A bijective proof of Lassalle's partition identity

Jiang Zeng

Institut Girard Désargues, Université Claude Bernard Lyon 1

43, boulevard du 11 Novembre 1918, F-69622 Villeurbanne Cedex

e-mail: zeng@desargues.univ-lyon1.fr

October 15, 2002

We give a bijective proof of the following form of a conjecture in [2, 3]. The special $s = 1$ case was proved in the same vein by Simion [2]. For other proofs of this conjecture, see [4]. Our notations are the same as in [2, 3]. For an integer partition μ let the length $l(\mu)$ be the number of the parts of μ, $m_i(\mu)$ the number of parts equal to i, $z_\mu = \prod_{i\geq 1} i^{m_i(\mu)} m_i(\mu)!$ and $\langle{\mu \atop r}\rangle$ the number of ways to choose r different cells from the diagram of the partition μ taking at least one cell from each row. For any positive integer n let $(a)_n = a(a+1)\cdots(a+n-1)$ and $[n] = \{1, 2, \ldots, n\}$.

Theorem 1 *We have*

$$\sum_{|\mu|=n} \frac{n!}{z_\mu} \langle{\mu \atop r}\rangle X^{l(\mu)-1} \sum_{i=1}^{l(\mu)} \mu_i \binom{\mu_i + s - 1}{s-1} = \binom{n}{r} (s+r)_{n-r} \left[(X+s)_r - (X)_r\right]. \tag{1}$$

Recall that the functional graph of an injection $f : [n] \to [n+r]$ consists of disjoint cycles and paths. Each path is of form $a_0 \to a_1 \to \cdots \to a_l$, where $f(a_i) = a_{i+1}$ for $0 \leq j < l$, with $f^{-1}(a_0)$ empty, and $a_l \in [n+r] \setminus [n]$. Each cycle is of the form $a_0 \to a_1 \to \cdots \to a_l$, where $f(a_i) = a_{i+1}$ for $0 \leq j \leq l$, with $a_{l+1} = a_0$. Let $\operatorname{cyc} f$ be the number of cycles in f. Then the corresponding generating function reads as follows:

$$\sum_{f:[n]\to[n+s]} X^{\operatorname{cyc} f} = (X+s)_n \tag{2}$$

where f is an injection. The above identity seems to be first published by Foata and Strehl [1] and can be readily verified by induction on n as in the well-known case of permutations, which corresponds to $s = 0$.

We shall prove that the two sides of (1) are the generating function of pairs (f, R) with respect to the number of cycles of f, where f is an injection from $[n]$ to $[n+s]$ and R is a r-subset of $[n]$ such that each cycle of f has at least one element in R and the restriction of f on R is not a permutation of R.

Indeed, to construct such a pair (f, R) we can first choose r elements of $[n]$ to form a r-subset R, there are $\binom{n}{r}$ ways to do so; and construct an injection $g : R \to R \cup \{n+1, \ldots, n+s\}$ such that its restriction on R is not a permutation of R, the generating function of such injections is then $(X+s)_r - (X)_r$; finally insert succesively the $n-r$ elements of $[n] \setminus R$ into the above injection g to obtain the injection $f : [n] \to [n+s]$, there are $(s+r)_{n-r}$ ways to insert the $n-r$ elements into any above injection g. This gives the right-hand side of (1).

On the other hand, we can construct such a pair (f, R) as follows: construct first a permutation σ on $[n]$, for any partition μ of n there are $n!/z_\mu$ such permutations of type μ; choose r elements of $[n]$ to form the subset R such that each cycle of σ has at least one element in R, there are $\left\langle {\mu \atop r} \right\rangle$ ways to do so; choose any cycle $(a, \sigma(a), \cdots, \sigma^{\mu_i-1}(a))$ of σ with length μ_i to form s paths:

$$\begin{array}{rcl}
a \to \sigma(a) \to \cdots \sigma^{l_1-1}(a) & \to & n+1, \\
\sigma^{l_1}(a) \to \sigma^{l_1+1}(a) \to \cdots \sigma^{l_2-1}(a) & \to & n+2, \\
\cdots\cdots & \to & \cdots \\
\sigma^{l_{s-1}}(a) \to \sigma^{l_{s-1}+1}(a) \to \cdots \sigma^{l_s-1}(a) & \to & n+s,
\end{array}$$

where $l_1 + l_2 + \cdots l_s = \mu_i$ and by convention the corresponding path is empty if $l_i = 0$. Obviously there are $\mu_i \binom{\mu_i+s-1}{s-1}$ ways to form such s paths starting from any cycle of length μ_i. Finally we complete the construction of $f : [n] \to [n+s]$ by using the s paths above and the remaining $l(\mu) - 1$ cycles of σ. This gives the left-hand side of (1).

The theorem follows then from the two above countings.

References

[1] Foata (D.) and Strehl (V.) *Combinatorics of Laguerre polynomials*, in : Enumeration and Design, Waterloo Jubilee Conference (Academic Press, New York, 1984) 123–140.

[2] Lassalle (M.) *Quelques conjectures combinatoires relatives à la formule classique de Chu-Vandermonde*, Adv. in Appl. Math. 21 (1998), 457-472.

[3] Lassalle (M.) *Une conjecture en théorie des partitions*, http://xxx.lanl.gov/math.CO/9901040, 1999.

[4] Lassalle (M.) *Une identité en théorie des partitions*, Journal of Combinatorial Theory, Series A, to appear.