The k-Extensions of some new Mahonian statistics

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Abstract

In previous work by the authors, new Mahonian statistics ENV, MAD and MAK were defined on words and it was shown that ENV is equal to the classical statistic INV and that the triple statistics $(\mathrm{des}, \mathrm{MAK}, \mathrm{MAD})$ and $(\mathrm{exc}, \mathrm{DEN}, \mathrm{ENV})$ are equidistributed over any rearrangement class of words. Here, exc and des are the classical Eulerian statistics, while DEN is Denert's statistic. In addition, a bijection between the symmetric group and sets of weighted Motzkin paths was used to give a continued fraction expression for the generating function of $(\mathrm{exc}, \mathrm{INV})$ or $(\mathrm{des}, \mathrm{MAD})$ on the symmetric group. These results are extended to the case in which the letters of the alphabet used are divided into two classes, large and small, with corresponding changes to the definitions of the above statistics.

1 Introduction

Let $\mathbf{c} = (c_1, c_2, \ldots, c_m)$ be a sequence of non-negative integers with $n = c_1 + c_2 + \cdots + c_m$. Denote by $R(\mathbf{c})$ the set of all rearrangements of the (non-decreasing) word $1^{c_1}2^{c_2}\ldots m^{c_m}$. In [3], some new Mahonian statistics MAD, MAK and ENV on $R(\mathbf{c})$ were introduced and the following results were proved.

Theorem 1 *For all $w \in R(\mathbf{c})$, we have $\mathrm{ENV}\, w = \mathrm{INV}\, w$.*

Theorem 2 *The triples $(\mathrm{des}, \mathrm{MAD}, \mathrm{MAK})$ and $(\mathrm{exc}, \mathrm{INV}, \mathrm{DEN})$ are equidistributed on $R(\mathbf{c})$.*

In fact, a bijection Φ_{W} was constructed on $R(\mathbf{c})$ such that for all $w \in R(\mathbf{c})$,

$$(\mathrm{des}, \mathrm{MAD}, \mathrm{MAK})\ w = (\mathrm{exc}, \mathrm{INV}, \mathrm{DEN})\ \Phi_{\mathrm{W}}(w).$$

In [1, 2, 4, 5], the statistics des, exc, MAJ, DEN and INV were extended to the case in which the letters in $[m] := \{1, 2, \ldots, m\}$ are divided into two classes. Namely, let k be a non-negative integer such that $k \le m$ and put $\ell = m - k$. The letters $1, 2, \ldots, \ell$ are called *small* and the letters $\ell + 1, \ldots, m$ are called *large*. In counting descents and excedances, one counts strict inequalities between small letters, but equalities and inequalities between large letters—see below for the details. Then k-extended statistics des_k, exc_k, MAJ_k, DEN_k and INV_k are defined, which reduce to their more familiar Eulerian and Mahonian counterparts in the case $k = 0$. Further, the pairs

$(\mathrm{des}_k, \mathrm{MAJ}_k)$ and $(\mathrm{exc}_k, \mathrm{DEN}_k)$ are equidistributed on $R(\mathbf{c})$. It is the purpose of this paper to k-extend the results of [3]. We will define statistics MAD_k, MAK_k and ENV_k and prove the following results.

Theorem 3 *For all $w \in R(\mathbf{c})$, we have $\mathrm{ENV}_k\, w = \mathrm{INV}_k\, w$.*

Theorem 4 *The triples $(\mathrm{des}_k, \mathrm{MAD}_k, \mathrm{MAK}_k)$ and $(\mathrm{exc}_k, \mathrm{INV}_k, \mathrm{DEN}_k)$ are equidistributed on $R(\mathbf{c})$.*

We will begin by recalling the definitions of k-descent and k-excedance from [1]. Let $1 \le i \le m$ and $w = a_1 a_2 \dots a_n \in R(\mathbf{c})$. Let $\overline{w} = b_1 b_2 \dots b_n$ be the non-decreasing rearrangement of w. It is convenient to introduce a new small letter $*$ such that $\ell < * < \ell + 1$, and to put $w* = a_1 a_2 \dots a_n *$. Thus we write $a_{n+1} = *$. (We use this convention throughout the paper.) We also introduce the following partial ordering on $[m]* = [m] \cup \{*\}$.

Definition 1 *If $a, b \in [m]$ then $a \prec b$ means either $a < b$ or $a = b$ with b large. Thus $a \not\prec b$ means either $b < a$ or $a = b$ with b small.*

Definition 2 *Let $w = a_1 a_2 \dots a_n$ be a word with $\overline{w} = b_1 b_2 \dots b_n$. A k-*descent* in the word w is a triple (i, a_i, a_{i+1}), where $1 \le i \le n$, such that $a_{i+1} \prec a_i$. Here i is called the k-*descent place*, a_i is called the k-*descent top* and a_{i+1} is called the k-*descent bottom*. A k-*excedance* in w is a triple (i, a_i, b_i), where $1 \le i \le n$, such that $b_i \prec a_i$. Here i is called the k-*excedance place*, a_i is called the k-*excedance top* and b_i is called the k-*excedance bottom*. The numbers of k-descents and k-excedances in w are denoted by* $\mathrm{des}_k\, w$ *and* $\mathrm{exc}_k\, w$ *respectively.*

*The k-*descent set* of w, $D_k(w)$, is the set of k-descent places. The k-*excedance set* of w, $E_k(w)$, is the set of k-excedance places.*

This is a slight change of wording from [1].

Definition 3 *The k-*major index* of w is*

$$\mathrm{MAJ}_k\, w = \sum_{i \in D_k(w)} i.$$

Recall the k-extensions of Imv and INV from [4] and [5]:

Definition 4

$$\begin{aligned} \text{INV}_k\, w &= \#\{\,(i,j) \mid i<j, a_j \prec a_i\,\} + \#\{\, i \mid a_i > \ell\}; \\ \text{Imv}_k\, w &= \#\{\,(i,j) \mid i<j, a_i \not\prec a_j\,\}. \end{aligned}$$

Here, $\#S$ denotes the number of elements in the set S.

Let $i_1 < i_2 < \cdots < i_e$ be the increasing sequence of elements of $E_k(w)$ and let $j_1 < j_2 < \cdots < j_{n-e}$ be the increasing complementary sequence in $[n]$. We form the *excedance subword* $w_{\mathrm{E}} = a_{i_1} a_{i_2} \dots a_{i_e}$ and the *non-excedance subword* $w_{\mathrm{N}} = a_{j_1} a_{j_2} \dots a_{j_{m-e}}$ of w. Put

$$\text{Ebot}_k\, w = \sum_{i \in E_k(w)} i,$$

that is, $\text{Ebot}_k\, w$ is the sum of the k-excedance places in w.

Remark 1 Note that $\text{Ebot}_k\, w$ is equal to the sum of the values of the k-excedance bottoms in $\bar{w}$ if there are no large letters in w as in [3], but this is no longer true if there are large letters in w, because of the $*$ and because we code equal large letters from right to left (See Section 2.1 below). However, we use the notation Ebot_k to be compatible with [3].

The following definition of $\text{DEN}_k\, w$, from [5], is equivalent to the definition of [2].

Definition 5

$$\text{DEN}_k\, w = \text{Ebot}_k\, w + \text{Imv}_k\, w_{\mathrm{E}} + \text{INV}_k\, w_{\mathrm{N}}.$$

We can write the definition of $\text{DEN}_k\, w$ in a slightly different way, using the following definition.

Definition 6 *Let $w = a_1 a_2 \dots a_n$ be a word. If i is a k-excedance place in w then the k-*imversion bottom number *of i is the number of places j such that $1 \le j < i$ and $a_j \not\prec a_i$. If i is not a k-excedance place in w then the k-*inversion top number *of i is the number of places j such that $i < j \le n+1$ and $a_j \prec a_i$. The k-*side number *s_i of i (in w) is the k-imversion bottom number or k-inversion top number of i as appropriate.*

Clearly, the sum of the k-imversion bottom numbers for the letters in w_{E} equals $\mathrm{Imv}_k\, w_{\mathrm{E}}$, while the sum of the k-inversion top numbers for the letters in w_{N} equals $\mathrm{INV}_k\, w_{\mathrm{N}}$. We write

$$\begin{aligned} \mathrm{Ine}_k\, w &= \mathrm{Imv}_k\, w_{\mathrm{E}} + \mathrm{INV}_k\, w_{\mathrm{N}} \\ &= s_1 + \cdots + s_n. \end{aligned}$$

Thus $\mathrm{DEN}_k\, w = \mathrm{Ebot}_k\, w + \mathrm{Ine}_k\, w$.

Example 1 Let

$$w = 2\ 4\ 4\ 1\ 3\ 1\ 1\ 3, \qquad \text{so that} \qquad \overline{w} = 1\ 1\ 1\ 2\ 3\ 3\ 4\ 4.$$

Take $k = 2$. Thus 1 and 2 are small, while 3 and 4 are large. There are 2-descent places at $i = 2,\ 3,\ 5$ and 8. Thus $\mathrm{des}_2\, w = 4$ and $\mathrm{MAJ}_2\, w = 2+3+5+8 = 18$.

There are 2-exedance places at $i = 1,\ 2,\ 3$ and 5. Thus $\mathrm{exc}_2\, w = 4$ and $\mathrm{Ebot}_2\, w = 11$. Now, one readily checks that the sequence of side numbers of w is $(0,0,0,0,2,0,0,1)$. Hence $\mathrm{Ine}_2\, w = 3$ and $\mathrm{DEN}_2\, w = 14$.

2 Further Mahonian statistics

2.1 Height and value

For each letter a, we define the *height* of a in w as

$$h_k(a) = h_{k,w}(a) = 1 + \#\{\, j \mid 1 \le j \le n, a_j \prec a \,\}.$$

It is easy to see that

$$i \in E_k(w) \text{ if and only if } i < h_{k,w}(a_i).$$

If $1 \le i \le n+1$, we define the *value* of the i-th letter in w by

$$\begin{aligned} v_{k,i} = v_{k,i}(w) &= \#\{\, j \mid j \le n+1, a_j < a_i \,\} \\ &\quad + \begin{cases} \#\{\, j \mid 1 \le j \le i, a_j = a_i \,\}, & \text{if } a_i \text{ small;} \\ \#\{\, j \mid i \le j \le n, a_j = a_i \,\}, & \text{if } a_i \text{ large.} \end{cases} \end{aligned}$$

Thus the height of a letter in a word is a function of the letter, not of its position in the word, whereas the value of a letter does depend on its position in the word. Hence we can strictly only speak of the value of a position in a word.

We will normally supress the argument w to v_k and h_k.

Using the relation $\prec$ we can write

$$v_{k,i} = 1 + \#\{\, j \mid i < j \le n+1, a_j \prec a_i \,\} + \#\{\, j \mid 1 \le j < i, a_i \not\prec a_j \,\}. \quad (1)$$

Notice that the word

$$v_k(w) = v_{k,1} v_{k,2} \ldots v_{k,n} v_{k,n+1}$$

is a permutation. In fact, our definition of $v_k(w)$ amounts to "coding" $w*$ to a permutation by coding equal small letters from left to right and coding equal large letters from right to left. The equation

$$D_k(w) = D(v_k(w)), \quad (2)$$

which is easily verified, is the motivation for this coding. Note that, if $w \in \mathcal{S}_n$ is a permutation and $1 \le i \le n$, then

$$v_{k,i} = h_k(a_i) = \begin{cases} a_i, & \text{if } 1 \le a_i \le \ell, \\ a_i + 1, & \text{if } \ell < a_i \le n. \end{cases}$$

Further, $v_{k,n+1} = \ell + 1$.

Example 2 For the word $w = 2\ 4\ 4\ 1\ 3\ 1\ 1\ 3$ considered in example 1 with $k = 2$, we have $v_2(w) = 4\ 9\ 8\ 1\ 7\ 2\ 3\ 6\ 5$. Also $h_2(1) = 1$, $h_2(2) = 4$, $h_2(3) = 7$ and $h_2(4) = 9$.

2.2 The statistic ENV_k

Now we put

$$\operatorname{Edif}_k w = \sum_{i \in E_k(w)} (h_k(a_i) - i)$$

and we define a new statistic $\text{ENV}_k\, w$ by

Definition 7

$$\text{ENV}_k\, w = \operatorname{Edif}_k w + \operatorname{Imv}_k w_{\text{E}} + \text{INV}_k\, w_{\text{N}}.$$

We can also write $\text{ENV}_k\, w = \operatorname{Edif}_k w + \operatorname{Ine}_k w$. For $k = 0$, when there are no large letters, we have $\text{ENV}_k = \text{ENV}$, the statistic defined in [3].

Example 3 For the word $w = 2\ 4\ 4\ 1\ 3\ 1\ 1\ 3$ considered above, we have $\mathrm{Edif}_2\, w = (4-1)+(9-2)+(9-3)+(7-5) = 18$ and $\mathrm{ENV}_2\, w = 18+3 = 21$.

One can easily check that $\mathrm{INV}_2\, w = 21$.

Lemma 5 *For any word w we have*

$$\mathrm{ENV}_k\, w = \sum_{i\in E_k(w)} (v_{k,i} - i) + \mathrm{INV}_k\, w_{\mathrm{E}} + \mathrm{INV}_k\, w_{\mathrm{N}} - \delta,$$

where δ is the number of large letters in w_{E}.

Proof: Suppose that w_{E} contains d_a letters equal to a, for $1 \le a \le m$. Then one can easily verify that

$$\mathrm{Imv}_k\, w_{\mathrm{E}} - \mathrm{INV}_k\, w_{\mathrm{E}} = \sum_{a=1}^{l} \binom{d_a}{2} - \sum_{a=l+1}^{m} \binom{d_a+1}{2}$$

and

$$\sum_{i\in E_k(w)} (v_{k,i} - h_k(a_i)) = \sum_{a=1}^{l} \binom{d_a}{2} - \sum_{a=l+1}^{m} \binom{d_a}{2},$$

bearing in mind that the d_a letters 'a' occurring in w_{E} will be the left-most d_a letters 'a' that occur in w. Hence the result follows. $\square$

We can now prove that ENV_k and INV_k are identical.

Proof of Theorem 3: From Lemma 5 we must show that

$$\begin{aligned}
\sum_{i\in E_k(w)} (v_{k,i} - i) &= \delta + \#\{\,(i,j) \mid i < j \le n, a_j \prec a_i, i \in E_k(w), j \notin E_k(w)\,\} \\
&= \delta + \sum_{i\in E_k(w)} \#\{\, j \mid i < j \le n, a_j \prec a_i, j \notin E_k(w)\,\} \\
&= \delta + \sum_{i\in E_k(w)} (\#\{\, j \mid i < j \le n, a_j \prec a_i\,\} \\
&\qquad -\#\{\, j \mid i < j \le n, a_j \prec a_i, j \in E_k(w)\,\}). \qquad (3)
\end{aligned}$$

Now, since $i = \#\{\, j \mid j \le i\,\}$, it follows from equation (1) that

$$\begin{aligned}
\sum_{i\in E_k(w)} (v_{k,i} - i) &= \sum_{i\in E_k(w)} (\#\{\, j \mid n+1 \ge j > i, a_j \prec a_i\,\} \\
&\qquad -\#\{\, j \mid j < i, a_i \prec a_j\,\}) \\
&= \delta + \sum_{i\in E_k(w)} (\#\{\, j \mid n \ge j > i, a_j \prec a_i\,\} \\
&\qquad -\#\{\, j \mid j < i, a_i \prec a_j\,\}). \qquad (4)
\end{aligned}$$

Hence, comparing equations (3) and (4), we need only to show that

$$\sum_{i\in E_k(w)} \#\{\, j \mid i < j \le n, a_j \prec a_i, j \in E_k(w)\,\} = \sum_{i\in E_k(w)} \#\{\, j \mid j < i, a_i \prec a_j \,\}.$$

But each of the sums in the above equation is $\textsc{inv}_k\, w_{\mathrm{E}} - \delta$. $\square$

2.3 The statistics $\textsc{mad}_k$ and $\textsc{mak}_k$

Let $w = a_1a_2 \ldots a_n$ be a word, with a_{n+1} as usual. By analogy with $\mathrm{Ebot}_k\, w$ and $\mathrm{Edif}_k\, w$ we put

$$\mathrm{Dbot}_k\, w = \sum_{i\in D_k(w)} v_{k,i+1}$$

and

$$\mathrm{Ddif}_k\, w = \sum_{i\in D_k(w)} (h_k(a_i) - v_{k,i+1}).$$

Definition 8 *Let $w = a_1a_2 \ldots a_n$ be a word. For $1 \le i \le n$, the* (right) k-embracing number *e_i in w is*

$$e_i = \begin{cases} \#\{\, j \mid i < j \le n, a_{j+1} < a_i \le a_j \,\}, & \textit{if } a_i \textit{ is small;} \\ \#\{\, j \mid i < j \le n, a_{j+1} \le a_i < a_j \,\}, & \textit{if } a_i \textit{ is large.} \end{cases}$$

The (right) k-embracing sum *of w is*

$$\mathrm{Res}_k\, w = e_1 + \cdots + e_n.$$

Alternatively, we may if we wish define e_i for both small and large letters a_i by

$$e_i = \#\{\, j \mid i < j \le n, a_{j+1} \prec a_i, a_j \not\prec a_i \,\}.$$

Definition 9 *Let $w = a_1a_2 \ldots a_n$ be a word. The k-descent (j, a_j, a_{j+1})* embraces *a_i if $a_{j+1} \prec a_i$ and $a_j \not\prec a_i$.*

So e_i is the number of k-descents in w to the right of a_i that embrace a_i.

The statistics $\textsc{mad}_k$ and $\textsc{mak}_k$ can now be defined as follows:

Definition 10

$$\begin{aligned} \textsc{mad}_k\, w &= \mathrm{Ddif}_k\, w + \mathrm{Res}_k\, w; \\ \textsc{mak}_k\, w &= \mathrm{Dbot}_k\, w + \mathrm{Res}_k\, w. \end{aligned}$$

For $k = 0$ we have $\text{MAD}_k = \text{MAD}$ and $\text{MAK}_k = \text{MAK}$ as defined in [3]. Note that, for any word w, it follows from Equation (2) and from

$$\operatorname{Res}_k w = \operatorname{Res}_k v_k(w) \tag{5}$$

that

$$\text{MAK}_k\, w = \text{MAK}_k\, v_k(w).$$

However, it is not in general true that $\text{MAD}_k\, w = \text{MAD}_k\, v_k(w)$.

In Section 3, we will produce a bijection Φ_k on $\mathcal{S}_n$ such that, for all $\pi \in \mathcal{S}_n$, we have

$$(\operatorname{des}_k, \text{MAD}_k, \text{MAK}_k)\ \pi = (\operatorname{exc}_k, \text{INV}_k, \text{DEN}_k)\Phi_k(\pi).$$

In Section 4, we extend the definition of Φ_k to words, thereby proving Theorem 4.

Example 4 Consider the word

$$w = 3\ 1\ 4\ 1\ 2\ 1\ 3\ 4,$$

with $k = 2$. Then

$$v_2(w) = 7\ 1\ 9\ 2\ 4\ 3\ 6\ 8\ 5$$

and $h_2(1) = 1$, $h_2(2) = 4$, $h_2(3) = 7$, $h_2(4) = 9$. Thus $\operatorname{Dbot}_2 w = 1+2+3+5 = 11$ and $\operatorname{Ddif}_2 w = (7-1)+(9-2)+(4-3)+(9-5) = 18$. The sequence of 2-embracing numbers of w is $(2,0,0,0,0,0,1,0,0)$, so $\operatorname{Res}_2 w = 3$. Therefore $\text{MAD}_2\, w = 18 + 3 = 21$ and $\text{MAK}_2\, w = 11 + 3 = 14$.

3 The bijection for permutations

Let $\pi = a_1 a_2 \ldots a_n \in \mathcal{S}_n$. Then the set of k-descents in π equals the set of descents in $\pi*$. Hence $\operatorname{Dbot}_k \pi = \operatorname{Dbot} \pi*$ and $\operatorname{Ddif}_k \pi = \operatorname{Ddif} \pi*$. Moreover, if $(e_1, e_2, \ldots, e_n)$ is the sequence of right k-embracing numbers of π, the sequence of right embracing numbers of $\pi*$ is $(e_1, e_2, \ldots, e_n, 0)$. Thus $\operatorname{Res}_k \pi = \operatorname{Res} \pi*$. Therefore

$$(\operatorname{des}_k, \text{MAD}_k, \text{MAK}_k)\ \pi = (\operatorname{des}, \text{MAD}, \text{MAK})\ \pi * . \tag{6}$$

Further, since $\operatorname{exc}_k \pi = \operatorname{exc}_k \tilde{\pi}$, where

$$\tilde{\pi} = \begin{pmatrix} 1 & 2 & \ldots & \ell & \ell+1 & \ldots & n \\ a_1 & a_2 & \ldots & a_\ell & a_{\ell+1} & \ldots & a_n \end{pmatrix}$$

and $\operatorname{exc}\pi* = \operatorname{exc}\widetilde{\pi*}$, where

$$\widetilde{\pi*} = \begin{pmatrix} 1 & 2 & \dots & \ell & * & \ell+1 & \dots & n-1 & n \\ a_1 & a_2 & \dots & a_\ell & a_{\ell+1} & a_{\ell+2} & \dots & a_n & * \end{pmatrix},$$

it follows that there is a one-to-one correspondence between the k-excedances of π and the excedances of $\pi*$. For, if $a_i = i > \ell$ then $a_i > i-1$. Thus the set of k-excedance places of π equals the set of excedance places of $\pi*$ and the set of k-excedance tops of π equals the set of excedance tops of $\pi*$. (The corresponding equality does *not* hold for the excedance bottoms, but they play no explicit part in our statistics.) So $\operatorname{Ebot}_k \pi = \operatorname{Ebot}\pi*$ and $\operatorname{Edif}_k \pi = \operatorname{Edif}\pi*$. Similarly, we have that $\operatorname{Ine}_k \pi = \operatorname{Ine}\pi*$, so that

$$(\operatorname{exc}_k, \mathrm{ENV}_k, \mathrm{DEN}_k)\ \pi = (\operatorname{exc}, \mathrm{ENV}, \mathrm{DEN})\ \pi*\,. \tag{7}$$

In [3], a bijection Φ was constructed on $\mathcal{S}_n$ such that, for all $\pi \in \mathcal{S}_n$,

$$(\operatorname{des}, \mathrm{MAD}, \mathrm{MAK})\ \pi = (\operatorname{exc}, \mathrm{ENV}, \mathrm{DEN})\ \Phi(\pi). \tag{8}$$

In order to prove Lemma 6 below, we briefly describe the bijection Φ. Let F, F', G and G' denote respectively the sets of descent bottoms, descent tops, non descent bottoms and non descent tops for π. Let f and g be the non-decreasing words formed from the letters of F and G respectively. Let f' and g' be the words formed from the letters of F' and G' respectively, whose imversion bottom and inversion top numbers respectively are the embracing numbers of those letters in π. Form the biword

$$\alpha = \begin{pmatrix} f & | & g \\ f' & | & g' \end{pmatrix}.$$

Then the columns to the left of the bar are excedances, while those to the right are non-excedances. Finally, rearrange the columns of α so that its first row is increasing and read off the permutation $\Phi(\pi)$ from the second row.

The bijection Φ has the following property.

Lemma 6 *Let $\pi = a_1a_2\dots a_n \in \mathcal{S}_n$ and let $\Phi(\pi) = b_1b_2\dots b_n$. Then $b_n = a_n$. (That is, the bijection Φ fixes the last letter of a permutation.)*

Proof: Let $\pi = a_1a_2\dots a_n \in \mathcal{S}_n$. Clearly, a_n is a non-descent top and its (right) embracing number is zero. Hence a_n will be a letter in g' and all the non-descent top letters smaller than a_n will be to the left of a_n in g'. It remains to show that any non-descent top letter larger than a_n will not be the rightmost letter in g'. If a_i $(i < n)$ is a non-descent top letter larger than a_n, then there is a letter a_j $(i < j < n)$ such that $a_i < a_j$ (hence $a_j > a_n$), for, otherwise, a_i would be a descent top. Let a_j be the smallest such letter. Thus $a_i > a_{j+1}$ and the embracing number of a_i must be larger than or equal to 1. This proves that a_i cannot be the rightmost letter in g'. $\square$

Hence we define a bijection Φ_k on $\mathcal{S}_n$ by

$$\pi \mapsto \pi* \mapsto \Phi(\pi*) = \pi'* \mapsto \pi',$$

where π' is a permutation. It follows from Equations (6), (7) and (8) and Theorem 3 that

$$(\mathrm{des}_k, \mathrm{MAD}_k, \mathrm{MAK}_k)\ \pi = (\mathrm{exc}_k, \mathrm{INV}_k, \mathrm{DEN}_k)\ \Phi_k(\pi). \tag{9}$$

This proves Theorem 4 in the case of permutations, that is, if all $c_i \leq 1$.

We note for future reference that Φ_k satisfies a stronger property than that specified in equation (9), namely that the set of k-excedance places of $\Phi_k(\pi)$ equals the set of k-descent bottoms of π, the set of k-excedance tops of $\Phi_k(\pi)$ equals the set of k-descent tops of π and the sequence of k-side numbers $s_1, s_2, \ldots, s_n$ of $\Phi_k(\pi)$ is a permutation of the sequence of k-embracing numbers $e_1, e_2, \ldots, e_n$ of π.

Example 5 Consider the permutation $\pi = 6\ 1\ 8\ 2\ 4\ 3\ 5\ 7$ with $k = 4$. Thus $\pi* = 6\ 1\ 8\ 2\ 4\ 3\ 5\ 7\ *$, where $4 < * < 5$. We may more conveniently write $\pi*$ as $\tau = 7\ 1\ 9\ 2\ 4\ 3\ 6\ 8\ 5$, a permutation on $\{1, \ldots, 9\}$. Now, applying the bijection of [3] gives $\Phi(\tau) = \tau' = 4\ 8\ 9\ 1\ 7\ 2\ 3\ 6\ 5$. Thus $\pi'* = 4\ 7\ 8\ 1\ 6\ 2\ 3\ 5\ *$ and $\Phi_2(\pi) = \pi' = 4\ 7\ 8\ 1\ 6\ 2\ 3\ 5$.

Although we can decompose Φ_k as the composition of two bijections between the *symmetric group* $\mathcal{S}_n$ and the *weighted Motzkin paths* as in [3], this decomposition does not seem to yield an interesting continued fraction expansion. Here we just record some partial results in this direction.

Let

$$A_m^k(t, q) = \sum_{\pi \in \mathcal{S}_n} t^{\mathrm{exc}_k \pi} q^{\mathrm{INV}_k \pi}$$

and

$$A_m(x, y, t, q) = \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} A_m^k(t, q).$$

The first values of the polynomials $A_m(x,y,t,q)$ are as follows:

$$\begin{aligned}
A_1(x,y,t,q) &= y + xtq;\\
A_2(x,y,t,q) &= y^2(1+tq) + 2xyt(q+q^2) + x^2(tq^3+t^2q^2);\\
A_3(x,y,t,q) &= y^3(1+(2q+q^2+q^3)t+t^2q^2)\\
&\quad +3xy^2((q+q^2+q^3+q^4)t+(q^2+q^3)t^2)\\
&\quad +3x^2y((q^3+q^4)t+(q^2+q^3+q^4+q^5)t^2)\\
&\quad +x^3(q^5t+(2q^4+q^5+q^6)t^2+q^3t^3).
\end{aligned}$$

It has been noted (see [1]) that

$$\sum_{m\geq 0}\frac{u^m}{m!}A_m(x,y,t,1) = \frac{(1-t)\exp(uy(1-t))}{1-t\exp(u(x+y)(1-t))}.$$

Therefore, by applying the addition formula of Rogers – Stieltjes (see [6]) we can derive the following result.

Proposition 7 *The ordinary generating function of $A_m(x,y,t,1)$ has the following Jacobi continued fraction expansion:*

$$\sum_{m\geq 0} A_m(x,y,t,1)u^m = \cfrac{1}{1-b_0u-\cfrac{\lambda_1u^2}{\ddots\,\cfrac{}{1-b_nu-\cfrac{\lambda_{n+1}u^2}{\ddots}}}},$$

where for $n\geq 0$,

$$\begin{aligned}
b_n &= (tx+y)(n+1)+(x+ty)n\\
\lambda_{n+1} &= (n+1)^2(x+y)^2t.
\end{aligned}$$

A combinatorial proof of the above result will eventually produce a q-analogue of Proposition 7. More precisely, set $[n]_q = 1+q+\cdots+q^{n-1}$ for $n\geq 1$ and $[0]_q=0$. Since

$$A_m^k(1,q) = \sum_{\pi\in\mathcal{S}_n} q^{\mathrm{INV}_k\,\pi} = q^k A_m^0(1,q),$$

hence

$$A_m(x,y,1,q) = \sum_{k=0}^{m} \binom{m}{k} (qx)^k y^{m-k} A_m^0(1,q) = (qx+y)^m A_m^0(1,q).$$

We derive then from [3, Theorem 10] the following result.

Proposition 8 *The ordinary generating function of $A_m(x,y,1,q)$ has the following Jacobi continued fraction expansion:*

$$\sum_{m\geq 0} A_m(x,y,1,q)u^m = \cfrac{1}{1-b_0u-\cfrac{\lambda_1 u^2}{\cfrac{\ddots}{1-b_nu-\cfrac{\lambda_{n+1}u^2}{\ddots}}}},$$

where for $n \geq 0$,

$$\begin{aligned} b_n &= q^n([n+1]_q+[n]_q)(qx+y), \\ \lambda_{n+1} &= q^{2n+1}[n+1]_q^2(qx+y)^2. \end{aligned}$$

However, the series $\sum_{m\geq 0} A_m(x,y,t,q)u^m$ does not seem to have a nice Jacobi continued fraction expansion. In fact, suppose

$$\sum_{m\geq 0} A_m(x,y,t,q)u^m = \cfrac{1}{1-b_0u-\cfrac{\lambda_1 u^2}{\cfrac{\ddots}{1-b_nu-\cfrac{\lambda_{n+1}u^2}{\ddots}}}},$$

then we find $b_0 = tqx+y$, $\lambda_1 = q\,(qx+y)^2\,t$ and

$$\begin{aligned} b_1 = {} & q(y^3q+y^3t+y^3+x^3q^3+tq^3x^3 \\ & +x^3tq^4-xy^2+3\,qxy^2+3\,xy^2q^2 \\ & +xy^2t+3\,xy^2tq+x^2yq+3\,yq^2x^2 \\ & +3\,tq^2x^2y+3\,x^2ytq^3-x^2ytq)/(qx+y)^2. \end{aligned}$$

4 Working with words

Let $w = a_1 a_2 \dots a_n \in R(\mathbf{c})$ and let $\pi * = v_k(w) = v_1 v_2 \dots v_n v_{n+1}$ as defined in Section 2.1. (Here, we are identifying $*$ with v_{n+1}.) Apply the bijection Φ_k of Section 3 to π to obtain a permutation $\pi' = v'_1 v'_2 \dots v'_n$. Finally, *decode* π' by replacing each letter x of π' by a letter a_i of w such that $v_{k,i}(w) = x$, to obtain a word $w' = a'_1 a'_2 \dots a'_n$. The mapping $\Phi_{k,\mathrm{w}}$ is then defined on $R(\mathbf{c})$ by $\Phi_{k,\mathrm{w}}(w) = w'$.

We introduce two pieces of notation to aid in the proof of Theorem 4. Firstly, let the rightmost occurence of the largest small letter ℓ in w be coded to the letter L. Then for any letter a_i in w, a_i is small if and only if $v_i \leq L$. Secondly, write θ as the decoding map. Thus $a_i = \theta(x)$ if $v_{k,i}(w) = x$.

Proof of Theorem 4: We shall show that $\Phi_{k,\mathrm{w}}$ is a bijection on $R(\mathbf{c})$ such that for all $w \in R(\mathbf{c})$, we have

$$(\mathrm{des}_k, \mathrm{MAD}_k, \mathrm{MAK}_k)\ w = (\mathrm{exc}_k, \mathrm{INV}_k, \mathrm{DEN}_k)\ \Phi_{k,\mathrm{w}}(w). \tag{10}$$

By equation (2), we have $D_k(\pi) = D_k(w)$, and moreover the sequence of k-embracing numbers of π equals the sequence of k-embracing numbers of w. By the remark following the proof of equation (9), it follows that the set of k-excedance places of π' equals the set of k-descent bottoms of w, the set of k-excedance tops of π' equals the set of k-descent tops of w and that the sequence of k-side numbers of π' is a permutation of the sequence of k-embracing numbers of w.

To complete the proof of equation (10), we must look in more detail at the construction of $\Phi_{k,\mathrm{w}}$, derived from the construction of Φ given above. Let F, F', G and G' denote respectively the sets of descent bottoms, descent tops, non descent bottoms and non descent tops for $\pi*$. We note that each of the above sets is the image under v_k of the corresponding k-set for $w*$. Let f and g be the non-decreasing words formed from the letters of F and G respectively. Let f' and g' be the words formed from the letters of F' and G' respectively, whose imversion bottom and inversion top numbers respectively are the embracing numbers of those letters in $\pi*$, that is, the k-embracing numbers of the corresponding letters in w. Form the biword

$$\alpha = \begin{pmatrix} f & | & g \\ f' & | & g' \end{pmatrix}.$$

Then the columns to the left of the bar are excedances, while those to the right are non-excedances. Finally, rearrange the columns of α to form the biword $\widetilde{\pi'*}$ and read off the permutation $\pi'*$ from the second row. We must show that the excedances in α correspond precisely to the k-excedances in $\widetilde{w'}$.

First we note that the column $\binom{a}{b}$ of α corresponds to the column $\binom{a}{b}$ of $\widetilde{\pi'*}$ and thence either to the column $\binom{a}{b}$ or to the column $\binom{a+1}{b}$ of $\widetilde{\pi}'$ according as $a \leq L$ or $a > L$.

Case 1: The column $\binom{a}{b}$ of α is an excedance.

Then a is a descent bottom and b is a descent top in $\pi*$.

Case 1(i): $b \leq L$.

Then $a < L$. So there is a column $\binom{a}{b}$ in $\widetilde{\pi}'$ corresponding to a column $\binom{\theta(a)}{\theta(b)}$ in $\widetilde{w'}$. Now the proof that $\binom{\theta(a)}{\theta(b)}$ is an excedance (which in this case is the same as a k-excedance, as $\theta(b)$ is small), follows exactly as in Section 4 of [3].

Case 1(ii): $b) > L$.

If $a \leq \ell$ then, as before, the corresponding column in $\widetilde{w'}$ is $\binom{\theta(a)}{\theta(b)}$, which is a k-excedance as $\theta(a) < \theta(b)$. If $a > L$ then, as $a < b$, $a+1 \leq b$ and so $\theta(a+1) \leq \theta(b)$. Therefore the corresponding column $\binom{\theta(a+1)}{\theta(b)}$ of $\widetilde{w'}$ is a k-excedance, as $\theta(b)$ is large.

Case 2: The column $\binom{a}{b}$ of α is a non-excedance.

Then a is a non descent bottom and b is a non descent top in $\pi*$.

Case 2(i): $b \leq L$.

Then $a \geq b$, so that $\theta(a) \geq \theta(b)$. Now if $a > L$ then the column $\binom{a+1}{b}$ occurs in $\widetilde{\pi}'$ and the corresponding column in $\widetilde{w'}$ is $\binom{\theta(a+1)}{\theta(b)}$, which is a non-$k$-excedance as $\theta(b)$ is small. If $a \leq L$ then the column $\binom{a}{b}$ occurs in $\widetilde{\pi}'$ and the corresponding column in $\widetilde{w'}$ is $\binom{\theta(a)}{\theta(b)}$, which is a non-$k$-excedance as $\theta(b)$ is small.

Case 2(ii): $b > L$.

Then also $a > L$, so the column in $\widetilde{w'}$ corresponding to the column $\binom{a}{b}$ of $\widetilde{\pi'*}$ is $\binom{\theta(a+1)}{\theta(b)}$, which we must show is a non k-excedance. Now, let b_1 be the smallest letter such that $b_1 > b$, $\theta(b_1) = \theta(b)$ and b_1 occurs to the right of b in the second row of α (that is, in the word g'). We will show that no such letter b_1 exists. Now such a b_1 must occur to the left of b in the word w, as large letters in w are coded from right to left. So the k-embracing numbers of b and b_1 in w satisfy $e(b_1) \geq e(b)$. But as b_1 occurs to the right of b in g and there is no letter c to the right of b in g in the range $b < c < b_1$, the inversion top numbers of b and b_1 in g satisfy $s(b_1) \leq s(b)$. By the construction of g we have $s(b_1) = e(b_1)$ and $s(b) = e(b)$. Hence $e(b_1) = e(b)$. Hence there can be no letter between b_1 and b in w that is greater than b. Hence b_1 is a descent top, and cannot be a letter of g. Hence we have shown that no letter $c > b$ with $\theta(c) = \theta(b)$ occurs to the right of b in g. Therefore, if b occurs in the p-th column from the right of g, we have

$$\begin{aligned} p &\leq 1 + e(b) + \#\{\, j \mid j \in G', \theta(a_j) > \theta(b)\,\} \\ &\leq 1 + \#\{\, j \mid j \in G, \theta(a_j) > \theta(b)\,\}. \end{aligned}$$

For to every k-descent of w to the right of b that embraces b there corresponds a non k-descent bottom of w to the right of w that is greater than b, and to every non k-descent top of w that is greater than b there corresponds a non k-descent bottom of w that is greater than b. Thus we have $\theta(a+1) > \theta(b)$ as required.

It now follows that the set of k-excedance places in w' equals the set of excedance places in $\pi'*$, that is, the set of k-excedance places in π'. Similarly, the set of k-excedance tops in w' equals the set of k-excedance tops in π'. Now, by the previous argument, $w'_{\mathrm{E}} = \theta(f')$ and $w'_{\mathrm{N}} = \theta(g'')$ (where g'' is obtained from g' by stripping off the final letter $*$.) By the proof of Case 2(ii) above, $\mathrm{INV}_k\, w'_{\mathrm{N}} = \mathrm{INV}\, \pi'*_{\mathrm{N}}$, and by a similar argument, $\mathrm{Imv}_k\, w'_{\mathrm{E}} = \mathrm{Imv}\, \pi'*_{\mathrm{E}}$. Thus $\mathrm{Ine}_k\, w' = \mathrm{Ine}\, \pi'* = \mathrm{Ine}_k\, \pi'$. Hence equation (10) follows.

To complete the proof, we must show that $\Phi_{k,\mathrm{W}}$ is a bijection. The proof is not essentially different from that presented in [3] for the mapping Φ_{W}, that is, for the case $k = 0$, but for completeness we set down the proof here.

It clearly suffices to show that $\Phi_{k,\mathrm{w}}$ is an injection. Suppose that for words w_1 and w_2 we have

$$w_t \mapsto \pi_t = v_k(w_t) \mapsto \pi'_t = \Phi(\pi_k) \mapsto w'$$

for $t = 1, 2$. Since the map $w_t \mapsto \pi'_t$ is clearly an injection, it suffices to prove that $\pi'_1 = \pi'_2$. We refer to the sets and words involved in the construction of $\Phi_{k,\mathrm{w}}(w_t)$ by F_t, f_t, G_t, etc., for $t = 1, 2$. Now $F_t = E(\pi'_t) = E(w'), = \{i_1, i_2, \ldots, i_r\}$, say, for $t = 1, 2$, and G_t is the complementary set. Hence the words f_t and g_t can be determined. Further, f'_t is an inverse image under θ of the word $a_{i_1} a_{i_2} \ldots a_{i_r}$, and we have a similar result for g'_t. We can calculate the side numbers of π'_t, as these equal the side numbers of w'. Now we can recover f'_t and g'_t, provided that we can determine the relative size of any two letters of these words that have the same image under θ. Suppose that $\theta(v'_i) = \theta(v'_j) = a$ for letters v'_i and v'_j of π'_t. We distinguish several cases.

1. Suppose that $s(v'_i) > s(v'_j)$. Then v'_i must occur to the left of v'_j in the word π_t. Hence, by the way in which the value $v(w_t)$ is defined, $v'_i > v'_j$ or $v'_i < v'_j$ according as a is large or small.

2. Suppose that $s(v'_i) = s(v'_j)$.

   (a) If $v'_i \in F'_t$ and $v'_j \in G'_t$ then v'_i is a descent top in π_t and v'_j is not a descent top in π_t. Hence v'_i must occur to the left of v'_j in the permutation π_t, otherwise $s(v'_j) = e(v'_j) > e(v'_i) = s(v'_i)$. Thus $v'_i > v'_j$ or $v'_i < v'_j$ as in the previous case.

   (b) Let $v'_i, v_j \in G'_t$ with $i < j$. We may assume that $\theta(v'_m) \neq a$ for any m between i and j. Then $v'_i < v'_j$, for otherwise (i, j) would be an inversion in π'_t and we would have $s(v'_i) \neq s(v'_j)$.

   (c) Let $v'_i, v'_j \in F'_t$ with $i < j$. Then both v'_i and v'_j are descent tops in π_t. Hence a must be a large letter. As in the previous case, we have $v'_i < v'_j$.

Hence f'_t and g'_t are completely determined by w'. Thus $\pi'_1 = \pi'_2$ and $\Phi_{k,\mathrm{w}}$ is an injection.

This completes the proof of Theorem 4. $\square$

Example 6 Consider the word $w = 3\ 1\ 4\ 1\ 2\ 1\ 3\ 4$ of example 4, with $k = 2$. Write $\pi * = v_2(w) = 7\ 1\ 9\ 2\ 4\ 3\ 6\ 8\ 5$. As in example 5 we have $\pi' * = \Phi_2(\pi) * = 4\ 8\ 9\ 1\ 7\ 2\ 3\ 6\ 5$. Decoding π then gives $w' = \Phi_{2,\mathrm{w}}(w) = 2\ 4\ 4\ 1\ 3\ 1\ 1\ 3$. From examples 1, 3 and 4 we have

$$\begin{aligned}(\mathrm{des}_2, \mathrm{MAD}_2, \mathrm{MAK}_2)\ w &= (\mathrm{exc}_2, \mathrm{INV}_2, \mathrm{DEN}_2)\ w' \\ &= (4, 21, 14).\end{aligned}$$

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