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On the powers of Motzkin paths

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Abstract

We give a simple combinatorial proof of a recent formula of Lascoux for the powers of continued fractions.

Given an alphabet $\mathbb{A}$ composed of commuting letters a's, let $\lambda_x(\mathbb{A}) = \prod_{a\in\mathbb{A}}(1+ax)$. For any $k \in \mathbb{R}$, let $\Lambda^n(k\mathbb{A})$ and $S^n(k\mathbb{A})$ be the coefficients of x^n in $(\lambda_x(\mathbb{A}))^k$ and $(\lambda_{-x}(\mathbb{A}))^{-k}$ respectively. Recall that the *weight* of a Motzkin path π, noted $w(\pi)$, is the product of the weights of its steps : 1 for a North-East step, α_i for an horizontal step at level i, ξ_i for a South-East step between level i and $i-1$.

Assuming that one has the following continued fraction expansion :

$$\sum x^n S^n(\mathbb{A}) = \cfrac{1}{(1-\alpha_0 x) - \cfrac{x^2\xi_1}{(1-\alpha_1 x) - \cfrac{x^2\xi_2}{(1-\alpha_2 x) - \cfrac{x^2\xi_3}{\ddots}}}}, \tag{1}$$

then Lascoux [2], generalizing a result of Lehner, proved the following :

Theorem 1 (Lascoux) *For any $n \in \mathbb{N}$, any $k \in \mathbb{R}$, one has*

$$\Lambda^n(k\mathbb{A}) = \sum_{\pi}(-1)^{n-\ell}\binom{k}{\ell}w(\pi) \tag{2}$$

where the sum is over all Motzkin paths π of length n, where ℓ is the number of ground points (other than the origin) of the path.

Lascoux's proof of (2) is of lots eruditions. Here we show how to derive it directly from Flajolet's classical result [1].

First we note that by definition $S^n(k\mathbb{A}) = (-1)^n\Lambda^n((-k)\mathbb{A})$, so (2) can be paraphrased as follows :

$$S^n(k\mathbb{A}) = \sum_{\pi} \binom{k+\ell-1}{\ell} w(\pi), \tag{3}$$

where the sum is the same as in (2). Since the two sides of (3) are polynomials of k, it suffices to prove (3) for $k \in \mathbb{N}$. In view of Flajolet's result [1], which corresponds to the $k = 1$ case of (3), the symmetric function $S^n(k\mathbb{A})$ is equal to $\sum w(\pi_1 \cdots \pi_k)$, where the sum is over all the k-tuples $(\pi_1, \ldots, \pi_k)$ of Motzkin paths such that the product (or juxtaposition) $\pi_1 \cdots \pi_k$ is a Motzkin path of length n. By convention, an empty Motzkin paths is of length 0. Clearly, each Motzkin path π having ℓ ground points (other than the origin) can be ontained in such a way from $\binom{k+\ell-1}{\ell}$ k-tuples $(\pi_1, \ldots, \pi_k)$ of Motzkin paths, which is exactly what (3) means.

References

[1] P. Flajolet, *Combinatorial aspects of continued fractions*, Discrete Math. **32** (1980) 125-161.

[2] A. Lascoux, *Motzkin paths and powers of continued fractions*, Séminaire Lotharingien de Combinatoire **44** (2000), Article B44e.