# A new decomposition of derangements

Dongsu Kim and Jiang Zeng

January 28, 2001

**Abstract**

We give a new decomposition of derangements, which gives a direct interpretation of a formula for their generating function. This decomposition also works for counting derangements by number of excedances.

## 1 Introduction

A permutation $\pi$ of $[n]=\{1,2,\ldots,n\}$ is a *derangement*, if $\pi(i) \neq i$, for all $i \in [n]$. A value $i \in [n]$ is an *excedance* of $\pi$ if $i < \pi(i)$. The number of excedances in $\pi$ is denoted by $\operatorname{exc}\pi$. Let $\mathcal{D}_n$ be the set of derangements of $[n]$, and $d_n(x)$ the polynomial

$$d_n(x) = \sum_{\pi \in \mathcal{D}_n} x^{\operatorname{exc}\pi}.$$

For example, $d_0(x) = 1$, $d_1(x) = 0$, $d_2(x) = x$, $d_3(x) = x + x^2$, $d_4(x) = x + 7x^2 + x^3$. The generating function of $d_n(x)$ can be written as follows [2, 5] :

$$\sum_{n\geq 0} d_n(x)\frac{t^n}{n!} = \frac{1}{1 - \sum_{n\geq 2}(x + x^2 + \cdots + x^{n-1})\, t^n/n!}\,. \tag{1}$$

Of course (1) can be proved by various methods, but, as pointed out by Gessel [4], it seems difficult to directly interpret (1) (even in the $x = 1$ case!) in terms of derangements. In [4] Gessel gave a direct proof of (1) in a different model with $x = 1$. His proof is actually based on a factorization of some *D-permutations*, and cannot be generalized in a straightforward way to prove (1). Our purpose is to give a decomposition of derangements which interprets (1) directly.

A sequence $\sigma = s_1 s_2 \cdots s_k$ of $k$ distinct integers $s_1, \ldots, s_k$ is called a *cycle* of length $k$ if $s_1 = \min\{s_1, \ldots, s_k\}$. A cycle $\sigma$ is called *unimodal* (resp. *prime*), if there exists $i$, $2 \leq i \leq k$, such that $s_1 < \cdots < s_{i-1} < s_i$ and $s_i > s_{i+1} > \cdots > s_k$ if $i < k$ (resp. in addition, $s_{i-1} < s_k$). Hence each unimodal (resp. prime) cycle is of length $\geq 2$.

Considering that $s_1$ is the smallest in our case, this definition is consistent with the usual definition of 'unimodal'. Clearly each cycle $\sigma = s_1 \dots s_k$ can be identified with the cyclic permutation $\sigma'$ of the set $\{s_1, \dots, s_k\}$ by $\sigma'(s_i) = s_{i+1}$ for $i \in [k]$, with $s_{k+1} = s_1$. We let $\operatorname{exc} \sigma$ denote the number of excedances of the associated cyclic permutation $\sigma'$.

Let $(l_1, \dots, l_m)$ be a composition of $n$. A *$P$-decomposition* of type $(l_1, \dots, l_m)$ of $[n]$ is a sequence of prime cycles $\tau = (\tau_1, \tau_2, \dots, \tau_m)$ such that $\tau_i$ is of length $l_i$ and the underlying sets of $\tau_i$, $i \in [m]$, form a partition of $[n]$.

Define the *excedance of* $\tau$ as the total number of excedances in its prime cycles, i.e. $\operatorname{exc} \tau = \operatorname{exc} \tau_1 + \cdots + \operatorname{exc} \tau_m$, and weight $\tau$ by $x^{\operatorname{exc} \tau}$. It turns out that the right-hand side of (1) is the excedance generating function of $P$-decompositions. Indeed, since the weight of prime cycles on any $l$-set is $x + x^2 + \cdots + x^{l-1}$, the generating function of $P$-decompositions of type $(l_1, \dots, l_m)$ is given by

$$\binom{l_1 + \cdots + l_m}{l_1, \dots, l_m} \prod_{i=1}^{m} (x + \cdots + x^{l_i - 1}) \frac{t^{l_1 + \cdots + l_m}}{(l_1 + \cdots + l_m)!}.$$

Summing on $l_1, \dots, l_m \geq 2$ and $m \geq 0$, we obtain the right hand side of (1).

In the next section we give an algorithm (or bijection), which maps each derangement into a $P$-decomposition with the same number of excedances, and thus prove (1). In section 3 we will apply a similar decomposition to give a direct interpretation of a generating function of *Eulerian polynomials*. Finally, in section 4 we indicate how to extend our algorithm to deal with similar problems in multipermutations.

## 2 Unimodal and prime decompositions

Given a derangement $\pi$ of $[n]$, we first factorize it into cycles of length $\geq 2$ :

$$\pi = (C_1, \dots, C_k),$$

sorted in the *decreasing* order of their minima. For each cycle $\sigma = s_1 s_2 \cdots s_k$ we define the following $U$-algorithm to decompose it into a sequence of unimodal cycles. For the algorithm we set $s_{k+1} = s_1$.

**$U$-Algorithm**

1. *If $\sigma$ is unimodal then $U(\sigma) = (\sigma)$.*
2. *Otherwise, let $i$ be the largest integer such that $s_{i-1} > s_i < s_{i+1}$, let $j$ be the unique integer greater than $i$ such that $s_j > s_i > s_{j+1}$, and set $U(\sigma) = (U(\sigma_1), \sigma_2)$, where $\sigma_1 = s_1 \cdots s_{i-1} s_{j+1} \cdots s_k$ and $\sigma_2 = s_i s_{i+1} \cdots s_j$, which is unimodal.*

**Example 2.1** *Let* $\sigma = 1\ 8\ 4\ 7\ 12\ 14\ 11\ 9\ 13\ 10\ 6\ 3\ 5\ 2$. *The* $U$*-algorithm runs as follows:*

$$\begin{aligned}
\sigma &\to (U(1\,8\,4\,7\,12\,14\,11\,9\,13\,10\,6\,2),\ 3\,5)\\
&\to (U(1\,8\,4\,7\,12\,14\,11\,6\,2),\ 9\,13\,10,\ 3\,5)\\
&\to (U(1\,8\,2), 4\,7\,12\,14\,11\,6,\ 9\,13\,10,\ 3\,5)\\
&\to (1\,8\,2, 4\,7\,12\,14\,11\,6,\ 9\,13\,10,\ 3\,5).
\end{aligned}$$

We extend $U$ to $\pi$ by applying $U$ to each of its cycles to obtain

$$U(\pi) = (U(C_1),\, U(C_2),\, \ldots,\, U(C_r)) = (u_1, \ldots, u_m),$$

which is called the *unimodal decomposition* of $\pi$.

Note that the first cycle $C_1$ of $\pi$ corresponds to the segment $(u_1, \ldots, u_i)$, where $i$ is the smallest integer satisfying $\min(u_1) > \min(u_{i+1})$, and the second to a segment of $(u_{i+1}, \ldots, u_m)$ in the same manner, etc., so that the underlying set of each cycle can be read off from the *unimodal decomposition* of $\pi$. The following result characterizes all the sequences of unimodal cycles obtained by the $U$-algorithm.

**Lemma 2.2** *A sequence of disjoint unimodal cycles,* $u = (u_1, \ldots, u_m)$*, is a unimodal decomposition of a derangement in* $\mathcal{D}_n$ *if and only if the underlying sets of* $u_i$, $i \in [m]$*, form a partition of* $[n]$ *and* $\max(u_{i-1}) > \min(u_i)$ *for each* $i = 2, \ldots, m$.

*Proof.* Clearly it suffices to show the 'if' part. Without loss of generality we may assume that $\min(u_1) < \min(u_i)$, for each $i = 2, \ldots, m$. We build $\pi$ step by step. Let $\pi^{(1)} = u_1$. For $i > 1$, assume that $\pi^{(i-1)}$ has been built and that $\pi^{(i-1)} = s_1 s_2 \cdots s_l$, where $s_1, \ldots, s_l$ is an appropriate rearrangement of elements in $u_1, u_2, \ldots, u_{i-1}$. Let $u_i = r_1 r_2 \cdots r_a$. Since $\max(u_{i-1}) > \min(u_i)$, there is an integer $j$ such that $s_j > \min(u_i)$, let $j_0$ be the largest such integer and set $\pi^{(i)} = s_1 s_2 \cdots s_{j_0} r_1 r_2 \cdots r_a s_{j_0+1} \cdots s_l$. Let $\pi = \pi^{(m)}$. Clearly $U(\pi) = u$. $\square$

For each unimodal cycle $\sigma = s_1 s_2 \cdots s_k$ we define the following $V$-algorithm to decompose it into a sequence of prime cycles.

**$V$-Algorithm**

1. *If* $\sigma$ *is prime then* $V(\sigma) = (\sigma)$.
2. *Otherwise, let* $j$ *be the smallest integer such that* $s_j > s_i > s_{j+1} > s_{i-1}$ *for some integer* $i$ *greater than 1 and set* $V(\sigma) = (V(\sigma_1), \sigma_2)$*, where* $\sigma_1 = s_1 \cdots s_{i-1} s_{j+1} \cdots s_k$ *and* $\sigma_2 = s_i s_{i+1} \cdots s_j$*, which is prime.*

We extend $V$-algorithm to $U(\pi)$ by applying $V$ to each of its components to obtain

$$V \circ U(\pi) = (V(u_1),\, V(u_2),\, \ldots,\, V(u_m)) = (\tau_1, \ldots, \tau_m),$$

which is called the *prime decomposition* of $\pi$.

The structure of the unimodal decomposition of $\pi$ can be easily obtained from its prime decomposition. The first unimodal cycle in $U(\pi)$ corresponds to the segment $(\tau_1, \ldots, \tau_i)$, where $i$ is the smallest integer satisfying $\max(\tau_i) > \min(\tau_{i+1})$, and the second to a segment of $(\tau_{i+1}, \ldots, \tau_m)$ in the same manner, etc.

**Example 2.3** *Let $\sigma$ be the same as the preceding example, whose unimodal decomposition is $U(\sigma) = (1\ 8\ 2,\ 4\ 7\ 12\ 14\ 11\ 6,\ 9\ 13\ 10,\ 3\ 5)$. Note that only the second cycle in $U(\sigma)$ is not prime. The $V$-algorithm applied to the second cycle runs as follows :*

$$4\ 7\ 12\ 14\ 11\ 6 \to (V(4\ 7\ 11\ 6),\ 12\ 14) \to (4\ 6,\ 7\ 11,\ 12\ 14).$$

*Therefore $V \circ U(\sigma) = (1\ 8\ 2,\ 4\ 6,\ 7\ 11,\ 12\ 14,\ 9\ 13\ 10,\ 3\ 5)$.*

It is clear that the composition $V \circ U$ maps any derangement of $[n]$ into a $P$-decomposition of $[n]$. The following result shows that this mapping is bijective.

**Theorem 2.4** *Any $P$-decomposition of $[n]$ is the prime decomposition of a unique derangement in $\mathcal{D}_n$.*

*Proof.* Let $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$ be a $P$-decomposition of $[n]$. We first construct a sequence of unimodal cycles as follows: starting from the right, if there is any pair of adjacent $\tau_i$ and $\tau_{i+1}$ such that $\max(\tau_i) < \min(\tau_{i+1})$, then we insert the elements of $\tau_{i+1}$ in $\tau_i$ just before the maximum of $\tau_i$ and obtain a new cycle $\tau_i * \tau_{i+1}$. Repeat this process with $(\tau_1, \ldots, \tau_i * \tau_{i+1}, \ldots, \tau_m)$, until there are no more such pairs. By lemma 2.2, the resulting sequence $\sigma$ is a unimodal decomposition of some $\pi \in \mathcal{D}_n$, i.e. $U(\pi) = \sigma$. It follows that $V \circ U(\pi) = V(\sigma) = \tau$. □

From the $U$-algorithm it is clear that the number of excedances in a cycle is the same as the sum of excedances in each unimodal component. Also the prime decomposition has the same property. Thus we have proved (1).

# 3 Application to Eulerian polynomials

If instead of derangements we let $A_n(x)$ denote the sum of $x^{\operatorname{exc}\pi}$ for all permutations $\pi$ of $[n]$, then the polynomials $xA_n(x)$ are the well-known *Eulerian polynomials* and have several other combinatorial interpretations in addition to counting permutations by number of excedances [6]. By virtue of classical theory of generating functions we see immediately that $A_n(x)$ are related to $d_n(x)$ by the following :

$$\sum_{n\geq 0} A_n(x)\frac{t^n}{n!} = e^t \sum_{n\geq 0} d_n(x)\,\frac{t^n}{n!}\,.$$

Hence it follows from (1) that

$$\sum_{n\geq 0} A_n(x)\frac{t^n}{n!} = \frac{1}{1-\sum_{n\geq 1}(x-1)^{n-1}\, t^n/n!}\,. \tag{2}$$

A similar proof can be given for (2), but in this case a *weight-preserving sign-reversing* involution is needed.

A sequence $\sigma = a_1a_2\ldots a_k$ of $k$ distinct integers $a_1,\, a_2,\, \ldots,\, a_k$ is called *unimodal* if $k=1$ or $k\geq 2$ and there exists an integer $i$, $1\leq i\leq k$, such that $a_1 < a_2 < \cdots < a_i$ and $a_i > a_{i+1} > \cdots > a_k$ if $i<k$. This is the usual definition of 'unimodal'. We define the weight of the unimodal sequence $\sigma$ by $x^{i-1}(-1)^{k-i}$, i.e. an ascent is given $x$ and a descent $-1$.

A $U$-*decomposition* (resp. $I$-*decomposition*) of $[n]$ is a sequence of unimodal (resp. increasing) sequences $(\tau_1,\, \tau_2,\, \ldots,\, \tau_m)$ such that the underlying sets of $\tau_i$, $i\in[m]$, form a partition of $[n]$ (resp. in addition, for $i>1$, if $\tau_i$ is a singleton then it is greater than the last entry of $\tau_{i-1}$). Hence the right side of (2) is the generating function of $U$-decompositions.

We now set up a weight-preserving sign-reversing involution on the $U$-decompositions to reduce the above generating function to that of $I$-decompositions. Given a $U$-decomposition $\pi = (\pi_1, \pi_2, \ldots, \pi_l)$, we call an integer $k$ *attachable*, if $k$ forms a singleton, i.e. $\pi_i = k$ for some $i>1$, and $k$ is smaller than the last entry of $\pi_{i-1}$; *detachable*, if there exists $\pi_j$ whose last entry is $k$ and whose penultimate entry is greater than $k$. The involution is then defined by detaching or attaching the smallest attachable or detachable integer (if any). It is clear that $\pi$ is fixed if and only if $\pi$ is an $I$-decomposition.

On the other hand, given a permutation $\pi$ of $[n]$, we can factorize it into ordered cycles $\pi = (s_1, \ldots, s_r, c_1, \ldots, c_t)$, where $s_1, \ldots, s_r$ are the singletons ordered in increasing order and $c_1, \ldots, c_t$ the cycles of length $\geq 2$ ordered in decreasing order of their minima. Applying $V\circ U$ algorithm to each cycle $c_i$ we obtain

$$\pi = (s_1, \ldots, s_r, V\circ U(c_1), \ldots, V\circ U(c_t)) = (\pi_1, \ldots, \pi_m),$$

where each $\pi_i$ is a prime or singleton cycle. Since each prime cycle $a_1\ldots a_{k-1}a_k\ldots a_l$ with $a_1 < \cdots < a_{k-1} < a_l < a_{l-1} < \cdots < a_k$ is in one-to-one correspondence with a sequence of increasing segments, $(a_1a_2\cdots a_{k-1}a_l,\, a_{l-1}, a_{l-2}, \ldots, a_k)$, which has no attachable or detachable element, we see that $\pi$ is in one-to-one correspondence with an $I$-decomposition of $[n]$. Note that the singletons in $\pi$ correspond to the singletons to the left of the first increasing sequence of length greater than one in an $I$-decomposition.

Therefore both sides of (2) are the generating functions of $I$-decompositions.

## 4 Remarks

Our decompositions work also for permutations of a multiset $\{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$. More precisely, let $w = w_1w_2\ldots w_n$ be such a permutation and $\delta(w) = p_1p_2\ldots p_n$ the non-decreasing rearrangement of the letters in $w$, where $n = n_1 + \cdots + n_m$. Then $w$ is a *multiderangement* if $p_i \neq w_i$ for each $i = 1, \ldots, n$, while the statistic of *excedance* of $w$ is defined by $\operatorname{exc} w = \#\{i : w_i > p_i\}$. Let $\mathcal{R}(\mathbf{n})$ be the set of all such permutations and define

$$d_{\mathbf{n}}(x) = \sum_{w \in \mathcal{R}(\mathbf{n})} x^{\operatorname{exc} w}.$$

Using Foata's factorization of multipermutations (see [3]) we can factorize each multiderangement as a product of cycles of length at least 2, combining with our two decompositions we get the following result:

$$\sum_{n_1,\ldots,n_m \geq 0} d_{\mathbf{n}}(x) x_1^{n_1} \ldots x_m^{n_m} = \frac{1}{1 - x\, e_2 - (x + x^2)\, e_3 - \cdots - (x + x^2 + \cdots + x^{m-1})\, e_m},$$

where $e_i$ $(2 \leq i \leq m)$ is the $i$-th elementary symmetric function of $x_1, \ldots, x_m$. The above result seems to be first proved by Askey and Ismail [1] using MacMahon's Master Theorem.

## Acknowledgment

This work was partially supported by KOSEF: 971-0106-038-2.

## References

[1] Askey, (R.) and Mourad, (I.), Permutation problems and special functions, Canad. J. Math. **28** (1976), no. 4, 853–874.

[2] Brenti, (F.), Unimodal polynomials arising from symmetric functions, Proc. Amer. Math. Soc. 108 (1990), no. 4, 1133–1141.

[3] Cartier, (P.) and Foata, (D.), Problèmes combinatoires de commutation et réarrangements, Lecture Notes in Mathematics **85**, Springer-Verlag, Berlin, 1969.

[4] Gessel, (I.M.), A coloring problem, Amer. Math. Monthly 98 (1991), no. 6, 530–533.

[5] Roselle, (D.P.), Permutations by number of rises and successions, Proc. Amer. Math. Soc. 19 (1968), 8–16.

[6] Stanley, (R.P.), Enumerative Combinatorics, Volume 1, Cambridge University Press, 1997.

Dongsu Kim, Department of Mathematics, KAIST, Taejon 305-701, Korea
Email address: dskim@math.kaist.ac.kr

Jiang Zeng, Institut Girard Desargues, Université Claude Bernard (Lyon I), France
Email address: zeng@desargues.univ-lyon1.fr