A_{\mathbf{k}-\mathbf{e}_i}(a+b_i-1,\mathbf{b}) \right) A_{\mathbf{n}-\mathbf{k}}(c,\mathbf{b}) \\ &= S_{\mathbf{n}}(a-1,c,\mathbf{b}) + \sum_{i=1}^{m} S_{\mathbf{n}-\mathbf{e}_i}(a+b_i-1,c,\mathbf{b}). \qquad (2.5) \end{aligned}$$

Eqs. (2.3)-(2.5) show that $S_{\mathbf{n}}(a,c,\mathbf{b})$ and $A_{\mathbf{n}}(a,c,\mathbf{b})$ satisfy the same recurrence and the initial conditions. Therefore they are equal.

Now we turn to (2.1). Note that (2.1) is equivalent to the following m identities:

$$\sum_{\mathbf{k}} k_j A_{\mathbf{k}}(a,\mathbf{b}) A_{\mathbf{n}-\mathbf{k}}(c,\mathbf{b}) = \frac{a\,n_j}{a+c} A_{\mathbf{n}}(a+c,\mathbf{b}), \qquad (1 \le j \le m). \qquad (2.6)$$

For $1 \le j \le m$, let $T_{\mathbf{n}}^{(j)}(a,c,\mathbf{b})$ and $S_{\mathbf{n}}^{(j)}(a,c,\mathbf{b})$ be respectively the left-hand and right-hand sides of (2.6). By definition, we first have

$$S_{\mathbf{n}}^{(j)}(a,c,\mathbf{b}) = (a-1)\binom{a-1+c+\mathbf{b}\cdot\mathbf{n}}{\mathbf{n}-\mathbf{e}_j} + \binom{a-1+c+\mathbf{b}\cdot\mathbf{n}}{\mathbf{n}-\mathbf{e}_j}. \qquad (2.7)$$

Now writing the first $\binom{a-1+c+\mathbf{b}\cdot\mathbf{n}}{\mathbf{n}-\mathbf{e}_j}$ as

$$\binom{a-2+c+\mathbf{b}\cdot\mathbf{n}}{\mathbf{n}-\mathbf{e}_j} + \sum_{i=1}^{m} \binom{a-2+c+\mathbf{b}\cdot\mathbf{n}}{\mathbf{n}-\mathbf{e}_i}$$

and the second one as

$$\begin{aligned} &\left(\frac{\mathbf{b}\cdot(\mathbf{n}-\mathbf{e}_j)}{a-1+c+\mathbf{b}\cdot\mathbf{n}} + \frac{a-1+c+b_j}{a-1+c+\mathbf{b}\cdot\mathbf{n}} \right) \binom{a-1+c+\mathbf{b}\cdot\mathbf{n}}{\mathbf{n}-\mathbf{e}_j} \\ &\quad = \sum_{i=1}^{m} b_i \binom{a-2+c+\mathbf{b}\cdot\mathbf{n}}{\mathbf{n}-\mathbf{e}_j-\mathbf{e}_i} + A_{\mathbf{n}-\mathbf{e}_j}(a-1+c+b_j,\mathbf{b}), \end{aligned}$$

we see that

$$\begin{aligned} S_{\mathbf{n}}^{(j)}(a,c,\mathbf{b}) = S_{\mathbf{n}}^{(j)}(a-1,c,\mathbf{b}) &+ \sum_{i=1}^{m} S_{\mathbf{n}-\mathbf{e}_i}^{(j)}(a-1+b_i,c,\mathbf{b}) \qquad (2.8) \\ &+ A_{\mathbf{n}-\mathbf{e}_j}(a-1+c+b_j,\mathbf{b}). \end{aligned}$$

Also we have the boundary conditions :

$$S^{(j)}_{\mathbf{n}}(0,c,\mathbf{b})=0 \qquad \text{and} \qquad S^{(j)}_{\mathbf{0}}(a,c,\mathbf{b})=0. \tag{2.9}$$

On the other hand, by (2.4) we verify easily that $T^{(j)}_{\mathbf{n}}(a,c,\mathbf{b})$ satisfies the same recurrence relation (2.8) and boundary condition (2.9). Thus completes proof of (2.6). $\square$

3 Multi-convolution polynomials

A family of polynomials $\{F_{\mathbf{n}}(x)\}_{\mathbf{n}\geq\mathbf{0}}$ forms a *multinomial convolution family* if $F_{\mathbf{n}}(x)$ has degree $\leq |\mathbf{n}|$ and if the convolution condition

$$F_{\mathbf{n}}(x+y)=\sum_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}} F_{\mathbf{k}}(x)\ F_{\mathbf{n}-\mathbf{k}}(y) \tag{3.1}$$

holds for all x and y and for all $\mathbf{n}\geq\mathbf{0}$. In the case that $m=1$ we recover the monomial *convolution family* studied by Knuth [9]. We should note that this definition is different from the one of "higher dimentional polynomials of binomial type" studied by Joni [10]. Many such families are known, and they appear frequently in applications. For example, we can let $F_{\mathbf{n}}(x)=x^{|\mathbf{n}|}/\mathbf{n}!$, the condition (1.7) is equivalent to the binomial theorem for integer exponents. Or we can let $F_{\mathbf{n}}(x)$ be the multinomial coefficient $\binom{x}{\mathbf{n}}$; the corresponding identity (1.8) may be called multi-Vandermonde's convolution.

Knuth showed that convolution polynomials arise as coefficients when a power series of *one variable* is raised to the power x. Now we show that multi-convolution polynomials arise as coefficients when a power series of *several variables* is raised to the power x. Let $\mathbf{z}^{\mathbf{n}}=z_1^{n_1}\ldots z_m^{n_m}$. For a formal series $F(\mathbf{z})=\sum_{\mathbf{n}\geq\mathbf{0}}F_{\mathbf{n}}\mathbf{z}^{\mathbf{n}}$ let $[\mathbf{z}^{\mathbf{n}}]F(\mathbf{z})$ be the coefficient of $\mathbf{z}^{\mathbf{n}}$ in $F(\mathbf{z})$.

Theorem 1 *Let*

$$F(\mathbf{z})=1+\sum_{\mathbf{n}\neq\mathbf{0}}F_{\mathbf{n}}\mathbf{z}^{\mathbf{n}} \tag{3.2}$$

be any power series with $F(\mathbf{0})=1$. Then the polynomials

$$F_{\mathbf{n}}(x)=[\mathbf{z}^{\mathbf{n}}]F(\mathbf{z})^x \tag{3.3}$$

form a multi-convolution family. Conversely, every convolution family arises in this way or is identically zero.

Proof: It is easy to verify that the coefficient of $\mathbf{z}^{\mathbf{n}}$ in $F(\mathbf{z})^x$ is indeed a polynomial in x of degree $\leq |\mathbf{n}|$ because

$$F(\mathbf{z})^x = 1 + \sum_{k\geq 1} \binom{x}{k} (\sum_{\mathbf{n}\neq \mathbf{0}} F_{\mathbf{n}} \mathbf{z}^{\mathbf{n}})^k$$

This construction produces a convolution family because of the rule for forming coefficients of the product $F(\mathbf{z})^{x+y} = F(\mathbf{z})^x F(\mathbf{z})^y$. Conversely, suppose the polynomials

$$F_{\mathbf{n}}(x) = (f_{\mathbf{n}0} + f_{\mathbf{n}1}x + \ldots + f_{\mathbf{n}|\mathbf{n}|}x^{|\mathbf{n}|})/\mathbf{n}!$$

form a convolution family. The condition $F_{\mathbf{0}}(x) = F_{\mathbf{0}}(x)^2$ can hold only if $F_{\mathbf{0}}(x) = 0$ or $F_{\mathbf{0}}(x) = 1$. In the former case it is easy to prove by induction that $F_{\mathbf{n}}(x) = 0$ for all $\mathbf{n}$ because $F_{\mathbf{n}}(x) = \sum_{\mathbf{0}<\mathbf{k}<\mathbf{n}} F_{\mathbf{k}}(x)\ F_{\mathbf{n}-\mathbf{k}}(0)$. Otherwise, the condition

$$F_{\mathbf{n}}(2\ x) = 2F_{\mathbf{n}}(x) + \sum_{\mathbf{0}<\mathbf{k}<\mathbf{n}} F_{\mathbf{k}}(x)\ F_{\mathbf{n}-\mathbf{k}}(x) \tag{3.4}$$

for $\mathbf{n} > \mathbf{0}$ implies that $F_{\mathbf{n}}(0) = f_{\mathbf{n}0} = 0$ for $\mathbf{n} > \mathbf{0}$ by induction. For $k > 1$, by equating coefficients of x^k on both sides of (3.4) we find that the coefficient $f_{\mathbf{n}k}$ is forced to have certain values based on the coefficients of $F_{\mathbf{i}}(x)$ with $\mathbf{i} < \mathbf{n}$, because $2^k f_{\mathbf{n}k}$ occurs on the left and $2f_{\mathbf{n}k}$ on the right. The coefficient $f_{\mathbf{n}1}$ can, however, be choosen freely. Any such choice must make

$$F_{\mathbf{n}}(x) = [\mathbf{z}^{\mathbf{n}}] \exp(x \sum_{\mathbf{n}\neq\mathbf{0}} f_{\mathbf{n}1} \mathbf{z}^{\mathbf{n}}),$$

by induction on $\mathbf{n}$. $\square$

Examples: The first example (1.3), that is $F_{\mathbf{n}}(x) = x^{|\mathbf{n}|}/\mathbf{n}!$, comes from the power series $F(\mathbf{z}) = e^{z_1+z_2+\ldots+z_m}$; the second example (1.4), that is $F_{\mathbf{n}}(x) = \binom{x}{\mathbf{n}}$, comes from $F(\mathbf{z}) = 1 + z_1 + z_2 + \ldots + z_m$.

As in the case of one-variable convolution polynomials [9], every multi-convolution family $\{F_{\mathbf{n}}(x)\}$ satisfies another general convolution formula in addition to the one we began with. In deed, we have the following result.

Lemma 2 *(A derived convolution). If $\{F_{\mathbf{n}}(x)\}$ is a set of multi-convolution polynomilas, then*

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} k_i F_{\mathbf{k}}(x) F_{\mathbf{n}-\mathbf{k}}(y) = x n_i F_{\mathbf{n}}(x+y) \qquad (1 \leq i \leq m).$$

or equivalently for all $\mathbf{q} \in \mathbf{C}^m$ *we have*

$$(x+y)\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}}(\mathbf{q}\cdot\mathbf{k})F_{\mathbf{k}}(x)F_{\mathbf{n}-\mathbf{k}}(y) = x(\mathbf{q}\cdot\mathbf{n})F_{\mathbf{n}}(x+y).$$

Proof: The alternative convolution formula is proved by differentiating the basic identity $F(\mathbf{z})^x = \sum_{\mathbf{n}\geq\mathbf{0}} F_{\mathbf{n}}(x)\mathbf{z}^{\mathbf{n}}$ with respect to z_i $(1 \leq i \leq m)$ and

$$xz_i\frac{\partial}{\partial z_i}F(\mathbf{z})F(\mathbf{z})^{x-1} = \sum_{\mathbf{n}} n_i F_{\mathbf{n}}(x)\mathbf{z}^{\mathbf{n}}.$$

Now observe that $\sum k_i F_{\mathbf{k}}(x)F_{\mathbf{n}-\mathbf{k}}(y)$ is the coefficient of $\mathbf{z}^{\mathbf{n}}$ in the product $xz_i(\partial F(\mathbf{z})/\partial z_i)F(\mathbf{z})^{x+y-1}$ while $n_iF_{\mathbf{n}}(x+y)$ is the coefficient of $\mathbf{z}^{\mathbf{n}}$ in $(x+y)z_i(\partial F(\mathbf{z})/\partial z_i)F(\mathbf{z})^{x+y-1}$. □

Lemma 3 *If* $F_{\mathbf{n}}(x) = [\mathbf{z}^{\mathbf{n}}]F(\mathbf{z})^x$*, then*

$$\frac{x}{x+n_i}F_{\mathbf{n}}(x+n_i) = [\mathbf{z}^{\mathbf{n}}]G_i(\mathbf{z})^x, \qquad 1 \leq i \leq m$$

where $G_i(\mathbf{z}) = F(z_1, \ldots, z_iG_i(\mathbf{z}), \ldots, z_m)$.

Proof: Since $F_{\mathbf{n}}(x)$ is a polynomial of degree $\leq |\mathbf{n}|$, so it is sufficient to prove the above identity for integer values k of x. Let $w(\mathbf{z}) = z_i/F(\mathbf{z})$, then $w(z_1, \ldots, z_iG_i(\mathbf{z}), \ldots, z_m) = z_i$. By Lagrange's inversion formula (see [3, p. 159]), we have

$$[z_i^{n_i}](z_i^kG_i^k(\mathbf{z})) = \frac{k}{n_i}[z_i^{n_i-k}]F(\mathbf{z})^{n_i}, \qquad 1 \leq i \leq m.$$

Consequently,

$$[z_i^{n_i}](G_i^k(\mathbf{z})) = \frac{k}{n_i+k}[z_i^{n_i}]F(\mathbf{z})^{n_i+k}, \qquad 1 \leq i \leq m.$$

It follows that for $1 \leq i \leq m$ we have

$$[\mathbf{z}^{\mathbf{n}}]G_i(\mathbf{z})^x = \frac{k}{n_i+k}[\mathbf{z}^{\mathbf{n}}]F(\mathbf{z})^{n_i+k} = \frac{k}{n_i+k}F_{\mathbf{n}}(n_i+k).$$

This completes the proof. □

Theorem 4 *Let $F_{\mathbf{n}}(x)$ be any family of polynomials in x such that $F_{\mathbf{n}}(x)$ has degree $\leq |\mathbf{n}|$. If*

$$F_{\mathbf{n}}(2\ x) = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} F_{\mathbf{k}}(x)\ F_{\mathbf{n}-\mathbf{k}}(x) \tag{3.5}$$

holds for all $\mathbf{n}$ and x, then the following identities hold for all $\mathbf{n}$, x, y and $\mathbf{t}$

$$\begin{aligned}
&\frac{(x+y)\ F_{\mathbf{n}}(x+y+\mathbf{t}\cdot\mathbf{n})}{x+y+\mathbf{t}\cdot\mathbf{n}} \\
&= \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \frac{x\ F_{\mathbf{k}}(x+\mathbf{t}\cdot\mathbf{k})}{x+\mathbf{t}\cdot\mathbf{k}} \cdot \frac{y\ F_{\mathbf{n}-\mathbf{k}}(y+\mathbf{t}\cdot(\mathbf{n}-\mathbf{k}))}{y+\mathbf{t}\cdot(\mathbf{n}-\mathbf{k})} \\
&\frac{(\mathbf{q}\cdot\mathbf{n})\ F_{\mathbf{n}}(x+y+\mathbf{t}\cdot\mathbf{n})}{x+y+\mathbf{t}\cdot\mathbf{n}} \\
&= \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \frac{(\mathbf{q}\cdot\mathbf{k})\ F_{\mathbf{k}}(x+\mathbf{t}\cdot\mathbf{k})}{x+\mathbf{t}\cdot\mathbf{k}} \cdot \frac{y\ F_{\mathbf{n}-\mathbf{k}}(y+\mathbf{t}\cdot(\mathbf{n}-\mathbf{k}))}{y+\mathbf{t}\cdot(\mathbf{n}-\mathbf{k})}.
\end{aligned}$$

Proof: The proof of Theorem 1 shows that the multinomial convolution condition could be weakened to (3.4), or (3.5). Hence (3.5) implies that polynomials $F_{\mathbf{n}}(x)$ form a set of convolution polynomials. Now, for $1 \leq i \leq m$, the process of going from $F_{\mathbf{n}}(x)$ to $xF_{\mathbf{n}}(x+n_i)/(x+n_i)$ of Lemma 3 can be iterated : another replacement gives $xF_{\mathbf{n}}(x+2n_i)/(x+2n_i)$, and after t_i iterations we discover that the polynomials $xF_{\mathbf{n}}(x+t_in_i)/(x+t_in_i)$ also form a convolution family. This holds for all positive integers t_i $(1 \leq i \leq m)$, and the convolution condition is expressible as a set of polynomial relations in t_i $(1 \leq i \leq m)$; therefore $xF_{\mathbf{n}}(x+t_in_i)/(x+t_in_i)$ is a convolution family for all complex numbers t_i $(1 \leq i \leq m)$. Thus by applying the above process we obtain succesfully a sequence of convolution polynomials families

$$\frac{x}{x+t_1n_1+\ldots+t_in_i} F_{\mathbf{n}}(x+t_1n_1+\ldots+t_in_i)$$

for $i = 1, 2, \ldots, m$ and the last family corresponds to the first identity. The second one follows by combining the first one with the derived convolution formula of Lemma 2. $\square$

Example 1: Let $F_{\mathbf{n}}(x) = \binom{x}{\mathbf{n}}$, then it follows from (1.4) and Theorem 4 that

$$\frac{x+y}{x+y+\mathbf{t}\cdot\mathbf{n}} \binom{x+y+\mathbf{t}\cdot\mathbf{n}}{\mathbf{n}}$$

$$= \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \frac{x}{x+\mathbf{t}\cdot\mathbf{k}} \binom{x+\mathbf{t}\cdot\mathbf{k}}{\mathbf{k}} \frac{y}{y+\mathbf{t}\cdot(\mathbf{n}-\mathbf{k})} \binom{y+\mathbf{t}\cdot(\mathbf{n}-\mathbf{k})}{\mathbf{n}-\mathbf{k}},$$

$$\frac{(\mathbf{q}\cdot\mathbf{n})}{x+y+\mathbf{t}\cdot\mathbf{n}} \binom{x+y+\mathbf{t}\cdot\mathbf{n}}{\mathbf{n}}$$

$$= \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \frac{\mathbf{q}\cdot\mathbf{k}}{x+\mathbf{t}\cdot\mathbf{k}} \binom{x+\mathbf{t}\cdot\mathbf{k}}{\mathbf{k}} \frac{y}{y+\mathbf{t}\cdot(\mathbf{n}-\mathbf{k})} \binom{y+\mathbf{t}\cdot(\mathbf{n}-\mathbf{k})}{\mathbf{n}-\mathbf{k}}.$$

Combining the above two identies yields (1.2).

Example 2: Let $F_{\mathbf{n}}(x) = x^{\mathbf{n}}/\mathbf{n}!$, then it follows from (1.3) and Theorem 4 that

$$\frac{(x+y)(x+y+\mathbf{t}\cdot\mathbf{n})^{|\mathbf{n}|-1}}{\mathbf{n}!}$$

$$= \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \frac{x(x+\mathbf{t}\cdot\mathbf{k})^{|\mathbf{k}|-1}}{\mathbf{k}!} \frac{y(y+\mathbf{t}\cdot(\mathbf{n}-\mathbf{k}))^{|\mathbf{n}-\mathbf{k}|-1}}{(\mathbf{n}-\mathbf{k})!},$$

$$\frac{(\mathbf{q}\cdot\mathbf{n})(x+y+\mathbf{t}\cdot\mathbf{n})^{|\mathbf{n}|-1}}{\mathbf{n}!}$$

$$= \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \frac{(\mathbf{q}\cdot\mathbf{k})(x+\mathbf{t}\cdot\mathbf{k})^{|\mathbf{k}|-1}}{\mathbf{k}!} \frac{y(y+\mathbf{t}\cdot(\mathbf{n}-\mathbf{k}))^{|\mathbf{n}-\mathbf{k}|-1}}{(\mathbf{n}-\mathbf{k})!}.$$

Combining the above two identies we obtain

$$\sum_{\mathbf{k}} (p+\mathbf{q}\cdot\mathbf{k}) B_{\mathbf{k}}(x,\mathbf{t}) B_{\mathbf{n}-\mathbf{k}}(y,\mathbf{t}) = \frac{p(x+y)+x(\mathbf{q}\cdot\mathbf{n})}{x+y} B_{\mathbf{n}}(x+y,\mathbf{t}), \quad (3.6)$$

where $B_{\mathbf{n}}(x,\mathbf{t}) = x(x+\mathbf{t}\cdot\mathbf{n})^{|\mathbf{n}|-1}/\mathbf{n}!$. If we take $\mathbf{q}=\mathbf{t}$ and $p=1$ in the above formula we obtain a multi-extension of Abel's identity :

$$(x+y+\mathbf{n}\cdot\mathbf{t})^{|\mathbf{n}|} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} (x+\mathbf{k}\cdot\mathbf{t})^{|\mathbf{k}|} y(y+(\mathbf{n}-\mathbf{k})\cdot\mathbf{t})^{|\mathbf{n}-\mathbf{k}|-1},$$

where $\binom{\mathbf{n}}{\mathbf{k}} = \prod_{i=1}^{m} \binom{n_i}{k_i}$, or equivalently

$$(x+y)^{|\mathbf{n}|} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} (x-\mathbf{k}\cdot\mathbf{t})^{|\mathbf{n}-\mathbf{k}|} y(y+\mathbf{k}\cdot\mathbf{t})^{|\mathbf{k}|-1}.$$

We can also derive (3.6) from the first example as follows. If we substitute a, c and $\mathbf{b}$ by xa, xc and $x\mathbf{b} = (xb_1, \cdots, xb_m)$ respectively in (1.2), we get a polynomial identity in x. Equating the coefficients of $x^{|\mathbf{n}|}$ in both sides of this identity yields (3.6).

Note that formula (3.6) generalizes also Hurwitz's *multinomial* extension of Abel's identity [8]. To see this, setting $\mathbf{n} = \mathbf{1} = (1, \ldots, 1)$, $\mathbf{q} = \mathbf{0}$ and $p = 1$, then (3.6) reduces to

$$\sum_{\mathbf{k} \in \{0,1\}^m} x(x + \mathbf{k} \cdot \mathbf{t})^{|\mathbf{k}|-1} y(y + (\mathbf{1} - \mathbf{k}) \cdot \mathbf{t})^{|\mathbf{1}-\mathbf{k}|-1} = (x + y)(x + y + |\mathbf{t}|)^{m-1}.$$

Exemple 3 Consider the set $[\mathbf{n}] = \{1\} \times [n_1] + \{2\} \times [n_2] + \ldots + \{m\} \times [n_m]$. For each element $\alpha = (i, j)$ of $[\mathbf{n}]$ we call i the *color* of α. A partition π of $[\mathbf{n}]$ is called *colored partition* (cf. [19]) if each block of π is only composed of elements with different colors. Let

$$F(\mathbf{z}) = \exp(e_1(\mathbf{z}) + e_2(\mathbf{z}) + \ldots + e_m(\mathbf{z})),$$

where $e_k(\mathbf{z})$ is the k-th $(1 \leq k \leq m)$ elementary symmetric function of $z_1, z_2, \ldots, z_m$. Then the corresponding family of convolution polynomials is

$$\begin{aligned} F_{\mathbf{n}}(x) &= [\mathbf{z}^{\mathbf{n}}] e^{x(e_1(\mathbf{z}) + e_2(\mathbf{z}) + \ldots + e_m(\mathbf{z}))} \\ &= \frac{\left\{ {\mathbf{n} \atop 0} \right\} + \left\{ {\mathbf{n} \atop 1} \right\} x + \ldots + \left\{ {\mathbf{n} \atop |\mathbf{n}|} \right\} x^{|\mathbf{n}|}}{\mathbf{n}!}, \end{aligned}$$

where $\left\{ {\mathbf{n} \atop k} \right\}$ is the number of colored partitions in k blocks of $[\mathbf{n}]$ (see [19]). We note that if $\mathbf{n} = (1, 1, \ldots, 1)$, the number $\left\{ {\mathbf{n} \atop k} \right\}$ is nothing but Stirling number of second kind $S(m, k)$, that is, the number of partitions in k blocks of $[m]$.

We have shown how to create new multinomial convolution families from given ones exactly as in the monomial case. The addiional constructions mentioned by Knuth [9] can also be extended. For example, if $F_{\mathbf{n}}(x)$ and $G_{\mathbf{n}}(x)$ are convolution families, then so is the family $H_{\mathbf{n}}(x)$ defined by

$$H_{\mathbf{n}}(x) = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} F_{\mathbf{k}}(x) G_{\mathbf{n}-\mathbf{k}}(x).$$

as did by Knuth [9] in the case $m = 1$.

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