Annals of Combinatorics **4** (1997) 313-327



# A combinatorial interpretation of the Seidel generation of $q$-derangement numbers

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Received September 15, 1997

*AMS Subject Classification*: 05A15, 05A30

**Abstract.** In [8] Dumont and Randrianarivony have given several combinatorial interpretations for the coefficients of the Euler-Seidel matrix associated to $n!$. In this paper we consider a $q$-analogue of their results, which leads to the discovery of a new mahonian statistic "maf" on the symmetric group. We then give new proofs and generalizations of some results of Gessel and Reutenauer [12] and Wachs [17].

*Keywords*: Mahonian statistics, permutations, $q$-derangement numbers, Seidel matrices

## 1. Introduction

Euler (see [8]) considered the *difference table* $(d_n^k)_{0\le k\le n}$, where the generic coefficients $d_n^k$ are defined by

$$d_n^n = n! \quad \text{and} \quad d_n^k = d_n^{k+1} - d_{n-1}^k \qquad (1 \le k \le n-1). \tag{1.1}$$

Let $a_n^k = d_{n+k}^k$ $(n, k \ge 0)$. Then the above relations can be written as

$$a_0^k = k! \quad \text{and} \quad a_n^k = a_n^{k-1} + a_{n+1}^{k-1} \qquad (n, k \ge 0).$$

The matrix $(a_n^k)_{n,k\ge 0}$ is also called the *Seidel matrix* associated to the sequence $a_n^0$ in the literature (see [7, 9]). The first terms of these matrices are as follows:

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | | | | | |
| 1 | 0 | 1 | | | | |
| 2 | 1 | 1 | 2 | | | |
| 3 | 2 | 3 | 4 | 6 | | |
| 4 | 9 | 11 | 14 | 18 | 24 | |
| 5 | 44 | 53 | 64 | 78 | 96 | 120 |

$(d_n^k)$

$\longrightarrow$

| $k\backslash n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 2 | 9 | 44 |
| 1 | 1 | 1 | 3 | 11 | 53 | |
| 2 | 2 | 4 | 14 | 64 | | |
| 3 | 6 | 18 | 78 | | | |
| 4 | 24 | 96 | | | | |
| 5 | 120 | | | | | |

$(a_n^k)$

Iterating the difference equation (1.1) we derive

$$a_n^0 = d_n^0 = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right), \tag{1.2}$$

which is the *classical derangement number $d_n$*, that is, the number of derangements on $\{1,2,\cdots,n\}$ (cf. [16, p. 67]).

In several recent papers [4, 6, 12, 17], the $q$-maj counting of the derangements on $\{1,2,\cdots,n\}$ has been studied. Consider the *$q$-derangement numbers $d_n(q)$* defined by

$$d_n(q) = \sum_{\sigma\in\mathcal{D}_n} q^{\operatorname{maj}\sigma}, \tag{1.3}$$

where $\mathcal{D}_n$ is the set of all derangements on $\{1,2,\cdots,n\}$. Then the following $q$-analogue of equation (1.2) has been obtained:

$$d_n(q) = [n]_q! \sum_{i=0}^{n} (-1)^i \frac{q^{\binom{i}{2}}}{[i]_q!} \qquad (n \geq 1). \tag{1.4}$$

Here, $[n]_q = 1+q+\cdots+q^{n-1}$ is the $q$-analogue of the nonnegative integer $n$ and $[n]_q! = [1]_q[2]_q\cdots[n]_q$ is the $q$-analogue of $n!$.

In this paper, we shall put the $q$-derangement numbers in the context of a Seidel matrix as Dumont and Randrianarivony [8] did for the ordinary derangement numbers. To this end, in section 2 we introduce the notion of $q$-Seidel matrix. In section 3 we define a new statistic "maf" on permutations and then prove bijectively that this is a mahonian statistic. In section 4 we consider the $q$-Seidel matrix associated to the $q$-derangement numbers and give combinatorial interpretations for all of the coefficients in this matrix in terms of the new statistic "maf". As a consequence we get a new proof of a formula of Gessel and Reutenauer [12] and of Wachs [17]. Finally we close this paper with some remarks and open questions.

We will need the following notations and results of $q$-calculus (see [11]). The $q$-binomial coefficients are defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \qquad (n \geq k \geq 0).$$

Define also $(t;q)_n = (1-t)(1-qt)\cdots(1-q^{n-1}t)$ and $(t;q)_\infty = \lim_{n\to\infty}(t;q)_n$. Then the two $q$-analogues of the exponential series $e^t = \sum_{n\geq 0} t^n/n!$ are defined by

$$e_q(t) = \sum_{n\geq 0} \frac{t^n}{[n]_q!} = \frac{1}{((1-q)t;q)_\infty}, \tag{1.5}$$

$$E_q(t) = \sum_{n\geq 0} \frac{q^{\binom{n}{2}} t^n}{[n]_q!} = (-(1-q)t;q)_\infty. \tag{1.6}$$

Notice that $e_q(t)\cdot E_q(-t) = 1$.

## 2. $q$-Seidel matrices

Let us introduce the following generalization of Seidel matrix.

**Definition 2.1.** *Given a sequence $(a_n(x,q))$ $(n\geq 0)$ of elements in a commutative ring, we call the q-Seidel matrix asssociated to $(a_n(x,q))$ the double sequence $(a_n^k(x,q))$ $(n\geq 0, k\geq 0)$ given by the recurrence*

$$\begin{cases} a_n^0(x,q) = a_n(x,q), & (n\geq 0) \\ a_n^k(x,q) = xq^n a_n^{k-1}(x,q) + a_{n+1}^{k-1}(x,q). & (k\geq 1, n\geq 0) \end{cases} \tag{2.1}$$

*Moreover $(a_n^0(x,q))$ is called the initial sequence and $(a_0^n(x,q))$ the final sequence of the q-Seidel matrix.*

**Lemma 2.1.** *We have*

$$a_n^k(x,q) = \sum_{i=0}^{k} (xq^n)^{k-i} \binom{k}{i}_q a_{n+i}^0(x,q). \tag{2.2}$$

**Proof:** Recall that
$$\binom{n}{k}_q = q^{n-1}\binom{n-1}{k-1}_q + \binom{n-1}{k}_q.$$
We proceed by induction on $k$. Clearly (2.2) is valid for $k=1$. Suppose (2.2) is true for $k-1$. We then have

$$\begin{aligned} a_n^k(x,q) &= \sum_{i=0}^{k-1} \binom{k-1}{i}_q \Big( (xq^n)^{k-i} a_{n+i}^0(x,q) + (xq^{n+1})^{k-1-i} a_{n+1+i}^0(x,q) \Big) \\ &= (xq^n)^k a_n^0(x,q) + \sum_{i=1}^{k-1} (xq^n)^{k-i} \binom{k-1}{i}_q a_{n+i}^0(x,q) \\ &\qquad + \sum_{i=0}^{k-2} (xq^{n+1})^{k-1-i} \binom{k-1}{i}_q a_{n+1+i}^0(x,q) + a_{n+k}^0(x,q) \\ &= (xq^n)^k a_n^0(x,q) + \sum_{i=1}^{k-1} (xq^n)^{k-i} \binom{k}{i}_q a_{n+i}^0(x,q) + a_{n+k}^0(x,q). \end{aligned}$$

This completes the proof. ∎

In particular we pass from the initial sequence to the final sequence and conversely by the *Gauss inversion formula* [2, p. 96]:

$$a_0^n(x,q) = \sum_{i=0}^{n} x^{n-i} \binom{n}{i}_q a_i^0(x,q), \tag{2.3}$$

$$a_n^0(x,q) = \sum_{i=0}^{n} (-x)^{n-i} q^{\binom{n-i}{2}} \binom{n}{i}_q a_0^i(x,q). \tag{2.4}$$

Define the generating functions as follows:

$$a(t) = \sum_{n\geq 0} a_n^0(x,q)t^n, \qquad \bar{a}(t) = \sum_{n\geq 0} a_0^n(x,q)t^n,$$

and

$$A(t) = \sum_{n\geq 0} a_n^0(x,q)\frac{t^n}{[n]_q!}, \quad \bar{A}(t) = \sum_{n\geq 0} a_0^n(x,q)\frac{t^n}{[n]_q!}.$$

**Proposition 2.1.** *The generating functions of the initial and final sequences are related by the following equations:*

$$\bar{a}(t) = \sum_{n\geq 0} a_n^0(x,q)\frac{t^n}{(xt;q)_{n+1}}; \tag{2.5}$$

$$\overline{A}(t) = e_q(xt)A(t). \tag{2.6}$$

**Proof:** Note that

$$\frac{1}{(t;q)_{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k}_q t^k.$$

Hence

$$\begin{aligned}
\sum_{n\geq 0} a_n^0(x,q)\frac{t^n}{(xt;q)_{n+1}} &= \sum_{n,k\geq 0} \binom{n+k}{k}_q a_n^0(x,q)x^k t^{n+k} \\
&= \sum_{m\geq 0} t^m \sum_{n=0}^{m} \binom{m}{n}_q x^{m-n} a_n^0(x,q) \\
&= \sum_{m\geq 0} a_0^m(x,q)t^m.
\end{aligned}$$

By (1.5) we have

$$\begin{aligned}
e_q(xt)A(t) &= \sum_{i,j\geq 0} \frac{a_i^0(x,q)t^i}{[i]_q!}\cdot\frac{x^j t^j}{[j]_q!} \\
&= \sum_{i,j\geq 0} \binom{i+j}{i}_q a_i^0(x,q)x^j \frac{t^{i+j}}{[i+j]_q!} \\
&= \sum_{n\geq 0} \Big(\sum_{i=0}^{n} x^{n-i}\binom{n}{i}_q a_i^0(x,q)\Big)\frac{t^n}{[n]_q!},
\end{aligned}$$

which completes the proof of (2.6) in view of (2.3). ∎

**Remark:** If $x = q = 1$ we get the classical formulas [7,9]:

$$\bar{a}(t) = \frac{1}{1-t} a\left(\frac{t}{1-t}\right) \quad \text{and} \quad \bar{A}(t) = e^t A(t).$$

If $x = 0$ we have $\bar{A}(t) = A(t)$.

## 3. A new mahonian statistic "maf"

Let $S_n$ be the set of permutations on $[n] = \{1, 2, \ldots, n\}$. Recall that $i \in [n]$ is a *fixed point* of $\sigma \in S_n$ if $\sigma(i) = i$. Let $\operatorname{fix}\sigma$ denote the number of fixed points of $\sigma$. The permutation $\sigma$ has a *descent* at $i \in \{1, 2, \ldots, n-1\}$ if $\sigma(i) > \sigma(i+1)$ and we call $i$ the *descent place* of $\sigma$. The *major index* of $\sigma$, denoted $\operatorname{maj}\sigma$, is the sum of all the descent places of $\sigma$. Let $\mathrm{FIX}(\sigma) = \{i \mid \sigma(i) = i\}$ be the set of all fixed points of $\sigma$ and $\tilde{\sigma}$ the *restriction* of $\sigma$ to $\{1, 2, \ldots, n\} \setminus \mathrm{FIX}(\sigma)$.

**Definition 3.1.** *If $\sigma \in S_n$ with $\mathrm{FIX}(\sigma) = \{i_1, i_2, \ldots, i_l\}$, then the statistic "maf" is defined by*

$$\operatorname{maf}\sigma = \sum_{j=1}^{l} (i_j - j) + \operatorname{maj}\tilde{\sigma}.$$

**Example 3.1.** *Let $\sigma = 3\,2\,1\,6\,5\,9\,4\,8\,7$. Then $\mathrm{FIX}(\sigma) = \{2, 5, 8\}$ and $\tilde{\sigma} = 3\,1\,6\,9\,4\,7$. Hence $\operatorname{fix}\sigma = 3$, $\operatorname{maj}\sigma = 1+2+4+6+8 = 21$ and $\operatorname{maf}\sigma = (2-1)+(5-2)+(8-3)+(1+4) = 14$.*

We now show that the bistatistics $(\mathrm{fix}, \mathrm{maf})$ and $(\mathrm{fix}, \mathrm{maj})$ are equidistributed on the symmetric group $S_n$ (Corollary 3.1). In particular, this shows that maf is a Mahonian statistic.

Let $\sigma = x_1 x_2 \ldots x_n \in S_n$. For convenience we put $x_0 = -\infty$ and $x_{n+1} = +\infty$. For $0 \le i \le n$, a pair $(i, i+1)$ of positions is the *$j$-th slot* of $\sigma$ *provided that $x_i \neq i$*, i.e., $i$ is not a fixed point of $\sigma$ and that $\sigma$ has $i - j$ fixed points $f$ such that $f < i$. Clearly we can insert a fixed point into the $j$-th slot to obtain the permutation

$$(\sigma, j) = x_1' x_2' \ldots x_i' \; (i+1) \; x_{i+1}' \ldots x_n', \tag{3.1}$$

where $x' = x$ if $x \le i$ and $x' = x + 1$ if $x > i$.

More generally, if $\sigma$ is a derangement in $S_n$ and $(i_1, i_2, \ldots, i_m)$ a sequence of integers such that $0 \le i_1 \le i_2 \le \cdots \le i_m \le n$, we can insert $m$ fixed points into the derangement $\sigma$ successively, finally obtaining

$$(\sigma, i_1, \ldots, i_m) = ((\sigma, i_1, \ldots, i_{m-1}), i_m).$$

Note that the fixed points of this last permutation are $i_1 + 1, i_2 + 2, \ldots, i_m + m$.

**Example 3.2.** *Let $\sigma = 2143$ and $(i_1, i_2, \ldots, i_m) = (0, 1, 1, 4)$. Then we have $(\sigma, 0) = 1\,3\,2\,5\,4$, $(\sigma, 0, 1) = 1\,4\,3\,2\,6\,5$, $(\sigma, 0, 1, 1) = 1\,5\,3\,4\,2\,7\,6$ and $(\sigma, 0, 1, 1, 4) = 1\,5\,3\,4\,2\,7\,6\,8$.*

We can of course undertake the reverse operation. That is, if a permutation $\sigma$ in $S_{m+n}$ has $m$ fixed points we can find a unique derangement $\mathrm{dp}(\sigma) \in S_n$, called (following Wachs [17]) the *derangement part* of $\sigma$, and a unique sequence of integers $i_1 \le \cdots \le i_m$, which we call the *fixed point sequence* of $\sigma$, such that

$$\sigma = (\mathrm{dp}(\sigma), i_1, \ldots, i_m).$$

It is easy to see that

$$\mathrm{maf}\,\sigma = \mathrm{maj}\,\mathrm{dp}(\sigma) + i_1 + \cdots + i_m. \tag{3.2}$$

Consider a permutation $\sigma$ with $n$ slots. The $j$-th slot $(i, i+1)$ of $\sigma$ is said to be *green* if $\mathrm{des}(\sigma, j) = \mathrm{des}\,\sigma$, *red* if $\mathrm{des}(\sigma, j) = \mathrm{des}\,\sigma + 1$. We assign *values* to the green slots of $\sigma$ from right to left, from 0 to $g$, and to the red slots from left to right, from $g+1$ to $n$. Denote the value of the $j$-th slot by $g_j$. (When we refer to the "largest" slot, we will mean largest in terms of $j$.)

**Example 3.3.** *Let* $\sigma = 2\,1\,4\,3$. *Then* $(\sigma,0) = 1\,3\,2\,5\,4$, $(\sigma,1) = 3\,2\,1\,5\,4$, $(\sigma,2) = 2\,1\,3\,5\,4$, $(\sigma,3) = 2\,1\,5\,4\,3$, $(\sigma,4) = 2\,1\,4\,3\,5$. *Hence slots 0, 2 and 4 are green, while 1 and 3 are red. Therefore*

$$(g_0, \ldots, g_n) = (2,3,1,4,0). \tag{3.3}$$

It is easy to see that every slot is either green or red. In fact, one can see that $(i, i+1)$ is green if either $x_{i+1} < x_i \le i$, or $i < x_{i+1} < x_i$, or $x_i \le i < x_{i+1}$. So $(i, i+1)$ is red if either $x_{i+1} \le i < x_i$, or $i < x_i < x_{i+1}$ or $x_i < x_{i+1} \le i$. (Expressed in terms of cyclic intervals (*cf.* [13]), slot $(i, i+1)$ is green if $i+1 \in \rrbracket x_i, x_{i+1} \rrbracket$.)

Denote by $d_j$ the number of descents of $(\sigma, j)$ that lie to the right of $x_i'$ in (3.1).

**Lemma 3.1.** *Let $\sigma$ be a permutation in $S_n$. If the $j$-th slot $(i, i+1)$ is green then* $\mathrm{maj}(\sigma, j) - \mathrm{maj}\,\sigma = d_j$, *if $(i, i+1)$ is red then* $\mathrm{maj}(\sigma, j) - \mathrm{maj}\,\sigma = d_j + i$.

**Proof:** Let $(i, i+1)$ be a green slot. Since no new descents are formed by inserting a fixed point into the $j$-th slot of $\sigma$, $\mathrm{maj}(\sigma, j) - \mathrm{maj}\,\sigma$ equals the number of descents of $\sigma$ that are displaced to the right when this fixed point is inserted. This number equals $d_j$. The case in which $(i, i+1)$ is red is dealt with similarly. ∎

**Remark:** If $\sigma$ is a derangement in $S_n$, the $j$-th slot of $\sigma$ is just $(j, j+1)$ for $0 \le j \le n$.

**Lemma 3.2.** *If $\sigma$ is a derangement in $S_n$, then*

$$\mathrm{maj}(\sigma, j) = \mathrm{maj}\,\sigma + g_j \quad \textit{for} \quad 0 \le j \le n.$$

**Proof:** Let $i$ and $j$ be slots. It follows from Lemma 3.1 that if $i$ and $j$ are both green and $i < j$ then $\mathrm{maj}(\sigma,i) \ge \mathrm{maj}(\sigma, j)$, while if $i$ and $j$ are both red and $i < j$ then $\mathrm{maj}(\sigma,i) \le \mathrm{maj}(\sigma, j)$. Therefore 0 is the green slot of $\sigma$ of highest value, and if $i$ is red and $j$ is green we have $\mathrm{maj}(\sigma,i) \ge \mathrm{maj}(\sigma, j)$. This is because for any red slot $i$ we have $\mathrm{maj}(\sigma,i) \ge \mathrm{maj}(\sigma,0)$ by Lemma 3.1. Hence, if $m$ is the largest red slot of $\sigma$, i.e., $g_m = n$, for any two slots $i$ and $j$ with $g_i < g_j$ we have

$$\mathrm{maj}\,\sigma \le \mathrm{maj}(\sigma,i) \le \mathrm{maj}(\sigma, j) \le \mathrm{maj}(\sigma,m).$$

It therefore suffices to show that

$$\operatorname{maj}(\sigma, m) = \operatorname{maj}\sigma + n.$$

Now, consider a green slot $(i, i+1)$. If $i+1$ is a non-excedance place, i.e., $x_{i+1} \leq i+1$, then, as $\sigma$ is a derangment, $x_{i+1} \leq i$. Hence $x_{i+1} < x_i \leq i$. Thus $i$ is a non-excedance place. Since $n$ is a non-excedance place and $m+1$, $m+2$, $\dots$, $n$ are green slots, we have

$$m+1 > x_{m+1} > \cdots > x_n.$$

As the slot $m$ is red, either $m$ is a non-excedance place and $m$ is a non-descent or $m$ is an excedance place and $m$ is a descent. In each case, inserting a fixed point into the $m$-th slot introduces a new descent for $i = m+1$ and moves $n-(m+1)$ descents one place further to the right. Hence

$$\operatorname{maj}(\sigma, m) = \operatorname{maj}\sigma + (m+1) + (n-m-1) = \operatorname{maj}\sigma + n,$$

as required. ∎

**Remark:** Suppose that $\sigma$ is a derangement in $S_n$ and $0 \leq i \leq n$. It follows from Lemmas 3 and 4 that $d_i = g_i$ if $i$ is green and $d_i = g_i - i$ if $i$ is red. If $i$ is green then

$$\operatorname{maj}(\sigma, i, i) = \operatorname{maj}(\sigma, i) + g_i.$$

Hence, if $j \leq i$, it follows from Lemma 3.2 that

$$\operatorname{maj}(\sigma, j, i) = \operatorname{maj}(\sigma, i) + g_j.$$

On the other hand, if $i$ is red, then

$$\operatorname{maj}(\sigma, i, i) = \operatorname{maj}(\sigma, i) + g_i - i.$$

Now one can easily see that, if $k$ is the largest green slot to the left of slot $i$, $g_k = g_i - i$. Hence, if $j < i$, it follows again from Lemma 3.2 that

$$\operatorname{maj}(\sigma, j, i) = \operatorname{maj}(\sigma, i) + g_j + 1.$$

We are now ready to state the key result of this section. Let $S(\sigma, m)$ denote the set of permutations in $S_{n+m}$ with derangement part $\sigma \in \mathcal{D}_n$. Note that

$$S(\sigma, m) = \{(\sigma, \mathbf{i}) \,|\, \mathbf{i} = (i_1, \dots, i_m) \text{ and } 0 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n\}.$$

**Theorem 3.1.** *There is a bijection $\Psi$ on $S(\sigma, m)$ such that if $\Psi(\sigma, \mathbf{i}) = (\sigma, \mathbf{j})$ then*

$$\operatorname{maj}(\sigma, \mathbf{i}) = \operatorname{maf}(\sigma, \mathbf{j}). \tag{3.4}$$

**Proof:** We divide the proof into two parts.

**The definition of $\Psi$.** We will define such a bijection $\Psi$ by induction on $m \geq 0$.

First, $\Psi$ is the identity mapping on $S(\sigma,0)$. Next, we define $\Psi$ on $S(\sigma,1)$ by

$$\Psi(\sigma,i)=(\sigma,g_i).$$

Then using equation (3.2) and Lemma 3.2 we see that $\Psi$ satisfies equation (3.4).

Let $m>1$ and suppose that $\Psi$ has been defined on $S(\sigma,k)$ for $0\le k\le m-1$. Consider $\tau=(\sigma,i_1,\ldots,i_m)$. Suppose that the $i_m$-th slot of $(\sigma,i_1,\ldots,i_{m-1})$ is green. Then, if

$$\Psi(\sigma,i_1,\ldots,i_{m-1})=(\sigma,j_2,\ldots,j_m),$$

we define

$$\Psi(\tau)=(\sigma,g_{i_m},j_2,\ldots,j_m).$$

Suppose that the $i_m$-th slot of $(\sigma,i_1,\ldots,i_{m-1})$ is red. Then the slots $i_1,\ldots,i_m$ cannot be all the same. Let $k$ be the smallest positive integer such that $i_{m-k}<i_m$. Thus $i_{m-k}<i_{m-k+1}=\cdots=i_m$. Then, if

$$\Psi(\sigma,i_1,\ldots,i_{m-k})=(\sigma,j_1,\ldots,j_{m-k}),$$

we define

$$\Psi(\tau)=(\sigma,\underbrace{g_{i_m}-i_m,\ldots,g_{i_m}-i_m}_{k-1\text{ terms}},j_1+1,\ldots,j_{m-k}+1,g_{i_m}).$$

The following lemma is easily proved by induction.

**Lemma 3.3.** *Let $\tau=(\sigma,i_1,\ldots,i_m)$ and $\Psi(\tau)=(\sigma,j_1,\ldots,j_m)$.*

*Suppose that at least one of the slots $i_1,\ldots,i_m$ is either green or is repeated. Let $i_l$ be the largest such slot. If $i_l$ is green then $j_1=g_{i_l}$. If $i_l$ is red and is repeated then $j_1=g_{i_l}-i_l$.*

*If on the other hand all of the slots $i_1,\ldots,i_m$ are red and are distinct, then $j_1=g_{i_1}$.*

*Suppose that at least one of the slots $i_1,\ldots,i_m$ is red. If $i_l$ is the largest red slot then $j_m=g_{i_l}$.*

*If on the other hand all of these slots are green then $j_m=g_{i_1}$.*

It follows from this lemma that $j_1,\ldots,j_m$ as defined above are in ascending order.

We now show by induction on $m$ that $\Psi$ satisfies equation (3.4).

If $i_m$ is green, then using Lemma 3.2 we have

$$\begin{aligned}\mathrm{maj}(\sigma,i_1,\ldots,i_m) &= \mathrm{maj}(\sigma,i_1,\ldots,i_{m-1})+g_{i_m}\\ &= \mathrm{maf}(\sigma,j_2,\ldots,j_m)+g_{i_m}\\ &= \mathrm{maf}(\sigma,g_{i_m},j_2,\ldots,j_m).\end{aligned}$$

If $i_m$ is red, let $k$ be the smallest positive integer such that $i_{m-k}<i_m$, then

$$\begin{aligned}&\mathrm{maj}(\sigma,i_1,\ldots,i_m)\\ &= \mathrm{maj}(\sigma,i_1,\ldots,i_{m-k})+(m-k)+(k-1)(g_{i_m}-i_m)+g_{i_m}\\ &= \mathrm{maf}(\sigma,j_1,\ldots,j_{m-k})+(m-k)+(k-1)(g_{i_m}-i_m)+g_{i_m}\\ &= \mathrm{maf}(\sigma,\underbrace{g_{i_m}-i_m,\ldots,g_{i_m}-i_m}_{k-1\text{ terms}},j_1+1,\ldots,j_{m-k}+1,g_{i_m}).\end{aligned}$$

This is because inserting the first fixed point $i_m$ into $(\sigma, i_1, \ldots, i_{m-k})$ adds a descent and increases maj by $g_{i_m} + (m-k)$. Inserting each of the remaining fixed points $i_m$ has the same affect as inserting a fixed point into a green slot of value $g_{i_m} - i_m$.

**$\Psi$ is a bijection.** It remains to show that $\Psi$ is a bijection on $S(\sigma, m)$. It suffices to show that $\Psi$ is an injection.

We use induction on $m$. The result is clearly true for $m = 0$ and $m = 1$.

Let $\tau = (\sigma, i_1, \ldots, i_m)$ and $\Psi(\tau) = (\sigma, j_1, \ldots, j_m)$. Suppose that $\Psi(\tau) = \Psi(\tau')$, where $\tau' = (\sigma, i'_1, \ldots, i'_m)$.

If both $i_m$ and $i'_m$ are green or red then it is easy to show using the induction hypothesis that $\tau = \tau'$. So suppose that $i_m$ is green and $i'_m$ is red. Thus $j_1 = g_{i_m}$, $j_m = g_{i'_m}$.

Suppose that $i_1, \ldots, i_m$ are all green. Then $j_m = g_{i_1}$. Hence $i_1 = i'_m$, contradiction.

Let $i_u$ be the largest red slot amongst $i_1, \ldots, i_m$. Then $j_m = g_{i_u}$. Hence $i'_m = i_u < i_m$.

*Case 1:* Suppose that one of the slots $i'_1, \ldots, i'_m$ is either green or is repeated. Let $i'_v$ be the largest such slot. If $i'_v$ is green, then

$$\Psi(\tau') = (\sigma, g_{i'_v} + (m-v), \ldots, g_{i'_m}).$$

Hence $j_1 = g_{i_m} = g_{i'_v} + (m-r)$. Since $i_m$ and $i'_v$ are both green, this means that $i_m \leq i'_v < i'_m$, contradiction.

If $i'_v$ is red, then

$$\Psi(\tau') = (\sigma, g_{i'_v} - i'_v + (m-v), \ldots, g_{i'_m}).$$

Hence $j_1 = g_{i_m} = g_{i'_v} - i'_v + (m-r)$. But $g_{i'_v} - i'_v$ is the value of the largest green slot $i_w$ less than $i'_v$. As $i_m$ is green this means that $i_m \leq i_w < i'_v \leq i'_m$, contradiction.

*Case 2:* Suppose that all of the slots $i'_1, \ldots, i'_m$ are red and distinct. Then

$$\Psi(\tau') = (\sigma, g_{i'_1} + (m-1), \ldots, i'_m).$$

Hence $j_1 = g_{i_m} = g'_{i_1} + (m-1) > g'_{i-1}$. This is a contradiction, since $i_m$ is green and $i'_1$ is red. ∎

**Example 3.4.** *Let $\sigma = 2\,1\,4\,3$ and consider $(\sigma, 0, 1, 1, 4) \in S(\sigma, 4)$. Then the values of the slots of $\sigma$ have been calculated in (3.3). The bijection $\Psi$ goes as follows:*
*since slot 0 is green in $\sigma$ we have*

$$\Psi(\sigma, 0) = (\sigma, g_0) = (\sigma, 2);$$

*since slot 1 is red we have*

$$\Psi(\sigma, 0, 1) = (\sigma, 2+1, g_1) = (\sigma, 3, 3);$$

*again, since slot 1 is red we have*

$$\Psi(\sigma, 0, 1, 1) = (\sigma, g_1 - 1, 2+1, g_1) = (\sigma, 2, 3, 3);$$

*Finally, since slot 4 is green we obtain*

$$\Psi(\sigma, 0, 1, 1, 4) = (\sigma, g_4, 2, 3, 3) = (\sigma, 0, 2, 3, 3) \in S(\sigma, 4).$$

*Let $\tau = (\sigma, 0, 1, 1, 4)$ and $\tau' = (\sigma, 0, 2, 3, 3)$. Then $\tau = 1\,5\,3\,4\,2\,7\,6\,8$ and $\tau' = 1\,3\,2\,4\,8\,6\,7\,5$. It is easy to see that $\operatorname{maj}\tau = 12$ and $\operatorname{maf}\tau' = 12$. Hence we have checked equation (3.4).*

Using theorem 3.1, we obtain the following result.

**Corollary 3.1.** $(a)$ *There is a bijection* $\phi : S_n \to S_n$ *such that for any* $\sigma \in S_n$ *we have*

$$(\text{fix}, \text{maf})\sigma = (\text{fix}, \text{maj})\phi(\sigma).$$

$(b)$ *The bi-statistic* $(\text{fix}, \text{maf})$ *is equidistributed with the bi-statistic* $(\text{fix}, \text{maj})$ *on the symmetric group* $S_n$.

The following result was first proved by Wachs [17, corollary 3].

**Corollary 3.2.** *Let* $\sigma$ *be a derangement in* $S_n$ *and* $m \geq 0$. *We have*

$$\sum_{\pi \in S(\sigma, m)} q^{\text{maj}\,\pi} = q^{\text{maj}\,\sigma} \binom{m+n}{n}_q .$$

**Proof:** By theorem 5 we have

$$\begin{aligned} \sum_{\pi \in S(\sigma, m)} q^{\text{maj}\,\pi} &= \sum_{\pi \in S(\sigma, m)} q^{\text{maf}\,\pi} \\ &= q^{\text{maj}\,\sigma} \sum_{0 \leq i_1 \leq \cdots \leq i_m \leq n} q^{i_1 + i_2 + \cdots i_m} \\ &= q^{\text{maj}\,\sigma} \binom{m+n}{n}_q . \end{aligned}$$

The last line follows from a well-known result [1, p. 33]. ∎

## 4. $q$-derangement matrices

We first prove the following result.

**Proposition 4.1.** *Let* $(a_n^k(x,q))$ *be a q-Seidel matrix. Then the following three conditions are equivalent:*

$$a_n^0(x,q) = [n]_q! \sum_{i=0}^{n} (-1)^i \frac{q^{\binom{i}{2}}}{[i]_q!}, \tag{4.1}$$

$$a_0^n(x,q) = [n]_q! \left( 1 + \sum_{i=1}^{n} \frac{(x-1)(x-q)\cdots(x-q^{i-1})}{[i]_q!} \right), \tag{4.2}$$

$$a_0^n(1,q) = [n]_q! \quad \textit{and} \quad a_n^0(x,q) \quad \textit{is independent of} \quad x. \tag{4.3}$$

**Proof:** By the $q$-binomial formula [11, p.7]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} t^n = \frac{(at;q)_\infty}{(t;q)_\infty},$$

we have in view of (1.5) and (1.6),

$$1 + \sum_{n=1}^{\infty} \frac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q!} t^n = e_q(xt) E_q(-t).$$

Therefore the generating functions of (4.1), (4.2) and (4.3) are respectively the following:

$$A(t) = \sum_{n\geq 0} a_n^0(x,q)\frac{t^n}{[n]_q!} = \frac{E_q(-t)}{1-t}, \tag{4.4}$$

$$\overline{A}(t) = \sum_{n\geq 0} a_0^n(x,q)\frac{t^n}{[n]_q!} = \frac{e_q(xt)E_q(-t)}{1-t}, \tag{4.5}$$

$$\overline{A}(t)|_{x=1} = \sum_{n\geq 0} a_0^n(1,q)\frac{t^n}{[n]_q!} = \frac{1}{1-t}. \tag{4.6}$$

So, it suffices to prove that the equivalence of (4.4), (4.5) and (4.6). Indeed,

(4.4) $\Longleftrightarrow$ (4.5): this follows from proposition 2;

(4.5) $\Longrightarrow$ (4.6): this is obvious;

(4.6) $\Longrightarrow$ (4.4): since $A(t)$ is independent of $x$, equation (4.4) follows then from (2.6) by setting $x=1$. ∎

**Definition 4.1.** *A q-derangement matrix is the q-Seidel matrix satisfying any of the three conditions of proposition 8.*

If $x=1$, then $a_0^n(x,q) = [n]_q!$ and the $q$-derangement matrix is as follows :

| $k\backslash n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | $1$ | $0$ | $q$ | $q+q^2$ | $\left(\begin{smallmatrix}q+2q^2+2q^3\\+2q^4+q^5+q^6\end{smallmatrix}\right)$ |
| 1 | $1$ | $q$ | $q+q^2+q^3$ | $\left(\begin{smallmatrix}q+2q^2+2q^3\\+3q^4+2q^5+q^6\end{smallmatrix}\right)$ | |
| 2 | $1+q$ | $q+2q^2+q^3$ | $\left(\begin{smallmatrix}q+2q^2+3q^3\\+4q^4+3q^5+q^6\end{smallmatrix}\right)$ | | |
| 3 | $[3]_q!$ | $\left(\begin{smallmatrix}q+3q^2+5q^3\\+5q^4+3q^5+q^6\end{smallmatrix}\right)$ | | | |
| 4 | $[4]_q!$ | | | | |

$(a_n^k(1,q))$

Denote by $S_n^k$ the set of permutations on $[n+k]$ of which all the fixed points are included in $\{n+1, n+2, \ldots, n+k\}$. In particular $S_n^0$ is the set of permutations without fixed points on $[n]$ and $S_0^k$ the set of all permutations on $[k]$. The following result generalizes a result of Dumont and Randrianarivony [8].

**Theorem 4.1.** *The coefficients $a_n^k(x,q)$ ($n, k \geq 0$) in a q-derangement matrix have the following combinatorial interpretation:*

$$a_n^k(x,q) = \sum_{\sigma\in S_n^k} x^{\mathrm{fix}\,\sigma} q^{\mathrm{maf}\,\sigma}. \tag{4.7}$$

**Proof:** Notice that $S_{n+1}^{k-1} \subset S_n^k$. Set

$$\Delta_n^k = S_n^k \setminus S_{n+1}^{k-1} = \{\sigma \in S_n^k \mid \sigma(n+1) = n+1\}.$$

We construct a bijection $\varphi : \Delta_n^k \to S_n^{k-1}$ such that for all $\sigma \in \Delta_n^k$,

$$\begin{aligned} \operatorname{maf}\sigma &= n + \operatorname{maf}(\varphi(\sigma)), \\ \operatorname{fix}\sigma &= 1 + \operatorname{fix}(\varphi(\sigma)). \end{aligned}$$

Indeed, if $\sigma \in \Delta_n^k$ we define $\varphi(\sigma)$ as the word obtained from $\sigma$ by deleting $n+1$ and reduce all the values strictly bigger than $n+1$. It is readily verified that $\varphi$ is the desired bijection. Therefore

$$\sum_{\sigma \in S_n^k} x^{\operatorname{fix}\sigma} q^{\operatorname{maf}\sigma} = xq^n \sum_{\sigma \in S_n^{k-1}} x^{\operatorname{fix}\sigma} q^{\operatorname{maf}\sigma} + \sum_{\sigma \in S_{n+1}^{k-1}} x^{\operatorname{fix}\sigma} q^{\operatorname{maf}\sigma}, \tag{4.8}$$

which is the recurrence (2.1). So it remains to check the initial condition. Now $S_0^n = S_n$ and it is well-known [14] that $\sum_{\sigma \in S_n} q^{\operatorname{maj}\sigma} = [n]_q!$, so it follows from corollary 3.1 that

$$a_0^n(1,q) = \sum_{\sigma \in S_n} q^{\operatorname{maf}\sigma} = \sum_{\sigma \in S_n} q^{\operatorname{maj}\sigma} = [n]_q!.$$

The theorem follows then from proposition 4.1, since $a_n^0(x,q)$ is clearly independent of $x$. ∎

**Remark:** Since (fix, maf) and (fix, maj) are not equidistributed on $S_1^2$ we cannot replace maf by maj in the above theorem.

From Corollary 3.1, proposition 4.1 and theorem 4.1 we derive the following result.

**Corollary 4.1.** *The final sequence of the q-derangement matrix has the following interpretation:*

$$\begin{aligned} a_0^n(x,q) &= \sum_{\sigma \in S_n} x^{\operatorname{fix}\sigma} q^{\operatorname{maf}\sigma} && (4.9) \\ &= \sum_{\sigma \in S_n} x^{\operatorname{fix}\sigma} q^{\operatorname{maj}\sigma} && (4.10) \\ &= [n]_q! \left( 1 + \sum_{i=1}^{n} \frac{(x-1)(x-q)\cdots(x-q^{i-1})}{[i]_q!} \right). && (4.11) \end{aligned}$$

Note that the last equation has been obtained by Gessel and Reutenauer [12] and by Wachs [17] in the special $x = 0$ case using different methods.

## 5. An open problem about $q$-succession numbers

Let $\sigma$ be a permutation in $S_n$. For convenience put $\sigma(0) = 0$. We say that an element $p$ (with $1 \le p \le n$) is a *succession* of $\sigma$ if $\sigma(p) = \sigma(p-1)+1$. The $p$ is called the *succession position*, while $\sigma(p)$ is called the *succession value*. Let $\operatorname{SUC}(\sigma)$ be the set of succesion values of $\sigma$ and let $\operatorname{suc}\sigma$ be the number of successions of $\sigma$. For example, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 4 & 3 & 8 & 9 & 5 & 6 & 7 & 2 \end{pmatrix},$$

then $\mathrm{SUC}(\sigma) = \{1,9,6,7\}$ and $\operatorname{suc}\sigma = 4$.

We use a variant of Foata's first fundamental transformation [10] to show that the statistics fix and suc are equidistributed on $\mathcal{S}_n$.

Given a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in S_n$ we set $\sigma^d = \sigma(2)\cdots\sigma(n)\sigma(1)$. We call the *standard form* of the factorization into cycles of $\sigma$ the unique writing $\bar{\sigma}$ such that in each cycle $(a, \sigma(a), \ldots, \sigma^l(a))$ the maximum $\sigma^l(a)$ is in the last position and the cycles of $\sigma$ are decreasingly ordered according to their maxima. (Note that this is *not* the usual definition of standard form.) We define $\varphi(\sigma)$ as the permutation obtained by erasing the parentheses in the standard form of $\bar{\sigma}^d$.

The following lemma is easy to verify.

**Lemma 5.1.** *The mapping $\varphi$ is a bijection on $S_n$ such that for all $\sigma \in S_n$, $FIX(\sigma) = \mathrm{SUC}(\varphi(\sigma))$ and $\operatorname{fix}\sigma = \operatorname{suc}\varphi(\sigma)$. Hence the statistics* fix *and* suc *are equidistributed on $S_n$.*

For example, if $\sigma = 1\,4\,2\,8\,3\,6\,7\,5\,9 \in S_9$, then

$$\sigma^d = 4\,2\,8\,3\,6\,7\,5\,9\,1 \quad \text{and} \quad \bar{\sigma}^d = (1\,4\,3\,8\,9)(5\,6\,7)(2).$$

Erasing the parentheses we obtain the permutation $\varphi(\sigma) = 1\,4\,3\,8\,9\,5\,6\,7\,2$. We have

$$\mathrm{FIX}(\sigma) = \mathrm{SUC}(\varphi(\sigma)) = \{1,6,7,9\}.$$

Define the statistic

$$\operatorname{suc}'\sigma = \begin{cases} \operatorname{suc}\sigma, & \text{if } \sigma(1) \neq 1, \\ \operatorname{suc}\sigma - 1, & \text{if } \sigma(1) = 1; \end{cases}$$

and let

$$F_n(x) = \sum_{\sigma\in S_n} x^{\operatorname{fix}\sigma}, \quad S_n(x) = \sum_{\sigma\in S_n} x^{\operatorname{suc}'\sigma}. \tag{5.1}$$

Then, using lemma 5.1, we obtain a bijective proof of the following known results (See [3, 15]).

**Proposition 5.1.** *We have*

$$S_{n+1}(x) = F_{n+1}(x) + (1-x)F_n(x), \tag{5.2}$$

*and in particular*

$$S_{n+1}(0) = d_{n+1} + d_n. \tag{5.3}$$

Setting $q = 1$ in (4.20) we see that

$$\sum_{n\geq 0} F_n(x)\frac{t^n}{n!} = \frac{e^{(x-1)t}}{1-t}. \tag{5.4}$$

Hence, from equation (5.2), we have

$$\sum_{n\geq 0} S_n(x)\frac{t^n}{n!} = \frac{e^{(x-1)t}}{1-t} + (1-x)\sum_{n\geq 1} F_{n-1}(x)\frac{t^n}{n!}, \tag{5.5}$$

in which by convention $S_0(x) = F_0(x) = 1$. Thus

$$\sum_{n\geq 0} S_n(x)\frac{t^n}{n!} = \frac{e^{(x-1)t}}{1-t} + (1-x)\int_0^t \frac{e^{(x-1)z}}{1-z}dz. \tag{5.6}$$

Let $\mathcal{L}$ be the formal Laplace transformation on the ring of formal power series, that is, $\mathcal{L}(\sum a_n x^n/n!) = \sum a_n x^n$. Then

$$\sum_{n\geq 0} F_n(x)t^n = \mathcal{L}\left(\frac{e^{(x-1)t}}{1-t}\right) = \sum_{n\geq 0} \frac{n!t^n}{[1-(x-1)t]^{n+1}}. \tag{5.7}$$

Therefore

$$\begin{aligned}
\sum_{n\geq 0} S_n(x)t^n &= \sum_{n\geq 0} F_n(x)t^n + (1-x)\sum_{n\geq 0} F_n(x)t^{n+1} \\
&= [1-(x-1)t]\sum_{n\geq 0} F_n(x)t^n \\
&= \sum_{n\geq 0} \frac{n!t^n}{[1-(x-1)t]^n}.
\end{aligned} \tag{5.8}$$

In the case of $q = 1$, using lemma 11, we can restate theorem 9 in terms of successions. Unfortunately, since the mapping $\varphi$ does not keep track of the maj statistic, we do not have a full interpretation in the last model.

The distribution of our statistics on $S_3$ is as follows:

| $\sigma\backslash$stat | maf | maj | suc | fix |
|---|---|---|---|---|
| 1 2 3 | 0 | 0 | 3 | 3 |
| 1 3 2 | 1 | 2 | 1 | 1 |
| 2 1 3 | 3 | 1 | 0 | 1 |
| 2 3 1 | 2 | 2 | 1 | 0 |
| 3 1 2 | 1 | 1 | 1 | 0 |
| 3 2 1 | 2 | 3 | 0 | 1 |

**Statistic distributions on** $S_3$

Finally we record two open problems related to our work.

1) Find a mahonian statistic "mag" such that $(\text{suc}, \text{mag})$ is equidistributed with $(\text{fix}, \text{maj})$ on the symmetric group $S_n$.

2) Generalize the statistic "maf" on permutations to *words* as in [5, 13] for other mahonian statistics.

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