New Euler–Mahonian statistics on permutations and words

Robert J. Clarke *
Pure Mathematics Department
University of Adelaide
Adelaide, South Australia 5005

Einar Steingrímsson †
Matematiska institutionen
University of Göteborg
412 96 Göteborg, Sweden

Jiang Zeng‡
Département de mathématique
Université Louis-Pasteur
67084 Strasbourg Cedex, France

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Abstract

We define new Mahonian statistics, called MAD, MAK and ENV, on words. Of these, ENV is shown to equal the classical INV, that is the number of inversions, while for permutations MAK has been already defined by Foata and Zeilberger [12]. It is shown that the triple statistics (des, MAK, MAD) and (exc, DEN, ENV) are equidistributed over the rearrangement class of an arbitrary word. Here, exc is the number of excedances, and DEN is Denert's statistic [12, 15]. In particular, this implies the equidistribution of (exc, INV) and (des, MAD). These bistatistics are not equidistributed with the classical Euler-Mahonian statistic (des,MAJ). The proof of the main result is by means of a bijection which, in the case of permutations, is essentially equivalent to several bijections in the literature (or inverses of these). These include bijections defined by Foata and Zeilberger, by Françon and Viennot and by Biane, between the symmetric group and sets of weighted Motzkin paths. These bijections are used to give a continued fraction expression for the generating function of (exc, INV) or (des, MAD) on the symmetric group.

1 Introduction

The subject of permutation statistics, it is frequently claimed, dates back at least to Euler [5]. However, it was not until MacMahon's extensive study [17] at the turn of the century that this became an established discipline of mathematics, and it was to take a long time before it developed into the vast field that it is today.

In the last three decades or so much progress has been made, both in discovering and analyzing new statistics, and in extending these, together with the classical permutation statistics, to arbitrary words with repeated letters. See for example [7, 8, 10, 11, 12, 15, 19, 21, 23]. Inroads have also been made in connecting permutation statistics to various geometric structures and to the classical theory of hypergeometric functions, as in [6, 13, 14, 18].

MacMahon considered four different statistics for a permutation π: The number of descents ($\operatorname{des}\pi$), the number of excedances ($\operatorname{exc}\pi$), the number of inversions ($\mathrm{INV}\,\pi$), and the major index ($\mathrm{MAJ}\,\pi$). These are defined as follows: A descent in a permutation $\pi = a_1a_2\cdots a_n$ is an i such that $a_i > a_{i+1}$, an excedance is an i such that $a_i > i$, an inversion is a pair (i, j) such that $i < j$ and $a_i > a_j$, and the major index of π is the sum of the descents in π.

In fact, MacMahon studied these statistics in greater generality, namely over the rearrangement class of an arbitrary word w. The rearrangement class $R(w)$ of a word $w = a_1 a_2 \cdots a_n$ is the set of all words obtained by permuting the letters of w. All of the above mentioned statistics generalize to words, and in each case, except for that of exc, the generalization is trivial.

MacMahon showed, algebraically, that exc is *equidistributed* with des, and that INV is equidistributed with MAJ, over $R(w)$ for any word w. That is to say,

$$\sum_{z \in R(w)} t^{\operatorname{exc} z} = \sum_{z \in R(w)} t^{\operatorname{des} z} \quad \text{and} \quad \sum_{z \in R(w)} q^{\mathrm{INV}\, z} = \sum_{z \in R(w)} q^{\mathrm{MAJ}\, z}.$$

Foata gave a combinatorial proof of these equidistribution results. (See [8].)

Any statistic that is equidistributed with des is said to be *Eulerian*, the reason being that Euler was apparently the first to discover the recurrence relation for the *Eulerian numbers* $A(n, k)$. These numbers count permutations in the symmetric group $\mathcal{S}_n$ with k descents and they are the coefficients of the *Eulerian polynomials* $A_n(t)$ defined by $A_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{1+\operatorname{des} \pi}$. The Eulerian polynomials satisfy the identity

$$\sum_{k \geq 0} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

Whether Euler was aware of that is less clear, but the polynomials $A_n(t)$ appear in his work [5, p. 373].

On the other hand, a statistic that is equidistributed with INV is said to be *Mahonian*. It was shown more than a hundred and fifty years ago, by Rodriguez [20], that, for permutations,

$$\sum_{\pi \in \mathcal{S}_n} q^{\mathrm{INV}\, \pi} = (1+q)(1+q+q^2) \cdots (1+q+\cdots+q^{n-1}).$$

This is easy to see. Namely, given a permutation $\pi = a_1 a_2 \cdots a_{n-1}$, if we insert n between a_{k-1} and a_k, then INV increases by $(n-k)$, whence the recurrence which yields the above identity.

All of the above statistics, and, in fact, most of the permutation statistics found in the literature, fall into one of these two categories; they are either Eulerian or Mahonian. Moreover, most of them can be generalized to the rearrangement class of an arbitrary word. As a rule, equidistribution results extend to these generalizations.

Curiously, new Eulerian statistics have not become prominent since MacMahon's definition of des and exc, whereas new Mahonian statistics are constantly entering the scene. Proving directly that a statistic is Mahonian is by no means always trivial, and there are still many such statistics for which no direct proof exists. What is more interesting, however, is the study of *pairs* of statistics, usually an Eulerian one and a Mahonian one, and equidistribution of such *bistatistics*, first developed in [7].

The first pair of equidistributed Euler-Mahonian bistatistics to be discovered was that of $(\mathrm{des}, \mathrm{INV})$ and $(\mathrm{des}, \mathrm{IMAJ})$, where $\mathrm{IMAJ}\,\pi$ is the major index of the *inverse* of the permutation π (see [11]). Although instrumental in some of the analytic developments of the subject, this discovery cannot be extended to words with repetition of letters. In addition, the purists among us are reluctant to admit to the Euler-Mahonian club a pair of pairs that really is only a triple. Thus, they would recommend that $(\mathrm{des}, \mathrm{INV})$ be accompanied by exc and a suitable Mahonian partner.

The first discovery of a *proper* pair of equidistributed Euler-Mahonian bistatistics is only a few years old, and it came from a rather unexpected direction. Denert [4], in 1990, conjectured that the pair $(\mathrm{des}, \mathrm{MAJ})$ was equidistributed with the pair $(\mathrm{exc}, \mathrm{DEN})$, where DEN is a Mahonian statistic somewhat related to, but crucially different from, INV. Shortly afterwards, her conjecture was proved by Foata and Zeilberger [12], who named the new statistic "Denert's statistic". In the process, Foata and Zeilberger defined yet another Mahonian statistic on permutations, which they called MAK, and which, when taken together with des, they showed to be equidistributed with $(\mathrm{exc}, \mathrm{DEN})$.

It is fair to say that the discovery of Denert's statistic paved the way to the more esoteric reaches of Mahonian statistics, because it was the first such statistic to be composed of "smaller" partial statistics. Since then, many such composite Mahonian statistics have been discovered, and most of these are treated here.

The pairs of bistatistics $(\mathrm{exc}, \mathrm{DEN})$, $(\mathrm{des}, \mathrm{MAJ})$ vs. $(\mathrm{exc}, \mathrm{DEN})$, $(\mathrm{des}, \mathrm{MAK})$ were the first proper pairs of Euler-Mahonian statistics to be shown equidistributed over the symmetric group, and they are, to the best of our knowledge, the only ones preceding the present paper. It is possible to vary the definition of MAK slightly, as will be made clear later, to obtain a new statistic for permutations, and four new statistics in the case of words. However, the bistatistics obtained are all equidistributed with each other, and this is easy to show.

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In the present paper, we define some new Mahonian statistics and redefine many of the existing ones, with an eye to illuminating their common properties and thus also their differences. Doing this allows us to recover some of the known instances of equidistribution among Euler-Mahonian pairs, and to prove the equidistribution of two new pairs introduced, as well as that of some similiar, but not equal, pairs of bistatistics. Moreover, we extend all these equidistribution results to arbitrary words, and we do this simultaneously for all the statistics involved, by means of a single, simply described bijection.

All of our constructions, and some of our statistics, have appeared previously, in the work of several authors and in many different guises. They have involved Motzkin paths, binary trees, and even more exotic structures, but have so far been limited to permutations. As we will show, the bijections in the literature pertaining to these statistics, those of Foata–Zeilberger, Françon–Viennot [13], de Médicis–Viennot [18], Simion–Stanton [21] and Biane [1], defined in different ways and for different purposes, are all essentially the same, or inverses of each other. In the case of permutations these bijections are equivalent to the bijection of this paper, but their relationships with each other have not before been elucidated.

Perhaps the most interesting fact to emerge is the equidistribution of the two bistatistics (des, MAD) and (exc, INV), where MAD is one of our new statistics. The latter bistatistic, whose components are classical, is *not* equidistributed with (des, MAJ) and might therefore, together with its equidistributed mates, be classified as an "Euler-Mahonian pair of the second kind." In fact there exist at least three different *families* of Euler-Mahonian statistics. The first one, containing (des, MAJ), (des, MAK), and (exc, DEN), has been extensively studied, both analytically and bijectively. For the family containing (des, INV) and (des, IMAJ), only the analytic branch has seen substantial development (see [9]). The bijective theory of the family with (des, MAD) and (exc, INV) is thoroughly analyzed in the present paper, but its analytic properties remain to be further elicited.

It is, of course, possible to define scores of different families of Euler-Mahonian statistics by arbitrarily combining an Eulerian statistic and a Mahonian one. Although some needles are sure to be found in that haystack, most of the possible such statistics seem rather unattractive, and unlikely to possess particularly interesting properties.

An essential feature of our bijection is that it simultaneously preserves each of several building blocks of the statistics involved. This allows us to derive the equidistribution of the triples of statistics (des, MAK, MAD) and (exc, DEN, INV), involving Mahonian statistics of both the first and second kind.

In contrast to the bijections mentioned above, the bijection by Han, in [15], seems to be singular. Han proved the equidistribution of (exc, DEN) and (des, MAJ) for words. Han's bijection employs a very interesting technique, called "contextual transposition", which will no doubt resurface many times in the literature on words. However, it does not preserve all the partial statistics vital to our present construction and thus does not extend to other pairs than (exc, DEN) and (des, MAJ). In fact, Han's bijection is the only one we are aware of which proves the equidistribution of a proper pair of Euler-Mahonian statistics and which can not be suitably modified to encompass the results presented here on the statistics (exc, INV), (des, MAD) and (exc, DEN), (des, MAK).

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We prove our main results first for permutations and then extend them to arbitrary words by *coding* a word into a permutation in a way which lends insight into how the statistics on the word are affected. It is possible to prove all our results directly for words, but this involves intricate labeling schemes for the letters, and we feel that it simply obscures the presentation.

In the next section we will present the formal definitions of our statistics and state the main results. These results will be proved in sections 3, 4 and 5. In section 6 we present some variations on our statistics and in section 7 we indicate precisely the relationship between our statistics and those previously defined, and indicate extensions of our results to the case of words with large and small letters.

2 Definitions and main results

We consider words $w = a_1a_2\cdots a_n$ on a totally ordered alphabet $\mathcal{A}$. Although it is not necessary, we always take $\mathcal{A}$ to be the interval $[m] = \{1, 2, \ldots, m\}$. For any word w, we will study statistics on its *rearrangement class* $R(w)$, that is, the set of all words that can be obtained by permuting the letters of w. If the letters of w are distinct, then w is a permutation and $R(w)$ is the set of elements of the symmetric group $\mathcal{S}_n$.

Let $w = a_1a_2\cdots a_n$ be a word. The *non-decreasing rearrangement* of w is the word $\overline{w} = b_1b_2\cdots b_n$, with $b_i \leq b_{i+1}$ and $\overline{w} \in R(w)$. The *biword associated to* w is $\widetilde{w} = \begin{pmatrix}\overline{w}\\ w\end{pmatrix}$. For example, if $w = 1\ 2\ 1\ 4\ 4\ 2\ 3\ 1\ 4$ then $\widetilde{w} = \begin{pmatrix}1 & 1 & 1 & 2 & 2 & 3 & 4 & 4 & 4\\ 1 & 2 & 1 & 4 & 4 & 2 & 3 & 1 & 4\end{pmatrix}$. This notation will be adhered to throughout, that is, if $w = a_1a_2\cdots a_n$ is a word, then $\overline{w}$ and $\widetilde{w}$ have the above meaning.

Definition 1 *Let $w = a_1a_2\cdots a_n$ be a word, with $\overline{w} = b_1b_2\cdots b_n$. A* descent *in w is a triple $(i,\ a_i,\ a_{i+1})$ such that $i \in [n-1]$ and $a_i > a_{i+1}$. Here i is called the* descent place*, a_i is called the* descent top *and a_{i+1} is called the* descent bottom*. An* excedance *in w is a triple $(i,\ a_i,\ b_i)$ such that $i \in [n]$ and $a_i > b_i$. Here i is called the* excedance place*, a_i is called the* excedance top *and b_i is called the* excedance bottom*. The* number of descents in w *is denoted by* $\operatorname{des} w$*, and the* number of excedances *is denoted by* $\operatorname{exc} w$*.*

The descent set of w*, $D(w)$, is the set of descent places. The* excedance set of w*, $E(w)$, is the set of excedance places.*

Traditionally, the *descent* has been identified with the *descent place*, and the *excedance* has been identified with the *excedance place*.

As an example, if $w = 3\ 2\ 2\ 1\ 3\ 2\ 1$ then the descent places of w are 1, 3, 5, 6, and, since $\widetilde{w} = \begin{pmatrix}1 & 1 & 2 & 2 & 2 & 3 & 3\\ 3 & 2 & 2 & 1 & 3 & 2 & 1\end{pmatrix}$, the excedance places are 1, 2, 5, so $\operatorname{des} w = 4$ and $\operatorname{exc} w = 3$.

If w is a permutation, an excedance place is simply an i such that $a_i > i$.

Given a word $w = a_1a_2\cdots a_n$, we separate w into its *descent blocks* by putting in dashes between a_i and a_{i+1} whenever $a_i \leq a_{i+1}$. A maximal contiguous subword of w which lies between two dashes is a descent block. A descent block is an *outsider* if it has only one letter, otherwise it is a *proper* descent block. The leftmost letter of a proper descent block is its *closer* and the rightmost letter is its *opener*. A letter which lies strictly inside a descent block is an *insider*. For example, the word 1 3 3 2 1 2 5 3 2 4 1 1 has descent block decomposition $1-3-3\ 2\ 1-2-5\ 3\ 2-4\ 1-1$, with closers $3,5,4$, corresponding openers $1,2,1$, outsiders $1,3,2,1$ and insiders $2,3$. We will frequently write a word w with its separating dashes to emphasize this structure.

Let B be a proper descent block of the word w and let $\mathrm{C}(B)$ and $\mathrm{O}(B)$ be the closer and opener, respectively, of B. If a is a letter of w, we say that *a is embraced by B* if $\mathrm{C}(B) \geq a > \mathrm{O}(B)$.

Definition 2 *The* (right) embracing numbers *of a word $w = a_1 a_2 \cdots a_n$ are the numbers $e_1, e_2, \ldots, e_n$ where e_i is the number of descent blocks in w that are strictly to the right of a_i and that embrace a_i. The* right embracing sum *of w, denoted by* $\mathrm{Res}\, w$*, is defined by*

$$\mathrm{Res}\, w = e_1 + e_2 + \cdots + e_n.$$

For instance, the embracing numbers of $w = 2\ 1-3-4\ 1-2-2\ 1$ are $2\ 0-1-0\ 0-1-0\ 0$, so $\mathrm{Res}\, w = 4$.

One can obviously define $\mathrm{Les}\, w$ in an analogous way, by simply replacing "right" by "left" in the above definition. (See section 6.)

Let $w = a_1 a_2 \cdots a_n$. The *height* of a letter a in w, $h_w(a)$ is one more than the number of letters in w that are strictly smaller than a. The (left) *value* of the i-th letter in π, denoted by $v_w(i)$, is defined by

$$v_w(i) = h_w(a_i) + \ell_w(i),$$

where $\ell_w(i)$ is the number of letters in w that are to the left of a_i and equal to a_i. Thus, the height of two identical letters in a word w is always the same, whereas the value of a letter depends on the letter's position in w, relative to other equal letters. For example, given $w = 1\ 2\ 1\ 4\ 4\ 2\ 3\ 1\ 4$, we have $\overline{w} = 1\ 1\ 1\ 2\ 2\ 3\ 4\ 4\ 4$, so the heights of $1,2,3,4$ are, respectively, $1,4,6,7$. The values of the letters of w are given by $1,4,2,7,8,5,6,3,9$, in the order in which they appear in w. Thus, replacing each letter of a word w by its value yields a permutation v_w which has the same number of inversions and descents as w does. If w is a permutation, then $v_w(i) = h_w(a_i) = a_i$.

Definition 3 *The* descent bottoms sum *of a word $w = a_1 a_2 \cdots a_n$, denoted by* $\mathrm{Dbot}\, w$*, is the sum of the values of the descent bottoms of w. The* descent tops sum *of w, denoted by* $\mathrm{Dtop}\, w$*, is the sum of the heights of the descent tops of w. The* descent difference *of w is*

$$\mathrm{Ddif}\, w = \mathrm{Dtop}\, w - \mathrm{Dbot}\, w.$$

As an example, $w = 2\,1\!-\!1\!-\!3\,2\,1\!-\!3$ has $\widetilde{w} = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 1 & 1 & 3 & 2 & 1 & 3 \end{pmatrix}$, so the heights of 1, 2 and 3 are 1, 4 and 6, respectively, and the values of the letters of w are given by $4\;1-2-6\;5\;3-7$. Consequently, $\mathrm{Dbot}\, w = v_w(2) + v_w(5) + v_w(6) = 1 + 5 + 3 = 9$, $\mathrm{Dtop}\, w = h_w(2) + h_w(3) + h_w(2) = 4 + 6 + 4 = 14$ and $\mathrm{Ddif}\, w = 14 - 9 = 5$.

It is easy to see that i is a descent place if and only if $h_w(a_i) > v_w(i+1)$. It follows that $\mathrm{Dtop}\, w \geq \mathrm{Dbot}\, w$, with equality only if there are no descents in w. If w is a permutation, then Dtop and Dbot are simply the sums of the descent tops and the descent bottoms, respectively. Thus, for a permutation π, $\mathrm{Ddif}\, \pi$ is the sum of closers minus the sum of openers of descent blocks.

Definition 4 *The* excedance bottoms sum *of a word $w = a_1 a_2 \cdots a_n$, denoted by* $\mathrm{Ebot}\, w$*, is the sum of the values, in the word $\bar{w}$, of the excedance bottoms of w. The* excedance tops sum *of w, denoted by* $\mathrm{Etop}\, w$*, is the sum of the heights of the excedance tops of w. The* excedance difference *of w is*

$$\mathrm{Edif}\, w = \mathrm{Etop}\, w - \mathrm{Ebot}\, w.$$

The excedance subword of w*, denoted by w_{E}, is the word consisting of all the excedance tops of w, in the order induced by w. The* non-excedance subword of w *is denoted by w_{N}, and consists of those letters of w that are not excedance tops.*

Notice that $v_{\bar{w}}(i) = i$ for all i. Thus the value of an excedance bottom equals the corresponding excedance place, and we could equally well define $\mathrm{Ebot}\, w$ to be the sum of the excedance places of w.

For example, if $w = 3\;2\;2\;1\;3\;2\;1$, so $\widetilde{w} = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 3 & 2 & 2 & 1 & 3 & 2 & 1 \end{pmatrix}$, then $w_{\mathrm{E}} = 3\;2\;3$ and $w_{\mathrm{N}} = 2\;1\;2\;1$.

The following lemma is straightforward to prove.

Lemma 1 *The integer i is an excedance place in $w = a_1 a_2 \cdots a_n$ if and only if $h_w(a_i) > i$.* $\square$

Definition 5 *An* inversion *in a word $w = a_1 a_2 \cdots a_n$ is a pair (i, j) such that $i < j$ and $a_i > a_j$. An* i*m*version *is a weak inversion, that is, a pair (i, j) such that $i < j$ and $a_i \geq a_j$. The* number of i*m*versions in w *is denoted by* $\mathrm{Imv}\, w$*. The* number of inversions in w *is denoted by* $\mathrm{INV}\, w$*.*

Note that we spell i*m*version with an italicized m to avoid confusion with the word "inversion". The reason we spell INV, but not Imv, with all capital letters is that INV is a Mahonian statistic, whereas Imv is not. We do this consistently throughout the paper, that is, all Mahonian statistics are spelled with uppercase letters. The two Eulerian statistics, exc and des, are spelled with lowercase letters, while "partial statistics" (such as Imv and Res), used in the definitions of Mahonian statistics, are merely capitalized.

Definition 6 *Let $w = a_1 a_2 \cdots a_n$ be a word and i an excedance place in w. We say that a_i* is the bottom of d i*m*versions *if there are exactly d letters in w to the left of a_i that are greater than or equal to a_i, and we call d the* i*m*version bottom number of i*. Similarly, if i is a non-excedance place in w and there are exactly d letters smaller than a_i and to the right of a_i in τ, then we say that d is the* inversion top number of i*. The* side number of a_i in w *is the imversion bottom number or the inversion top number of a_i in w, according as i is an excedance place or not in w. The* sequence of side numbers of w *is the sequence $s_1, s_2, \ldots, s_n$ where s_i is the side number of i.*

For example, let

$$w = 1\ 3\ 2\ 3\ 2\ 3\ 2\ 1\ 1.$$

Then $w_{\mathrm{E}} = 3\ 2\ 3\ 3$, so the i*m*version bottom numbers of the excedances in w are 0, 1, 1, 2. Also, $w_{\mathrm{N}} = 1\ 2\ 2\ 1\ 1$, so the inversion top numbers of the non-excedances in w are 0, 2, 2, 0, 0. Hence the sequence of side numbers of w is 0, 0, 1, 1, 2, 2, 2, 0, 0.

Note that if i is an excedance place of the word w, then any letter in w that is to the left of a_i and greater than or equal to a_i must also represent an excedance. Hence, the sum of the i*m*version bottom numbers of the letters in w_{E} equals the total number of imversions in w_{E}, that is, $\mathrm{Imv}\, w_{\mathrm{E}}$. Similarly, the sum of the inversion top numbers of the letters in w_{N} equals $\mathrm{INV}\, w_{\mathrm{N}}$.

Definition 7 *Let w be a word. Then*

$$\operatorname{Ine} w = \operatorname{Imv} w_{\mathrm{E}} + \mathrm{INV}\, w_{\mathrm{N}}.$$

Hence, from the remark preceding definition 7, we have

$$\operatorname{Ine} w = s_1 + \cdots + s_n. \tag{1}$$

We now define the four Mahonian statistics central to this paper.

Definition 8

$$\begin{aligned}
\mathrm{MAK}\, w &= \operatorname{Dbot} w \;+\; \operatorname{Res} w.\\
\mathrm{MAD}\, w &= \operatorname{Ddif} w \;+\; \operatorname{Res} w.\\
\mathrm{DEN}\, w &= \operatorname{Ebot} w \;+\; \operatorname{Ine} w.\\
\mathrm{ENV}\, w &= \operatorname{Edif} w \;+\; \operatorname{Ine} w.
\end{aligned}$$

As it turns out, our statistic ENV equals the classical INV. It may seem superfluous to redefine INV in this way, but it turns out that ENV's similarity in definition to MAD is crucial in proving our main results.

Let $w = a_1 \cdots a_n$. Since $\operatorname{Edif} w = \sum_{i \in E(w)} (h(a_i) - i)$, we have

$$\mathrm{ENV}\, w = \operatorname{Imv} w_{\mathrm{E}} + \mathrm{INV}\, w_{\mathrm{N}} + \sum_{i \in E(w)} (h(a_i) - i),$$

which clearly leads to

$$\mathrm{ENV}\, w = \mathrm{INV}\, w_{\mathrm{E}} + \mathrm{INV}\, w_{\mathrm{N}} + \sum_{i \in E(w)} (v(i) - i). \tag{2}$$

Theorem 2 *For any word $w = a_1 a_2 \cdots a_n$, we have $\mathrm{ENV}\, w = \mathrm{INV}\, w$.*

Proof: By equation (2) and the definition of INV, it suffices to show that

$$\sum_{i \in E(w)} (v(i) - i) = \#\{\, (i,j) \mid i < j, a_i > a_j, i \in E(w), j \notin E(w) \,\}.$$

The right hand side equals

$$\sum_{i \in E(w)} \#\{\, j \mid i < j, a_i > a_j, j \notin E(w) \,\},$$

which, in turn, is equal to

$$\sum_{i\in E(w)} \left(\#\{\, j \mid i<j, a_i>a_j\,\} - \#\{\, j \mid i<j, a_i>a_j, j\in E(w)\,\}\right). \qquad (3)$$

Now,

$$\begin{aligned} v(i)-i &= v(i) - \#\{\, j \mid j\le i\,\} \\ &= \#\{\, j \mid a_j<a_i\,\} - \#\{\, j \mid j<i, a_j\ne a_i\,\} \\ &= \#\{\, j \mid j>i, a_j<a_i\,\} - \#\{\, j \mid j<i, a_j>a_i\,\}. \qquad (4)\end{aligned}$$

Hence, comparing (3) and (4), we must show that

$$\sum_{i\in E(w)} \#\{\, j \mid i<j, a_i>a_j, j\in E(w)\,\} = \sum_{i\in E(w)} \#\{\, j \mid j<i, a_j>a_i\,\}. \qquad (5)$$

Clearly, each of the sums in equation (5) is $\mathrm{INV}\, w_{\mathrm{E}}$. $\square$

We now describe the main results of this paper.

In section 3 we will define a mapping Φ on $\mathcal{S}_n$ and prove the following result.

Proposition 3 *For any permutation π, we have*

$$\begin{aligned} (\mathrm{des}, \mathrm{Dbot}, \mathrm{Ddif}, \mathrm{Res})\ \pi &= (\mathrm{exc}, \mathrm{Ebot}, \mathrm{Edif}, \mathrm{Ine})\ \Phi(\pi), \\ (\mathrm{des}, \mathrm{MAD}, \mathrm{MAK})\ \pi &= (\mathrm{exc}, \mathrm{INV}, \mathrm{DEN})\ \Phi(\pi). \end{aligned}$$

By showing that Φ is a bijection, we deduce the following theorem.

Theorem 4 *The quadristatistics*

$$(\mathrm{des}, \mathrm{Dbot}, \mathrm{Ddif}, \mathrm{Res}) \quad \textit{and} \quad (\mathrm{exc}, \mathrm{Ebot}, \mathrm{Edif}, \mathrm{Ine})$$

are equidistributed over the symmetric group $\mathcal{S}_n$. That is,

$$\sum_{\pi\in\mathcal{S}_n} t^{\mathrm{des}\,\pi} x^{\mathrm{Dbot}\,\pi} y^{\mathrm{Ddif}\,\pi} q^{\mathrm{Res}\,\pi} = \sum_{\pi\in\mathcal{S}_n} t^{\mathrm{exc}\,\pi} x^{\mathrm{Ebot}\,\pi} y^{\mathrm{Edif}\,\pi} q^{\mathrm{Ine}\,\pi}.$$

Hence the triple $(\mathrm{des}, \mathrm{MAK}, \mathrm{MAD})$ *is equidistributed with* $(\mathrm{exc}, \mathrm{DEN}, \mathrm{INV})$ *over* $\mathcal{S}_n$.

In section 4, we will extend these results to words. In fact we will prove the following result.

Theorem 5 *For any word v there is a bijection Φ_{W} on the rearrangement class $R(v)$ of v such that, for any word $w = a_1a_2\cdots a_n \in R(v)$, we have*

$$(\mathrm{des}, \mathrm{MAD}, \mathrm{MAK})\ w = (\mathrm{exc}, \mathrm{INV}, \mathrm{DEN})\ \Phi_{\mathrm{W}}(w).$$

In section 5, we shall make evident the relation between our bijection Φ and some well-known bijections between the symmetric group $\mathcal{S}_n$ and weighted Motzkin paths. As a by-product, we will obtain a continued fraction expansion (equation 16) for the ordinary generating function of

$$D_n(x,q) = \sum_{\pi\in\mathcal{S}_n} x^{\mathrm{des}\,\pi} q^{\mathrm{MAD}\,\pi}.$$

3 The bijection Φ for permutations

Before proving Proposition 3 and Theorem 4, we describe the construction of a bijection $\Phi : \mathcal{S}_n \to \mathcal{S}_n$ which takes a permutation π to a permutation τ such that the set of descent tops in π equals the set of excedance tops in τ and the set of descent bottoms in π equals the set of excedance bottoms in τ. Moreover, the embracing numbers of π are preserved in a way that we now describe.

Observe that, since the letters of a permutation are distinct, we can refer to the i-th embracing number e_i of the permutation π as the embracing number of the *letter* a_i in π, and we will then denote e_i by $e(a_i)$. Similarly, we may if we wish denote the i-th side number of π by $d(a_i)$.

We will construct $\tau = \Phi(\pi)$ in such a way that the embracing number of a letter a_i in π is the side number of a_i in τ.

Given a permutation π, we first construct two biwords, $\begin{pmatrix} f \\ f' \end{pmatrix}$ and $\begin{pmatrix} g \\ g' \end{pmatrix}$, and then form the biword $\tau' = \begin{pmatrix} f\ g \\ f'\ g' \end{pmatrix}$ by concatenating f and g, and f' and g', respectively. The word f is defined as the subword of descent bottoms in π, ordered increasingly, and g is defined as the subword of non-descent bottoms in π, also ordered increasingly. The word f' is the subword of descent tops in π, ordered so that the i*m*version bottom number of a letter a in f' is the embracing number of a in π, and g' is the subword of non-descent tops in π, ordered so that the inversion top number of a letter b in g' is the embracing number of b in π. Rearranging the columns of τ', so that the top row is in increasing order, we obtain the permutation $\tau = \Phi(\pi)$ as the bottom row of the rearranged biword.

Example 1 Let $\pi = 4\ 1 - 2 - 7 - 9\ 6\ 5 - 8\ 3$, with embracing numbers 1, 0, 0, 2, 0, 1, 1, 0, 0. Then

$$\binom{f}{f'} = \binom{1\ 3\ 5\ 6}{8\ 4\ 6\ 9}, \quad \binom{g}{g'} = \binom{2\ 4\ 7\ 8\ 9}{1\ 2\ 7\ 5\ 3}, \quad \tau' = \binom{1\ 3\ 5\ 6\ 2\ 4\ 7\ 8\ 9}{8\ 4\ 6\ 9\ 1\ 2\ 7\ 5\ 3}$$

and thus $\Phi(\pi) = \tau = 8\ 1\ 4\ 2\ 6\ 9\ 7\ 5\ 3$.

It is easily checked that the descent tops and descent bottoms in π are the excedance tops and excedance bottoms in τ, respectively, and that the embracing number of each letter in π is the side number of the same letter in τ.

Proof of Proposition 3: Assuming that the construction of f' and g' can be carried out in the way described, and such that $f' = \tau_{\mathrm{E}}$ and $g' = \tau_{\mathrm{N}}$, it is clear that the excedance tops and excedance bottoms in τ are the descent tops and descent bottoms, respectively, in π, and that

$$\mathrm{Res}\,\pi = \mathrm{Imv}\,\tau_{\mathrm{E}} + \mathrm{INV}\,\tau_{\mathrm{N}} = \mathrm{Ine}\,\tau.$$

As a consequence, we have

$$(\mathrm{des}, \mathrm{Dbot}, \mathrm{Ddif}, \mathrm{Res})\ \pi = (\mathrm{exc}, \mathrm{Ebot}, \mathrm{Edif}, \mathrm{Ine})\ \Phi(\pi).$$

To complete the proof, we need to show two things. Firstly, that f' and g' can be constructed so that the i*m*version bottom numbers and the inversion top numbers of f' and g' respectively are those claimed, and, secondly, that $f' = \tau_{\mathrm{E}}$ (and thus $g' = \tau_{\mathrm{N}}$).

Let a be the least descent top in π. Then, if the embracing number of a in π is k, there are k descent blocks in π to the right of a that embrace a. Thus, there are at least k descent tops in π that are larger than a, namely the closers of the descent blocks embracing a. Also, there are at least $k+1$ descent bottoms in π that are smaller than a, namely the openers of the descent blocks embracing a, together with the opener of the descent block containing a. If we put a in the $(k+1)$–st place in f' from the left, then the *im*version bottom number of a in f' is k as desired, and the $(k+1)$–st place *does exist* in f', because, as pointed out, there are at least $(k+1)$ descent bottoms, and thus at least $(k+1)$ descent tops, in π if a has embracing number k.

Moreover, the same argument shows that a is larger than the $(k+1)$–st letter in f, because the first $(k+1)$ letters in f are descent bottoms in π that are smaller than a. Hence, a is an excedance in τ. If we now remove the letter a from f' and its corresponding descent bottom, the $(k+1)$–st letter of f, from f, then we can repeat the argument, appealing to induction, to show that f' can be constructed in the way described. That is, so that each letter x in f' is an excedance top in τ whose *im*version bottom number equals the embracing number of x in π. An analogous argument shows that g' can be constructed as claimed, and so that each letter in g' is not an excedance top in τ. $\square$

In order to prove Theorem 4, we must show that Φ is a bijection. Since Φ is a map from a finite set to itself, it suffices to show Φ injective, but in the process we will, in fact, construct an inverse to Φ.

We introduce the idea of the *skeleton* of a permutation. This is closely related to the idea of "gravid permutation" introduced by Foata and Zeilberger [7]. We first adjoin to the positive integers a symbol ∞ such that $a < \infty$ for any positive integer a.

Definition 9 *Let n be a positive integer. A* block *is a subset B of $[n] \cup \infty$ such that $B \cap [n] \neq \emptyset$. The block B is called* open *if $\infty \in B$,* closed *if $\infty \notin B$ and* improper *if $|B| = 1$. The* opener*, $\mathrm{O}(B)$, of B is the smallest element of B, the* closer*, $\mathrm{C}(B)$, of B is the largest element of B.*

A skeleton *is a sequence $S = B_1 - B_2 - \ldots - B_r$ of blocks such that any pair of blocks intersect in at most $\{\infty\}$. The skeleton S is* valid *if for each i with $1 \leq i < n$ we have $\mathrm{O}(B_i) < \mathrm{C}(B_{i+1})$.*

Definition 10 *Let $\pi \in \mathcal{S}_n$ be a permutation with descent block decomposition $B_1 - B_2 - \cdots - B_r$. Let $1 \leq a \leq n$. The a-*skeleton *of π is the sequence of blocks obtained by*

- *deleting any descent block B of π for which $\mathrm{O}(B) > a$;*

- *replacing any remaining letter of π that is greater than a by ∞;*
- *replacing each remaining descent block (which is a* sequence *of elements) by its underlying set.*

For example, if $\pi = 3\ 1 - 6 - 7\ 5 - 9\ 8\ 4\ 2$, the 4-skeleton of π is the sequence $\{1, 3\} - \{2, 4, \infty\}$, which we will write as $3\ 1 - \infty\ 4\ 2$.

It is clear that one can recover π from its n-skeleton.

Lemma 6 *Let $\pi \in \mathcal{S}_n$. For any a with $1 \leq a \leq n$, the a-skeleton of π is valid.*

Proof: We use downwards induction on a, knowing that the result is true for the n-skeleton. Suppose it is true for a. To obtain the $(a-1)$-skeleton from the a-skeleton, we merely replace a by ∞ in the block B in which it occurs and delete that block if it now contains only ∞. But as the a-skeleton is valid and a is the largest finite element occuring in it, a block $\{a, \infty\}$ or $\{a\}$ can only occur as the last block in the a-skeleton, in which case its deletion will not cause invalidity. $\Box$

The following result is easy to see.

Lemma 7 *The embracing number of the letter a in π equals the number of open blocks to the right of the block containing a in the a-skeleton of π.* $\Box$

Proof of Theorem 4: We show that Φ is an injection by showing that, given a permutation τ in the image of Φ, there is at most one permutation π such that $\Phi(\pi) = \tau$. Given such a τ, it is clear what the associated biword $\begin{pmatrix} f\ g \\ f'\ g' \end{pmatrix}$ must be. Namely, $\begin{pmatrix} f \\ f' \end{pmatrix}$ consists of those columns of $\tilde{\tau}$ that represent excedances in τ, and $\begin{pmatrix} g \\ g' \end{pmatrix}$ consists of the remaining columns of $\tilde{\tau}$. Now denote by F, F', G and G' the sets whose elements are the letters of f, f', g and g' respectively. Then we can identify the openers of π as the letters in $F \cap G'$, the closers as the letters in $F' \cap G$, the insiders as those in $F \cap F'$ and the outsiders as those in $G \cap G'$. As we can calculate the side numbers of τ, we know the embracing numbers of π. We will show how to construct successively the 1-, 2-, $\ldots$, n-skeletons of π.

The 1-skeleton of π is either $\{1\}$ or $\{1, \infty\}$, according as 1 is an outsider or an opener. Suppose that the $(a-1)$-skeleton $S = B_1 - B_2 - \ldots - B_r$ of π has been constructed. To construct the a-skeleton of π we must insert a in the correct place in S. Let the embracing number of a in π be e. Let B_i be the $(e+1)$-st open block from the right in S. If a is an insider or a closer in π then, by Lemma 7, a must be inserted into B_i, and if a is a closer then B_i must be closed by the removal of its ∞. Suppose that a is an outsider. Then the improper block $B = \{a\}$ must be inserted immediately to the left of B_i. For if B is inserted to the left of B_{i-1} then either the resulting skeleton will be invalid or Lemma 7 will be violated. Similarly, if a is an opener, the open block $\{a, \infty\}$ must be inserted immediately before B_i.

After the n-skeleton of π has been constructed, we can immediately construct π. Hence there is at most one permutation π such that $\Phi(\pi) = \tau$. Hence, Φ is injective, and so a bijection, and its inverse is defined by the construction just described.

The equidistribution of (des, Dbot, Ddif, Res) and (exc, Ebot, Edif, Ine) now follows from Proposition 3 and the fact that Φ is a bijection. As $\mathrm{INV} = \mathrm{ENV}$, the equidistribution of (des, MAK, MAD) with (exc, DEN, INV) follows from the definitions of the Mahonian statistics involved, since each is the sum of two of the partial statistics Dbot, Ddif, Res and Ebot, Edif, Ine, respectively. $\square$

Example 2 Let $\tau = 8\ 1\ 4\ 2\ 6\ 9\ 7\ 5\ 3$, so

$$\binom{f}{f'} = \binom{1\ 3\ 5\ 6}{8\ 4\ 6\ 9}, \quad \binom{g}{g'} = \binom{2\ 4\ 7\ 8\ 9}{1\ 2\ 7\ 5\ 3}, \quad \tau' = \binom{1\ 3\ 5\ 6\ 2\ 4\ 7\ 8\ 9}{8\ 4\ 6\ 9\ 1\ 2\ 7\ 5\ 3}.$$

For clarity, we now rewrite τ' with a bar separating f from g and f' from g', and we write the imversion bottom and inversion top numbers in f' and g' respectively as subscripts of their corresponding letters, omitting those that are zero.

$$\tau' = \begin{pmatrix} f & | & g \\ f' & | & g' \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 & 6 & | & 2 & 4 & 7 & 8 & 9 \\ 8 & 4_1 & 6_1 & 9 & | & 1 & 2 & 7_2 & 5_1 & 3 \end{pmatrix}.$$

The sequence of k-skeletons, for $k = 1, 2, \ldots, 9$, of our required permutation

π is:

$$\begin{aligned}
&\infty\ 1;\\
&\infty\ 1-2;\\
&\infty\ 1-2-\infty\ 3;\\
&4\ 1-2-\infty\ 3;\\
&4\ 1-2-\infty\ 5-\infty\ 3;\\
&4\ 1-2-\infty\ 6\ 5-\infty\ 3;\\
&4\ 1-2-7-\infty\ 6\ 5-\infty\ 3;\\
&4\ 1-2-7-\infty\ 6\ 5-8\ 3;\\
&4\ 1-2-7-9\ 6\ 5-8\ 3.
\end{aligned}$$

Hence

$$\pi = 4\ 1-2-7-9\ 6\ 5-8\ 3.$$

4 The extension to words of the bijection Φ

To extend the bijection Φ of the previous section to the rearrangement class of an arbitrary *word*, we *code* each word $w = a_1a_2\cdots a_n$ by the permutation $\pi = v_w = x_1x_2\cdots x_n$, where $x_i = v_w(i)$. For example, the word 2 1 3 3 2 1 2 is coded by 3 1 6 7 4 2 5. We then apply Φ to the permutation π to get a permutation π' such that $(\mathrm{des}, \mathrm{Dbot}, \mathrm{Ddif}, \mathrm{Res})\pi = (\mathrm{exc}, \mathrm{Ebot}, \mathrm{Edif}, \mathrm{Ine})\pi'$. Finally, we *decode* π' by replacing each letter x of π' by a letter a_i of w such that $v_w(i) = x$. (We point out that in general this "decoding" map is *not* the inverse of the coding map that replaces a_i by $v_w(i)$.)

Let $w = a_1a_2\cdots a_n$ be a word with descent set $D(w) = \{i_1,\ldots,i_s\}$ and embracing numbers $e_1,\ldots,e_n$. Let $\pi = x_1\cdots x_n$ be the coding of w described above. Thus $x_i < x_j$ if and only if either $a_i < a_j$ or $a_i = a_j$ with $i < j$. Let $\pi' = \Phi(\pi) = x_1'\cdots x_n'$. Finally, let $w' = a_1'\cdots a_n'$ be the decoding of π', where θ is the decoding map, so $a_i' = \theta(x_i')$. We define Φ_{W} on $R(\overline{w})$ by setting $\Phi_{\mathrm{W}}(w) = w'$.

Proof of Theorem 5: It is clear that $D(\pi) = D(w)$, so that the set of descent bottoms of π is $\{x_{i_1+1},\ldots,x_{i_s+1}\}$, and the set of descent tops of π is

$\{x_{i_1}, \ldots, x_{i_s}\}$. Furthermore, the sequence of embracing numbers of π equals the sequence of embracing numbers of w.

Now, if the letters a_i and a_j are equal for distinct i and j, then the following claims are easily checked:

i) If $e_i > e_j$ then $i < j$.

ii) If $e_i = e_j$ and $i \in D(w)$, $j \notin D(w)$ then $i < j$.

iii) If $i, j \in D(w)$ then $e_i \neq e_j$. In fact, if $i < j$ then $e_i > e_j$.

In the first two cases, for the corresponding letters of π we have $x_i < x_j$.

As in the previous section, we refer to the embracing numbers e_i of π as $e(x_i)$.

Let the biwords $\beta = \begin{pmatrix} f \\ f' \end{pmatrix}$ and $\gamma = \begin{pmatrix} g \\ g' \end{pmatrix}$ be as in the definition of Φ for permutations. Then, from the previous section, the set F' of letters of f', which is the set of excedance bottoms of π', equals the set of descent bottoms of π, and the set G' of letters of g', which is the set of excedance tops of π', equals the set of descent tops of π. Moreover, the side number of a letter x in π' equals the embracing number of x in π. Suppose now that the same letter a occurs in postions i and j of w', that is, that $\theta(x'_i) = \theta(x'_j) = a$. The following are immediate consequences of i-iii) above.

iv) If $e(x'_i) > e(x'_j)$ then $x'_i < x'_j$.

v) If $e(x'_i) = e(x'_j)$ and $x'_i \in F'$, $x'_j \in G'$ then $x'_i < x'_j$.

vi) If $e(x'_i) = e(x'_j)$ and $x'_i, x'_j \in G'$ with $i < j$ then also $x'_i < x'_j$ (for otherwise (i, j) would be an inversion in π, so that $e(x'_i) > e(x'_j)$.)

vii) If $x'_i, x'_j \in F'$ then $e(x'_i) \neq e(x'_j)$. In fact, if $x'_i < x'_j$ then $e(x'_i) > e(x'_j)$.

We claim that the sets of excedance places of w' and π' are equal. It is easy to check that a non-excedance place in π' is a non-excedance place in w'. Thus we need only show that no excedances "disappear" when going from π' to w', that is, that every column of the biword $\theta(\beta) = \begin{pmatrix} \theta(f) \\ \theta(f') \end{pmatrix}$ represents an excedance, where $\theta(f)$ is the word obtained by applying θ to each letter of f. Let the p-th entry of f' be the descent top x_j of π. Then

$$p \leq 1 + e_j + \#\{\, i \in D(w) \mid a_i < a_j \,\},$$

by vii) above. But the number q of letters in $\theta(f)$ strictly less than a_j satisfies

$$q \geq 1 + e_j + \#\{\, i \in D(w) \mid a_i < a_j \,\}. \tag{6}$$

For let B be one of the e_j descent blocks to the right of a_j in w that embrace a_j. Let a_k be the largest letter in B that is less than a_j. Then a_k is a descent bottom in w that is less than a_j. In addition, for each descent top a_i of w which is less than a_j, there is a descent bottom in w which is less than a_j. But as a_j itself is a descent top, its corresponding descent bottom a_{j+1} is less than a_j. Hence inequality (6) follows, so the p-th column of $\theta(\beta)$ represents an excedance, as claimed.

Now we show that the side number of a letter x'_i in π' equals the side number of the corresponding letter $a = \theta(x'_i)$ in w'. For let x'_i and x'_j be letters of f' such that $\theta(x'_i) = \theta(x'_j) = a$. We may assume that $i < j$ and that there is no letter x'_k in f, with $i < k < j$, such that $\theta(x'_k) = a$. Then by vii) above, $e(x'_i) \neq e(x'_j)$ and, since $e(x'_i)$ is the i*m*version bottom number of x'_i in f' and x'_i occurs to the left of x'_j in f', we must have $x'_i > x'_j$. Hence, for every pair of equal letters in $\theta(f')$, there is an inversion between the corresponding letters of f'. Therefore the i*m*version bottom number of a in w' equals the i*m*version bottom number of x'_i in π'. Similarly, from vi) above it follows that if x'_i is a letter of g' then the inversion top number of $\theta(x'_i)$ in w' equals the inversion top number of x'_i in π'.

Thus

$$\operatorname{Ine} w' = \operatorname{Ine} \pi' = \operatorname{Res} \pi = \operatorname{Res} w.$$

Since the set of excedance tops and the set of excedance bottoms of w' equal, respectively, the set of descent tops and the set of descent bottoms of w, we have

$$(\mathrm{des}, \mathrm{MAD}, \mathrm{MAK})\ w = (\mathrm{exc}, \mathrm{ENV}, \mathrm{DEN})\ \Phi_{\mathrm{W}}(w).$$

To complete the proof of theorem 5, we need only show that Φ_{W} is an injection. Suppose that for words w_1 and w_2 we have $w_k \mapsto \pi_k \mapsto \pi'_k \mapsto w'$

for $k=1,2$. Let us refer to the sets and words involved in the construction of $\Phi_{\mathrm{W}}(w_k)$ by F_k, f_k, G_k, etc., for $k=1,2$. Suppose that $E(w')=E(\pi'_k)=\{i_1,\ldots,i_r\}$. Then $F_k=\{i_1,\ldots,i_r\}$ and G_k is the complementary set. Hence the words f_k and g_k can be determined. Further, f'_k is an inverse image under θ of the word $a'_{i_1}\cdots a'_{i_r}$, and we have a similar result for g'_k. We can compute the side numbers of π'_k, as these equal the side numbers of w'. Now we can recover f'_k and g'_k. For, suppose that the letter a occurs only in positions $j_1,\ldots,j_s$ of w'. Then from $e(x'_{j_1}),\ldots,e(x'_{j_s})$ and iv-vii) above, we can find $x'_{j_1},\ldots,x'_{j_s}$. Hence $\pi'_1=\pi'_2$. As Φ_{W} is injective on permutations, we have $\pi_1=\pi_2$, so that $w_1=w_2$. $\square$

Note that we can explictly describe an inverse to Φ_{W} with the aid of the inverse to Φ for the case of permutations and the proof of the preceding theorem.

Example 3 Consider the word

$$w = 1\ 3\ 2\ 2\ 3\ 2\ 1\ 3\ 1.$$

The coding of w gives

$$\pi = 1\ 7\ 4\ 5\ 8\ 6\ 2\ 9\ 3.$$

Now,

$$\pi' = \Phi(\pi) = 1\ 9\ 6\ 8\ 4\ 7\ 5\ 2\ 3.$$

Thus

$$w' = \Phi_{\mathrm{W}}(w) = 1\ 3\ 2\ 3\ 2\ 3\ 2\ 1\ 1.$$

Conversely, given the word

$$w' = \Phi_{\mathrm{W}}(w) = 1\ 3\ 2\ 3\ 2\ 3\ 2\ 1\ 1,$$

consider the biword

$$\begin{pmatrix} \overline{w'} \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 3 & 2 & 3 & 2 & 3 & 2 & 1 & 1 \end{pmatrix}.$$

The excedance places of w' are 2, 3, 4 and 6. Hence

$$\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 & 6 \\ 3' & 2' & 3'' & 3''' \end{pmatrix}$$

and

$$\binom{g}{g'} = \begin{pmatrix} 1 & 5 & 7 & 8 & 9 \\ 1' & 2'' & 2''' & 1'' & 1''' \end{pmatrix},$$

where $1'$, $1''$, $1'''$ are inverse images of 1, etc. Now

$$1' = 1,\ 1'' = 2,\ 1''' = 3,$$

since the corresponding side numbers satisfy $s(1') = s(1'') = s(1''') = 0$. Also,

$$2'' = 4,\ 2''' = 5,\ 2' = 6,$$

since $s(2'') = s(2''') = 2,\ s(2') = 1$, and

$$3''' = 7,\ 3'' = 8,\ 3' = 9,$$

since $s(3''') = 2,\ s(3'') = 1,\ s(3') = 0$. Hence we have

$$\pi' = \Phi(\pi) = 1\ 9\ 6\ 8\ 4\ 7\ 5\ 2\ 3$$

and we can recover π, and thus w, as before.

5 Motzkin paths and a continued fraction expansion

In this section we shall make evident the relation between our bijection Φ and some well-known bijections between the symmetric group $\mathcal{S}_n$ and weighted Motzkin paths. As a by-product we get the continued fraction expansion for the generating function of $\mathcal{S}_n$ with respect to some of our statistics.

A *Motzkin path*, informally, is a connected sequence of n line segments, or "steps," in the first quadrant of R^2, starting out from the origin in R^2 and ending at $(0, n)$. The steps are of four different types, northeast steps (N) going from (a, b) to $(a+1, b+1)$, southeast (S) going from (a, b) to $(a+1, b-1)$ and solid/dotted east steps (E,dE), from (a, b) to $(a+1, b)$ (see Figure 1). Formally, a Motzkin path is defined as follows.

Definition 11 *A* Motzkin path *is a word $w = c_1 c_2 \cdots c_n$ on the alphabet $\{\mathrm{N}, \mathrm{S}, \mathrm{E}, \mathrm{dE}\}$ such that for each i the* level h_i of the i-th step*, defined by*

$$h_i = \#\{j | j < i, c_j = \mathrm{N}\} - \#\{j | j < i, c_j = \mathrm{S}\},$$

is non-negative, and equal to zero if $i = n$.

Definition 12 *A* weighted Motzkin path of length n *is a pair* (c, d)*, where* $c = c_1 \cdots c_n$ *is a Motzkin path of length* n*, and* $d = (d_1, \ldots, d_n)$ *is a sequence of integers such that*

$$0 \leq d_i \leq \begin{cases} h_i & \textit{if } c_i \in \{\mathrm{N}, \mathrm{E}\}, \\ h_i - 1 & \textit{if } c_i \in \{\mathrm{S}, \mathrm{dE}\}. \end{cases}$$

The set of weighted Motzkin paths of length n *is denoted by* Γ_n*.*

Françon and Viennot [13] gave the first bijection Ψ_{FV} between $\mathcal{S}_n$ and Γ_n. Here we describe one *variant* of this bijection.

Definition 13 *Let* $\pi = a_1 \cdots a_n \in \mathcal{S}_n$ *and set* $a_0 = 0$ *and* $a_{n+1} = n + 1$*. For* $1 \leq i \leq n$ *we say that* a_i *is a*

- linear double ascent *(outsider) if* $a_{i-1} < a_i < a_{i+1}$*;*
- linear double descent *(insider) if* $a_{i-1} > a_i > a_{i+1}$*;*
- linear peak *(closer) if* $a_{i-1} < a_i > a_{i+1}$*;*
- linear valley *(opener) if* $a_{i-1} > a_i < a_{i+1}$*.*

The bijection Ψ_{FV} of Françon and Viennot

Given a permutation $\pi \in \mathcal{S}_n$, determine the right embracing number e_i for each $i \in [n]$. Form the weighted Motzkin path $(c, d) = \Psi_{\mathrm{FV}}(\pi)$ by setting $d_i = e_i$ and by defining c_i as follows:

- if i is a linear double descent, then $c_i = \mathrm{dE}$;
- if i is a linear double ascent then $c_i = \mathrm{E}$;
- if i is a linear peak then $c_i = \mathrm{S}$;
- if i is a linear valley then $c_i = \mathrm{N}$.

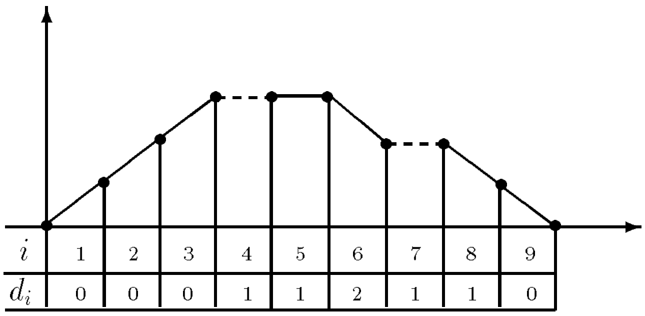


Figure 1

For example, if $\pi = 6\ 1 - 8\ 7\ 4\ 2 - 5 - 9\ 3$, then the corresponding weighted Motzkin path $\Psi_{\mathrm{FV}}(\pi) = (c, d)$ is shown in Figure 1.

The bijection Ψ_{FZ} of Foata and Zeilberger

In [12] Foata and Zeilberger gave another bijection from $\mathcal{S}_n$ to Γ_n, which can be described by the following example.

Start with a permutation π, say, $\pi = 9\ 4\ 7\ 6\ 1\ 2\ 8\ 5\ 3$, so

$$\widetilde{\pi} = \begin{pmatrix} 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9 \\ 9\ 4\ 7\ 6\ 1\ 2\ 8\ 5\ 3 \end{pmatrix}.$$

As in section 3, separate $\widetilde{\pi}$ into two biwords corresponding to π_E and π_N to get

$$\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 7 \\ 9 & 4 & 7 & 6 & 8 \end{pmatrix}, \quad \begin{pmatrix} g \\ g' \end{pmatrix} = \begin{pmatrix} 5 & 6 & 8 & 9 \\ 1 & 2 & 5 & 3 \end{pmatrix}.$$

Form the weighted Motzkin path $(c, d) = \Psi_{\mathrm{FZ}}(\pi)$ as follows: Let $s_1, s_2, \ldots, s_n$ be the sequence of side numbers of π (see Definition 6) and put

$$d_{\pi(i)} = s_i \text{ for } i = 1, 2, \ldots, n. \tag{7}$$

Let

$$c_i = \begin{cases} \text{dE}, & \text{if } i \in F \cap F', \\ \text{E}, & \text{if } i \in G \cap G', \\ \text{S}, & \text{if } i \in F' \cap G, \\ \text{N}, & \text{if } i \in F \cap G'. \end{cases}$$

Here we have $d = (0,\ 0,\ 0,\ 1,\ 1,\ 2,\ 1,\ 1,\ 0)$ and

$$F \cap F' = \{4,\ 7\}, \quad G \cap G' = \{5\}, \quad F' \cap G = \{6,\ 8,\ 9\}, \quad F \cap G' = \{1,\ 2,\ 3\}.$$

$$
\begin{tikzcd}
\mathcal{S}_n \arrow[rr, "\Phi"] \arrow[dr, "\Psi_{\mathrm{FV}}"'] & & \mathcal{S}_n \arrow[dl, "\Psi_{\mathrm{FZ}}"] \\
& \Gamma_n &
\end{tikzcd}
$$

Figure 2

Definition 14 *For $\pi \in \mathcal{S}_n$ and $i \in [n]$, we say that i is a*

- cyclic double ascent *if $\pi^{-1}(i) < i < \pi(i)$;*
- cyclic double descent *if $\pi^{-1}(i) \geq i \geq \pi(i)$;*
- cyclic peak *if $\pi^{-1}(i) < i > \pi(i)$;*
- cyclic valley *if $\pi^{-1}(i) > i < \pi(i)$.*

Note that the four sets $F \cap F'$, $G \cap G'$, $F' \cap G$ and $F \cap G'$ correspond respectively to cyclic double ascents, cyclic double descents, cyclic peaks and cyclic valleys of π. The corresponding weighted Motzkin path is the same as in Figure 1. We note that $\Psi_{\mathrm{FV}} = \Psi_{\mathrm{FZ}} \circ \Phi$. In other words, we have the commutative diagram in Figure 2.

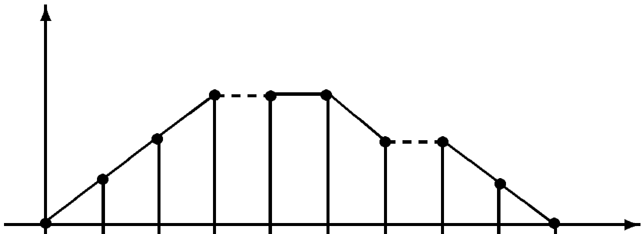
Biane's bijection

In [1], Biane gave a bijection similar to Ψ_{FZ} which we now describe.

Definition 15 *A* labeled path of length n *is a pair (c, ξ), where $c = c_1 \cdots c_n$ is a Motzkin path of length n, and $\xi = (\xi_1, \ldots, \xi_n)$ is a sequence such that*

$$
\xi_i \in \begin{cases} \{\Delta\}, & \text{if } c_i = \mathrm{N}, \\ \{0, \ldots, h_i\}, & \text{if } c_i = \mathrm{dE} \text{ or } \mathrm{E}, \\ \{0, \ldots, h_i - 1\}^2, & \text{if } c_i = \mathrm{S}. \end{cases}
$$

Biane's bijection is from the labeled paths of length n to $\mathcal{S}_n$. Using the same notation as in the description of Ψ_{FZ}, the inverse of Biane's bijection can be summarized as follows. Let $d_1, d_2, \ldots, d_n$ be the sequence of numbers calculated using equation (7) from the side numbers of π. Note that Biane



i	1	2	3	4	5	6	7	8	9
ξ_i	Δ	Δ	Δ	1	0	(0,2)	1	(1,1)	(0,0)

Figure 3

gave a recursive algorithm to compute these numbers but did not point out that they are actually the side numbers of π, that is the i*m*version bottom and inversion top numbers in f' and g' respectively. Form the labeled path (c, ξ) thus:

- if $i \in F \cap G'$ (valley), let $c_i = \mathrm{N}$ and $\xi_i = \Delta$;
- if $i \in F \cap F'$ (double ascent), let $c_i = \mathrm{dE}$ and $\xi_i = d_i$;
- if $i \in G \cap G'$ (double descent), let $c_i = \mathrm{E}$ and $\xi_i = d_{\pi(i)}$;
- if $i \in F' \cap G$ (peak), let $c_i = \mathrm{S}$ and $\xi_i = (d_{\pi(i)}, d_i)$.

The path is the same as for Ψ_{FZ}, the only difference being the distribution of the side numbers associated to each step of the path. If we apply Biane's bijection to the permutation π above, we get the labeled path in Figure 3.

In [12], Foata and Zeilberger's purpose with the bijection Ψ_{FZ} was to code the DEN statistic by weighted Motzkin paths, in order to show that (exc, DEN) was equidistributed with (des, MAJ). That Ψ_{FZ} also keeps track of the INV statistic was first remarked by de Médicis and Viennot [18, Proposition 5.2]. They proved that

$$\mathrm{INV}\,\pi = \sum_{i=1}^{n} h_i + \sum_{i=1}^{n} d_i. \tag{8}$$

In Biane's bijection, on the other hand, the INV statistic is seen to equal

$$\mathrm{INV}\,\pi = \sum_{i=1}^{n} (h_i + |\xi_i|),$$

where $|\xi| = x + y$ if $\xi = (x, y)$. This is obviously equivalent to (8).

The proof of (8) given in [18] was based on a new definition of INV, similar to that of ENV. This statistic of de Médicis and Viennot's, which we denote INV_{MV} can be defined in our notation by

$$\begin{aligned}\text{INV}_{\text{MV}}\,\pi = \text{INV}\,\pi_E &+ \text{INV}\,\pi_N + \#\{(i,j)|i \le j < \pi(i),\, \pi(j) > j\} \\ &+ \#\{(i,j)|\pi(i) < \pi(j) \le i,\, \pi(j) \le j\}.\end{aligned} \tag{9}$$

However, their proof that INV equals INV_{MV} is fairly complicated, and can be compared to that of the equivalence of the two definitions of DEN given in [12]. In [2], Clarke gave a short proof of the equivalence of the two definitions of DEN. Actually, the identity proved in [2] can also be used to prove the equivalence of the three definitions of INV mentioned above. Of course, we have already furnished in Theorem 2 a proof that $\text{ENV}\,\pi = \text{INV}\,\pi$.

Lemma 8 *For any permutation $\pi \in \mathcal{S}_n$, we have*

$$\#\{(i,j)|i \le j < a_i,\, a_j > j\} = \#\{(i,j)|a_i < a_j \le i,\, a_j > j\}, \tag{10}$$

$$\#\{(i,j)|i \le j < a_i,\, a_j \le j\} = \#\{(i,j)|a_i < a_j \le i,\, a_j \le j\}. \tag{11}$$

Proof: The equality (10) was proved by Clarke [2]. Since

$$\#\{(i,j)|i \le j < a_i\} = \#\{(i,j)|a_i < j \le i\} = \#\{(i,j)|a_i < a_j \le i\},$$

the identity (11) follows from (10). $\square$

Proposition 9 *For $\pi \in \mathcal{S}_n$, we have* $\text{INV}_{\text{MV}}\,\pi = \text{ENV}\,\pi = \text{INV}\,\pi$.

Proof: Notice that

$$\begin{aligned}\sum_{i \in E(\pi)} (a_i - i) &= \#\{(i,j)|i \le j < a_i\} && (12)\\ &= \#\{(i,j)|i < j < a_i,\, a_j \le j\} && (13)\\ &\quad + \#\{(i,j)|i \le j < a_i,\, a_j > j\}.\end{aligned}$$

Comparing with (9), the first equality follows from (11). On the other hand, since

$$\begin{aligned}\text{INV}\,\pi &= \text{INV}\,\pi_E + \text{INV}\,\pi_N + \#\{(i,j)|i < j,\, a_i > a_j,\, i \in \pi_E,\, j \in \pi_N\} \\ &= \text{INV}\,\pi_E + \text{INV}\,\pi_N + \#\{(i,j)|i < j < a_i,\, a_j \le j\} \\ &\qquad + \#\{(i,j)|i < a_i,\, a_j < a_i \le j\},\end{aligned}$$

it follows from (10) that $\text{ENV}\,\pi = \text{INV}\,\pi$. $\square$

Using the connections between Motzkin paths and permutations we have described, we now give a continued fraction expansion for the generating function $D_n(x,q) = \sum_{\pi \in \mathcal{S}_n} x^{\operatorname{des}\pi} q^{\text{MAD}\,\pi}$.

For $n \geq 0$ let $[n]_q = 1 + q + \cdots + q^{n-1}$ and let

$$f_n(x,p,q) = \sum_{\pi \in \mathcal{S}_n} x^{\operatorname{exc}\pi} q^{\operatorname{Edif}\pi} p^{\operatorname{Ine}\pi}.$$

Then, by Theorem 4, we also have

$$f_n(x,p,q) = \sum_{\pi \in \mathcal{S}_n} x^{\operatorname{des}\pi} q^{\operatorname{Ddif}\pi} p^{\operatorname{Res}\pi}.$$

Theorem 10 *The ordinary generating function of $f_n(x,p,q)$ has the following Jacobi continued fraction expansion:*

$$\sum_{n\geq 0} f_n(x,p,q)t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\ddots}{1 - b_n t - \cfrac{\lambda_{n+1} t^2}{\ddots}}}}},$$

where $b_n = q^n(x[n]_p + [n+1]_p)$ and $\lambda_{n+1} = xq^{2n+1}([n+1]_p)^2$ for $n \geq 0$.

Proof: For $\pi \in \mathcal{S}_n$, if $\Psi_{\text{FZ}}(\pi) = (c,d)$, then it is easy to see that

$$\begin{aligned}
\operatorname{exc}\pi &= \sum_{c_i=\text{N}} 1 + \sum_{c_i=\text{dE}} 1,\\
\operatorname{Edif}\pi &= \sum_{c_i=\text{S}} i - \sum_{c_i=\text{N}} i = \sum_{i=1}^{n} h_i,\\
\operatorname{Ine}\pi &= \sum_{i=1}^{n} d_i.
\end{aligned}$$

It follows that

$$f_n(x,p,q) \;=\; \sum_{(c,d)\in\Gamma_n} \prod_{c_i=\text{N}} xq^{h_i}p^{d_i} \prod_{c_i=\text{S}} q^{h_i}p^{d_i} \prod_{c_i=\text{dE}} xq^{h_i}p^{d_i} \prod_{c_i=\text{E}} q^{h_i}p^{d_i}$$

$$= \sum_{c\in M_n} \prod_{c_i=\mathrm{N}} xq^{h_i}[h_i+1]_p \prod_{c_i=\mathrm{S}} q^{h_i}[h_i]_p \prod_{c_i=\mathrm{dE}} xq^{h_i}[h_i]_p \prod_{c_i=\mathrm{E}} q^{h_i}[h_i+1]_p,$$

where M_n is the set of all Motzkin paths of length n. The theorem then follows by applying a result of Flajolet [6, Theorem 1]. □

Using the *contraction* formula

$$\cfrac{c_0}{1-\cfrac{c_1 t}{1-\cfrac{c_2 t}{\ddots}}} = c_0 + \cfrac{c_0 c_1 t}{1-(c_1+c_2)t-\cfrac{c_2 c_3 t^2}{1-(c_3+c_4)t-\cfrac{c_4 c_5 t^2}{\ddots}}}, \tag{14}$$

we immediately get the following Stieltjes continued fraction expansion for the same generating function.

Corollary 11 *We have*

$$\sum_{n\geq 0} f_n(x,p,q)t^n = \cfrac{1}{1-\cfrac{t}{1-\cfrac{xqt}{\cfrac{\ddots}{1-\cfrac{q^{n-1}[n]_p t}{1-\cfrac{xq^n[n]_p t}{\ddots}}}}}}. \tag{15}$$

In particular, if $D_n(x,q) = \sum_{\pi\in\mathcal{S}_n} x^{\operatorname{des}\pi} q^{\mathrm{MAD}\,\pi}$, *if follows from Corollary 11, by putting* $p=q$ *in the above equation, that*

$$\sum_{n\geq 0} D_n(x,q)t^n = \cfrac{1}{1-\cfrac{t}{1-\cfrac{xqt}{\cfrac{\ddots}{1-\cfrac{q^{n-1}[n]_q t}{1-\cfrac{xq^n[n]_q t}{\ddots}}}}}}. \quad □ \tag{16}$$

$k\backslash m$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1															
1		5	4	7	8	10	10	8	4	1						
2			10	12	24	32	41	44	43	36	27	18	10	4	1	
3				10	12	24	32	41	44	43	36	27	18	10	4	1
4					5	4	7	8	10	10	8	4	1			
5						1										

Table 1: $[x^k q^m]D_n(x,q)$ for $n=6$.

Note that the continued fraction expansion of the generating function of $\sum_{\pi\in\mathcal{S}_n} x^{\operatorname{des}\pi} q^{\operatorname{INV}\pi}$ can also be derived from [18, Theorem 6.5].

Corollary 12 *For $0 \le k \le n-1$ and $0 \le m \le \frac{n(n-1)}{2}$ we have*

$$[x^k q^{k+m}]D_n(x,q) = [x^{n-1-k} q^{n-1-k+m}]D_n(x,q), \tag{17}$$

where $[x^k q^m]D_n(x,q)$ is the coefficient of $x^k q^m$ in the polynomial $D_n(x,q)$.

Proof: Let $B_n(x,q) = D_n(xq^{-1},q)$. Then (17) is equivalent to

$$[x^k q^m]B_n(x,q) = [x^{n-1-k} q^m]B_n(x,q). \tag{18}$$

From (16) and (14) we derive

$$\sum_{n\ge 1} B_n(x,q)t^n = \cfrac{t}{1-(c_1+c_2)t-\cfrac{c_2c_3t^2}{1-(c_3+c_4)t-\cfrac{c_4c_5t^2}{\ddots}}}, \tag{19}$$

where $c_{2n-1} = q^{n-1}[n]_q$ and $c_{2n} = xq^{n-1}[n]_q$ for $n \ge 0$. Replacing x by $1/x$ and t by xt in (19) we get

$$\sum_{n\ge 1} x^{n-1}B_n(x^{-1},q)t^n = \cfrac{t}{1-(c_1+c_2)t-\cfrac{c_2c_3t^2}{1-(c_3+c_4)t-\cfrac{c_4c_5t^2}{\ddots}}}. \tag{20}$$

Comparing (19) and (20) then yields $B_n(x,q) = x^{n-1}B_n(x^{-1},q)$, which is clearly equivalent to (18). $\square$

To illustrate the above corollary, we give in Table 1 the number of permutations corresponding to the values of (des, MAD) when $n=6$ and for clarity we omit writing zeros.

6 Variations on our statistics

There are several variants on our statistics. First of all, two new statistics $\textsc{madl}$ and $\textsc{makl}$ are defined for a word w as follows.

Definition 16

$$\begin{aligned} \textsc{madl}\, w &= \operatorname{Ddif} w + \operatorname{Les} w. \\ \textsc{makl}\, w &= \operatorname{Dbot} w + \operatorname{Les} w. \end{aligned}$$

Recall that $\operatorname{Les} w$ is the sum of the left embracing numbers of w, defined by replacing "right" by "left" in the definition of $\operatorname{Res} w$. (See Definition 2.)

The relationship between these statistics and $\textsc{mad}$ and $\textsc{mak}$ for permutations is based on the following result, the proof of which will occupy the major part of this section.

Proposition 13 *There is an involution ϵ on $\mathcal{S}_n$ such that, for each $\pi \in \mathcal{S}_n$,*

$$(\operatorname{des}, \operatorname{Dbot}, \operatorname{Dtop}, \operatorname{Res}, \operatorname{Les})\ \pi = (\operatorname{des}, \operatorname{Dbot}, \operatorname{Dtop}, \operatorname{Les}, \operatorname{Res})\ \epsilon(\pi).$$

In particular,

$$\begin{aligned} \textsc{mad}\, \pi &= \textsc{madl}\, \epsilon(\pi), \\ \textsc{mak}\, \pi &= \textsc{makl}\, \epsilon(\pi). \end{aligned}$$

The involution ϵ can be informally described as follows. Let π be a permutation with descent block decompostion $B_1 - B_2 - \cdots - B_k$. Write the descent blocks of π down in reverse order to give a permutation $\pi' = B_k - B_{k-1} - \cdots - B_1$. Now this may not be the descent block decomposition of π', as new descents may have been introduced between adjacent descent blocks. If $\textsc{o}(B_2) > \textsc{c}(B_1)$, that is, if a new descent has been introduced between B_2 and B_1, then move block B_2 to the right of B_1 to get $B_k - B_{k-1} - \cdots - B_1 - B_2$, otherwise leave block B_2 where it is. Now consider block B_3. If there is a descent between B_3 and the block to its right, move B_3 past that block. Continue moving B_3 to the right until there is no descent between it and the block to its right. Continuing in this way we arrive at the permutation $\epsilon(\pi)$.

Example 4 If $\pi = 3\ 1 - 5\ 4\ 2 - 7\ 6$ with descent blocks $B_1 - B_2 - B_3$ then $\pi' = B_3 - B_2 - B_1 = 7\ 6 - 5\ 4\ 2 - 3\ 1$ and, as there is a descent between B_3 and B_2, we get successively $5\ 4\ 2 - 7\ 6 - 3\ 1$ and $\epsilon(\pi) = 5\ 4\ 2 - 3\ 1 - 7\ 6$.

To show that ϵ is an involution, we must proceed more formally. For the remainder of this section, π will be a permutation with descent block decomposition $B_1 - B_2 - \cdots - B_k$. We will write π either in this way or as the juxtaposition product $B_1 B_2 \cdots B_k$.

Definition 17 *Let $(i_1, i_2, \ldots, i_r)$ be a sequence of distinct integers in $[k]$. Then $(B_{i_1}, B_{i_2}, \ldots, B_{i_r})$ is a* valid *sequence of descent blocks if $\mathrm{O}(B_{i_j}) \leq \mathrm{C}(B_{i_{j+1}})$ for $1 \leq j < r$.*

Note that if $(B_{i_1}, B_{i_2}, \ldots, B_{i_r})$ is a valid sequence of descent blocks, then $B_{i_1} - B_{i_2} - \cdots - B_{i_r}$ is the descent block decomposition of the permutation $B_{i_1} B_{i_2} \ldots B_{i_r}$.

Definition 18 *For any two descent blocks B_i and B_j of π, write $B_i < B_j$ if $\mathrm{O}(B_j) > \mathrm{C}(B_i)$, that is, if (B_j, B_i) is not a valid sequence of descent blocks.*

Thus $B_i \not< B_j$ if and only if (B_j, B_i) *is* a valid sequence of descent blocks. It is easy to see that $<$ is a partial ordering on the descent blocks of π.

Example 5 If $\pi = 3\ 1 - 5\ 4\ 2 - 7\ 6$ with descent blocks $B_1 - B_2 - B_3$ then $B_1 < B_3$ and $B_2 < B_3$.

Note that, as $(B_1, B_2, \ldots, B_k)$ is a valid sequence of descent blocks, we have $B_{i+1} \not< B_i$ for $1 \leq i < k$.

We now describe more formally the involution ϵ. Given the permutation π, define the sequence of permutations $\pi_1, \pi_2, \ldots, \pi_k = \pi$ by

$$\pi_r = B_1 - B_2 - \cdots - B_r, \qquad 1 \leq r \leq k.$$

We shall construct a sequence of permutations $S_1, S_2, \ldots, S_k$, with

$$S_r = B_{r_1} - B_{r_2} - \cdots - B_{r_r}, \qquad 1 \leq r \leq k,$$

such that, for each r,

- the sequence $(B_{r_1}, B_{r_2}, \ldots, B_{r_r})$ is a valid sequence of descent blocks, and is a rearrangement of the sequence $(B_1, B_2, \ldots, B_r)$;
- $(\mathrm{des}, \mathrm{Dbot}, \mathrm{Dtop}, \mathrm{Res}, \mathrm{Les})\ \pi_r = (\mathrm{des}, \mathrm{Dbot}, \mathrm{Dtop}, \mathrm{Les}, \mathrm{Res})\ S_r$. $\qquad (*)$

Having done so we define

$$\epsilon(\pi) = S_k.$$

The algorithm for constructing the sequence $S_1, \ldots, S_k$ is as follows.

1. Put $S_1 = B_1$.

2. If $B_1 < B_2$ then put $S_2 = B_1 - B_2$, otherwise put $S_2 = B_2 - B_1$.

3. Suppose that S_r has been constructed as required for some $r < k$. Let j be the smallest integer such that $B_{r_j} \not< B_{r+1}$, if such an integer exists. Then form S_{r+1} from S_r by inserting block B_{r+1} immediately to the left of block B_{r_j}. If no such integer j exists, then form S_{r+1} by appending B_{r+1} to the right of S_r.

It is easy to verify that equation $(*)$ above holds. We need merely note that if B and B' are descent blocks of π with $\mathrm{O}(B) > \mathrm{C}(B')$ then no element of B is embraced by B' and no element of B' is embraced by B.

Example 6 If $\pi = 3\ 1 - 5\ 4\ 2 - 7\ 6$ then $S_1 = 3\ 1$, $S_2 = 5\ 4\ 2 - 3\ 1$ and $\epsilon(\pi) = S_3 = 5\ 4\ 2 - 3\ 1 - 7\ 6$.

The following lemma is easy to prove.

Lemma 14 *Let $1 \leq i < j \leq k$. Then the block B_i occurs to the right of B_j in $\epsilon(\pi)$ if and only if there is a sequence $i = i_0 < i_1 < \cdots < i_\ell = j$ such that $(B_j, B_{i_{\ell-1}}, \ldots, B_i)$ is a valid sequence of descent blocks.* $\square$

As a special case of the above lemma, if $1 \leq i < j \leq k$ and the block B_i occurs to the left of B_j in $\epsilon(\pi)$ then $B_i < B_j$.

Lemma 15 *The mapping ϵ is an involution.*

Proof: Let $1 \leq i < j \leq k$, so that B_i occurs to the left of B_j in π. Suppose first that B_i occurs to the right of B_j in $\epsilon(\pi)$. We show that B_i occurs to the left of B_j in $\epsilon^2(\pi)$. Without loss of generality, we may assume that j is the smallest integer greater than i such that B_i occurs to the right of B_j in $\epsilon(\pi)$. Suppose that B_i occurs to the right of B_j in $\epsilon^2(\pi)$. Then, by the remark after Lemma 14, we have $B_j < B_i$. We show by induction that for every t with $i \leq t < j$ we have $B_j < B_t$. This is true for $t = i$. Suppose it is true for $t-1$, where $i < t < j$. Then, from the definition of j, the block B_t does not occur between B_j and B_i in $\epsilon(\pi)$. If B_t occurs to the left of B_j then $B_t < B_j < B_{t-1}$, a contradiction. Hence B_t occurs to the right of B_i and we have $B_j < B_i < B_t$. But now $B_j < B_{j-1}$, a contradiction.

Suppose now that $1 \leq i < j \leq k$ and that B_i occurs to the left of B_j in $\epsilon(\pi)$. We show that B_i occurs to the left of B_j in $\epsilon^2(\pi)$. If not, by Lemma 14 applied to $\epsilon(\pi)$ there must be a sequence $i = i_0, i_1, \ldots, i_\ell = j$ such that

- for $1 \leq r < \ell$, B_{i_r} occurs to the left of $B_{i_{r+1}}$ in $\epsilon(\pi)$.
- the sequence $(B_j, B_{i_{\ell-1}}, \ldots, B_i)$ is a valid sequence of descent blocks.

Now, as $i < j$, for some r we must have $i_r < i_{r+1}$. But, as B_{i_r} occurs to the left of $B_{i_{r+1}}$ in $\epsilon(\pi)$, $B_{i_r} < B_{i_{r+1}}$, a contradiction as the above sequence will not be valid.

Thus we have shown that any two descent blocks of π occur in the same order in $\epsilon^2(\pi)$ as they do in π. Therefore $\epsilon^2(\pi) = \pi$. $\square$

Proof of Proposition 13: The equation

$$(\mathrm{des}, \mathrm{Dbot}, \mathrm{Dtop}, \mathrm{Res}, \mathrm{Les})\ \pi = (\mathrm{des}, \mathrm{Dbot}, \mathrm{Dtop}, \mathrm{Les}, \mathrm{Res})\ \epsilon(\pi)$$

follows from equation $(*)$ above and the fact that $\pi = \pi_k$ and $\epsilon(\pi) = S_k$. The equations involving MAD and MAK follow from the definitions of those statistics. $\square$

It follows from Proposition 13 that the triple (des, MADL, MAKL) is equidistributed with the triple (des, MAD, MAK) on $\mathcal{S}_n$.

The above involution ϵ is defined in precisely the same way for words as for permutations, but Proposition 13 is replaced by a result (Proposition 16 below) that is both more complicated and more interesting. Let $w = a_1 a_2 \cdots a_n$ be a word. If $1 \leq i \leq n$, define the *right value* of the i-th letter in w, $v'_w(i)$, by

$$v'_w(i) = h(a_i) + r_w(i),$$

where $r_w(i)$ is the number of letters in w that are to the *right* of a_i and equal to a_i. For example, if $w = 1\ 2\ 1\ 4\ 4\ 2\ 3\ 1\ 4$, the right values of the letters of w are given by 3, 5, 2, 9, 8, 4, 6, 1, 7, in the order in which they appear in w. Then we can define the *right* descent bottoms sum of w, denoted by $\mathrm{Dbot}'\,w$, as the sum of the right values of the descent bottoms of w and the *right* descent difference of w, denoted by $\mathrm{Ddif}'\,w$, by

$$\mathrm{Ddif}'\,w = \mathrm{Dtop}\,w - \mathrm{Dbot}'\,w.$$

Now we can define two new statistics corresponding to each of MAD and MAK:

Definition 19

$$\begin{aligned}
\textsc{mak}'\,w &= \mathrm{Dbot}'\,w + \mathrm{Res}\,w. \\
\textsc{mad}'\,w &= \mathrm{Ddif}'\,w + \mathrm{Res}\,w. \\
\textsc{makl}'\,w &= \mathrm{Dbot}'\,w + \mathrm{Les}\,w. \\
\textsc{madl}'\,w &= \mathrm{Ddif}'\,w + \mathrm{Les}\,w.
\end{aligned}$$

It is not hard to see that the effect of the involution ϵ above is to interchange Res and Les *and* to interchange value and right value. Thus we have the following result.

Proposition 16 *Let v be a word. Then there is an involution ϵ on the rearrangement class $R(v)$ such that for all $w \in R(v)$,*

$$(\mathrm{des}, \mathrm{Dbot}, \mathrm{Dtop}, \mathrm{Res}, \mathrm{Les})\ w = (\mathrm{des}, \mathrm{Dbot}', \mathrm{Dtop}, \mathrm{Les}, \mathrm{Res})\ \epsilon(w).$$

In particular,

$$\begin{aligned}
\textsc{mad}\,w &= \textsc{madl}'\,\epsilon(w), \\
\textsc{madl}\,w &= \textsc{mad}'\,\epsilon(w), \\
\textsc{mak}\,w &= \textsc{makl}'\,\epsilon(w), \\
\textsc{makl}\,w &= \textsc{mak}'\,\epsilon(w). \quad \square
\end{aligned}$$

It now immediately follows that the triple (des, MADL$'$, MAKL$'$) is equidistributed with the triple (des, MAD, MAK).

Consequently, it is easy to modify the bijection Φ_{W} of section 4 to show that the triple (des, MADL, MAKL) is stated that equidistributed with (exc, INV, DEN), and hence with (des, MAD, MAK). Hence the triple (des, MAD$'$, MAK$'$) is also equidistributed with (des, MAD, MAK).

7 Concluding remarks

7.1 Links to the past

Some Mahonian statistics on $\mathcal{S}_n$ equivalent to MAD have been given by Simion and Stanton [21] and by de Médicis and Viennot [18]. More precisely, de Médecis and Viennot [18, Proposition 6.2] gave a Mahonian statistic they called "lag" (but which we call LAG, for the sake of consistency). It can be defined as follows: Given a permutation $\pi = a_1 a_2 \cdots a_n$, let

$$\pi' = (n+1)a_1 a_2 \cdots a_n 0 \in \mathcal{S}_{n+2}$$

and let $\mathrm{Run}\,\pi$ be the number of descent blocks in π. Then

$$\textsc{lag}\,\pi = \mathrm{Ddif}\,\pi' + \mathrm{Les}\,\pi' - \mathrm{Run}\,\pi - n.$$

It is not hard to see that $\textsc{lag}\,\pi = \textsc{mad}\,\pi^r$, where $\pi^r = a_n a_{n-1} \cdots a_1$.

We define the *dual* of a permutation $\pi = a_1 a_2 \cdots a_n \in \mathcal{S}_n$ as the permutation $\pi^* = b_1 b_2 \cdots b_n$, where $b_i = n + 1 - a_i$ for $1 \le i \le n$. Simion and Stanton [21] use notions dual to ours, with ascents instead of descents, and embracing by ascent blocks, which they call "runs". They also use the notion of left embracing. Their statistic, translated into our dual setting, is

$$\textsc{sist}\ \pi = n - \mathrm{Run}\,\pi + 2\,\mathrm{Les}\,\pi + \mathrm{Res}\,\pi$$

(see Theorem 2 in [21]), so, since $n - \mathrm{Run}\,\pi = \mathrm{des}\,\pi$, we have

$$\textsc{sist}\ \pi = \mathrm{des}\,\pi + 2\,\mathrm{Les}\,\pi + \mathrm{Res}\,\pi.$$

A counterpart of this statistic, namely

$$\textsc{sist}'\ \pi = \mathrm{des}\,\pi + 2\,\mathrm{Res}\,\pi + \mathrm{Les}\,\pi,$$

whose dual was also defined by Simion and Stanton, is readily seen to equal MAD for permutations, because of the identity:

$$\mathrm{Ddif}\,\pi = \mathrm{des}\,\pi + \mathrm{Res}\,\pi + \mathrm{Les}\,\pi. \qquad (*)$$

However, $\textsc{sist}'$ is not Mahonian for words — because the identity $(*)$ does not hold for words — and neither are the other statistics defined by Simion and Stanton. Some of their other statistics are equivalent to the variations on MAD given in the last section.

7.2 Large and small letters

Various authors, for example [3, 16, 24], have considered statistics on words and permutations in the context of an alphabet $\mathcal{A} = [m]$ in which the letters are divided into two classes, *large* and *small*. Namely, for some k with $0 \leq k \leq m$ and for $\ell = m - k$, the letters $1, 2, \ldots, \ell$ are designated small and the letters $\ell{+}1, \ldots, m$ are designated large. Then, for a word $w = a_1 a_2 \cdots a_m$, a *k-descent* is an integer i such that one of the following conditions holds:

- $1 \leq i < m$ and $a_i > a_{i+1}$;
- $1 \leq i < m$ and $a_i = a_{i+1} > \ell$;
- $i = m$ and $a_i > \ell$.

Then $\operatorname{des}_k w$ equals the number of k-descents of w. One can similarly define k-extensions of the other Eulerian and Mahonian statistics. The results of the present paper can all be k-extended, as will be presented by the present authors in a subsequent note.

References

[1] P. Biane: Permutations suivant le type d'excédance et le nombre d'inversions et interprétation combinatoire d'une fraction continue de Heine, Europ. J. Combinatorics **14** (1993), 277–284.

[2] R. J. Clarke: A short proof of a result of Foata and Zeilberger, Adv. Appl. Math. **16** (1995), 129–131.

[3] R. J. Clarke and D. Foata: Eulerian Calculus I: Univariate statistics, Europ. J. Combinatorics **15** (1994), 345–362.

[4] M. Denert: The genus zeta function of hereditary orders in central simple algebras over global fields, Math. Comp. **54** (1990), 449–465.

[5] L. Euler: Institutiones calculi differentialis, in *Opera Omnia, Series Prima*, vol. X, Verlag von B.G. Teubner, Leipzig, 1913.

[6] P. Flajolet: Combinatorial aspects of continued fractions, Disc. Math. **41** (1982), 145–153.

[7] D. Foata: Distribution Eulériennes et Mahoniennes sur le groupe des permutations, in M. Aigner (ed.), *Higher Combinatorics*, 27–49, D. Reidel, Boston, Berlin Combinatorics Symposium, 1976.

[8] D. Foata: Rearrangements of words, in M. Lothaire, *Combinatorics on Words*, (ed.) G.-C. Rota, Vol. 17, Encyclopedia of Math. and its Appl., Addison-Wesley Publishing Company, 1983.

[9] D. Foata and G.-N. Han: Calcul basique des distributions statistiques sur les permutations signées, Preprint, 1995.

[10] D. Foata and M.-P. Schützenberger: *Théorie géométriques des polynômes eulériens*, Lecture Notes in Math., vol. 138, Springer-Verlag, Berlin, 1970.

[11] D. Foata and M.-P. Schützenberger: Major index and inversion number of permutations, Math. Nachr. **83** (1978), 143–159.

[12] D. Foata and D. Zeilberger: Denert's permutation statistic is indeed Euler-Mahonian, Studies in Appl. Math. **83** (1990), 31–59.

[13] J. Françon and X. G. Viennot: Permutations selon les pics, creux, doubles montées, doubles descentes, nombres d'Euler, nombres de Genocchi, Disc. Math. **28** (1979), 21–35.

[14] A. M. Garsia and I. M. Gessel: Permutations statistics and partitions, Adv. in Math. **31** (1979), 288–305.

[15] G.-N. Han: Une transformation fondamentale sur les réarrangements de mots, Adv. in Math., **105** (1994), 26–41.

[16] G.-N. Han: The k-extension of a Mahonian statistic, Adv. Appl. Math., **16** (1995),297–305.

[17] P.A. MacMahon: *Combinatory Analysis*, vols. 1 and 2. Cambridge Univ. Press, Cambridge, 1915 (reprinted by Chelsea, New York, 1955).

[18] A. de Médicis and X. G. Viennot: Moments des q-Polynômes de Laguerre et la bijection de Foata-Zeilberger, Adv. Appl. Math. **15** (1994), 262–304.

[19] D. Rawlings: Permutation and multipermutation statistics, Europ. J. Comb., **2** (1981), 67-78.

[20] O. Rodriguez: Note sur les inversions, ou dérangements produits dans les permutations, J. de Math. **4** (1839), 236–240.

[21] S. Simion and D. Stanton: Specializations of generalized Laguerre polynomials, SIAM J. Math. Anal. **25** (1994), 712–719.

[22] S. Simion and D. Stanton: Octabasic Laguerre polynomials and permutation statistics, J. Comp. Appl. Math., to appear.

[23] R. Stanley: Binomial posets, Möbius inversion and permutation enumeration, J. Comb. Theory, **A, 20** (1976), 712–719.

[24] E. Steingrímsson: Permutation statistics of indexed permutations, Europ. J. Combinatorics, **15** (1994), 187–205.