THE GENERATING FUNCTION FOR THE DIFFERENCE IN EVEN AND ODD THREE-LINE LATIN RECTANGLES

Jiang Zeng

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Résumé. On donne une fonction génératrice exponentielle pour la différence entre le nombre des rectangles latin pairs de taille $3 \times n$ et le nombre de ceux impairs. On montre ensuite que cette formmule équivaut à une formula, qui est apparemment différente, donnée dans [4].

Abstract. We give a simple exponential generating function for the difference between the number of even reduced Latin rectangles of size $3 \times n$ and the number of odd ones. We show that this formula is equivalent to an apparently different formula given in [4].

Let $\mathcal{S}_n$ be the set of permutations of $[n] = \{1, 2, \ldots, n\}$. For each $\sigma \in \mathcal{S}_n$, we denote by $\mathrm{cyc}\,\sigma$ (*resp.* $fix\ \sigma$) the number of cycles (*resp.* fixed points) of σ. Further let L_n be the generating polynomial for $\mathcal{S}_n \times \mathcal{S}_n$ defined by

$$L_n(\alpha, \beta, p, q, \gamma) = \sum_{\sigma, \tau \in \mathcal{S}_n} \alpha^{\mathrm{cyc}\,\sigma} \beta^{\mathrm{cyc}\,\tau} p^{fix\ \sigma} q^{fix\ \tau} \gamma^{fix\ \sigma\tau^{-1}}.$$

Let $(x)_n = x(x+1)\cdots(x+n-1)$. Using a generalization of Cauchy's summation theorem for Schur functions and some group representation theories Andrews et al. [1] obtained the following expression for the exponential generating function for the L_n :

$$1 + \sum_{n \geq 1} \frac{t^n}{n!} L_n(\alpha, \beta, p, q, \gamma) = e^{\alpha\beta t(2-p-q-\gamma+pq\gamma)} \sum_{m \geq 0} \frac{t^m}{m!} \frac{(\alpha)_m (\beta)_m}{[1+\beta(1-q)t]^{m+\alpha}[1+\alpha(1-p)t]^{m+\beta}[1+(1-\gamma)t]^{m+\alpha\beta}}. \tag{1}$$

In [5] the author has shown some applications of (1) and further derived a combinatorial proof of (1) by generalizing a standard method due to Gessel [3].

Recently Habsieger and Janssen [4] raised the problem to count *signed Latin rectangles*. Recall that a *Latin rectangle* of size $k \times n$ is an $k \times n$ array, such that each row is a permutation of $12 \dots n$, and the product $^{-1}$ of any two distinct row permutations , is *fixed-point* free, *i.e.* is a *derangement*. A *reduced* Latin rectangle has the additional property that the first row is the identity permutation. A Latin rectangle is called *row-even* if the product of the signs of its row permutations is $+1$ and *row-odd* if this product is -1. Let $S_{k,n}$ denote the difference between the number of even reduced Latin rectangles of size $k \times n$ and the number of odd ones. In [4], Habsieger and Janssen proved the following results :

$$\begin{aligned} &(1.0) \qquad & S_{1,n} &= 1, \\ &(1.1) \qquad & S_{2,n} &= (-1)^{n-1}(n-1), \\ &(1.2) \qquad & S_{3,n} &= (-1)^{n-1}\frac{(n-2)n!}{2}[t^n]e^{2(arctanh\, t - t)}, \end{aligned}$$

where $[t^n]F(t)$ stands for the coefficient of t^n in the series expansion of $F(t)$. Note that in [4] the series $arctanh\, t$ of (1.2) was *incorrectly* replaced by $\sinh t$ due to an obvious error at the end of its proof [4, p. 75], where $(2j+1)$ was *incorrectly* replaced by $(2j+1)!$.

Actually formulas (1.0) and (1.1) are straightforward. The purpose of this note is to show that (1.2) is a equivalent to a special case of (1).

Proposition 1 *We have*

$$(1.3) \qquad 1 + \sum_{n \geq 1} S_{3,n}\frac{t^n}{n!} = e^{2t}\left[\frac{(1-t)^2}{1+t} + \frac{t}{(1+t)^2}\right].$$

Proof: Let $\mathcal{D}\backslash$ be the set of derangements of $[n]$. By definition, we have

$$S_{3,n} = \sum_{\sigma, \tau, \sigma\tau^{-1} \in \mathcal{D}_n} \varepsilon(\alpha)\varepsilon(\beta).$$

As $\varepsilon(\alpha)\varepsilon(\beta) = (-1)^{n - \mathrm{cyc}\,\alpha}(-1)^{n-\mathrm{cyc}\,\beta} = (-1)^{\mathrm{cyc}\,\alpha}(-1)^{\mathrm{cyc}\,\beta}$, we conclude that

$$S_{3,n} = L_n(-1,-1,0,0,0), \qquad (n \geq 1).$$

Formula (1.3) follows then from (1). $\square$

Proposition 2 *Formulas (1.2) and (1.3) are equivalent.*

Proof: From (1.3) we derive

$$\sum_{n\geq 1}(-1)^{n-1}S_{3,n}\frac{t^n}{n!} = 1+\frac{e^{-2t}}{(1-t)^2}(t^2+t^3-1). \tag{1.4}$$

Note that

$$\begin{aligned}\frac{e^{-2t}}{(1-t)^2} &= \sum_{n\geq 0}\frac{(-2)^n t^n}{n!}\cdot\sum_{n\geq 0}(n+1)t^n \\ &= \sum_{n\geq 0}\left(\sum_{i=0}^{n}(n-i+1)\frac{(-2)^i}{i!}\right)t^n.\end{aligned}$$

Substituting this into (1.4) and equating the coefficients of t^n yield

$$(-1)^{n-1}\frac{S_{3,n}}{n!} = \sum_{i=0}^{n-3}(n-4-i)\frac{(-2)^i}{i!}+\frac{(-2)^{n-1}}{n(n-3)!}. \tag{1.5}$$

On the other hand, since

$$arctanh\, x = \sum_{n\geq 0}\frac{x^{2n+1}}{2n+1} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

therefore

$$[x^n]e^{2\,arctanh\, x-2x} = [x^n]\frac{1+x}{1-x}e^{-2x} = 2\sum_{i=0}^{n-1}\frac{(-2)^i}{i!}+\frac{(-2)^i}{n!},$$

and it follows from (1.2) that

$$(-1)^{n-1}\frac{S_{3,n}}{n!} = (n-2)\sum_{i=0}^{n-1}\frac{(-2)^i}{i!}+(-1)^n(n-2)\frac{2^{n-1}}{n!}. \tag{1.6}$$

It is easy to check that the right hand-sides of formulas (1.5) and (1.6) are equal. Indeed, if we subtract the right-hand side of (1.6) from that of (1.5), we obtain

$$\begin{aligned}\sum_{i=0}^{n-3}(2+i)\frac{(-2)^i}{i!}+\frac{(-2)^{n-2}}{(n-3)!} &= 2\sum_{i=0}^{n-3}\frac{(-2)^i}{i!}-2\sum_{i=0}^{n-3}\frac{(-2)^{i-1}}{(i-1)!}+\frac{(-2)^{n-2}}{(n-3)!} \\ &= 2\frac{(-2)^{n-3}}{(n-3)!}+\frac{(-2)^{n-2}}{(n-3)!} \\ &= 0,\end{aligned}$$

which was to be proved. □

Finally, by applying Darboux's method [Co, p. 121] we get from (1.4) a good asymptotic estimate for $(-1)^{n-1}S_{3,n}$:

$$(-1)^{n-1}S_{3,n} \sim \sqrt{2\pi}\; n^{n+3/2}e^{-n-2} \qquad (n \longrightarrow +\infty).$$

1 Résumé substantiel en français

Un rectangle latin est *pair* selon les lignes si le produit des signes des permutation de lignes vaut $+1$ et *impair* selon les lignes si ce produit vaut -1. Notons $S_{k,n}$ la différence entre le nombre de rectangle réduits pairs de taille $k \times n$ et le nombre de ceux impairs. Dans [4], Habsieger et Janssen ont établi des formules explicites pour $S_{k,n}$ lorsque $k = 1, 2, 3$:

$$\begin{array}{llcl} (1.0) & S_{1,n} & = & 1, \\ (1.1) & S_{2,n} & = & (-1)^{n-1}(n-1), \\ (1.2) & S_{3,n} & = & (-1)^{n-1}\dfrac{(n-2)n!}{2}[t^n]e^{2(arctanh\, t - t)}, \end{array}$$

Le but de cette note est de signaler une formule simple pour la fonction génératrice exponentielle de la suite $\{S_{3,n}\}_{n\geq 1}$:

$$(1.3) \qquad 1 + \sum_{n\geq 1} S_{3,n}\frac{t^n}{n!} = e^{2t}\left[\frac{(1-t)^2}{1+t} + \frac{t}{(1+t)^2}\right].$$

Cette formule est en fait un cas particulier d'une formule plus générale due à Gessel [3] ou de celle de Andrews et al. [1]. On montre ensuite que les deux formules (1.2) et (1.3) sont equivalentes.

References

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Jiang ZENG
UFR de Mathématiques et Informatique
Université Louis Pasteur
7, rue René Descartes
67084 Strasbourg Cedex, France
e-mail : jzeng@math.u-strasbg.fr