The q-variations of Sylvester's bijection between odd and strict partitions

Jiang Zeng

Institut Girard Desargues, Université Claude Bernard (Lyon 1)

43, boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France

E-mail: `zeng@desargues.univ-lyon1.fr`

Abstract

The q-identities corresponding to Sylvester's bijection between odd and strict partitions are investigated. In particular, we show that Sylvester's bijection implies the Rogers-Fine identity and give a simple proof of a partition theorem of Fine, which does not follow directly from Sylvester's bijection. Finally, the so-called (m,c)-analogues of Sylvester's bijection are also discussed.

1 Introduction

A *partition* is any sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ in weakly decreasing order and containing only finitely many non-zero terms, called *parts*. A partition is called *odd* if all the parts are odd, and *strict* if the parts are distinct each other. A partition of an integer n is a partition such that the sum of its parts is equal to n. Let $\mathcal{P}_{\mathcal{O}}(n)$ be the set of partitions into *odd* parts of an integer n and $\mathcal{P}_{\mathcal{D}}(n)$ be the set of *strict* partitions of n. Then Euler's partition theorem states that $\#\mathcal{P}_{\mathcal{O}}(n) = \#\mathcal{P}_{\mathcal{D}}(n)$. It is easy to see that Euler's result follows from the infinite product identity :

$$\prod_{i\geq 0} \frac{1}{1-q^{2i+1}} = \prod_{i\geq 0} (1+q^{i+1}). \tag{1}$$

In 1882 Sylvester [21] published an interesting bijective proof of Euler's result. Another classical bijection was given by Glaisher [13], but his proof is usually considered to be close to the above generating function (or g.f.) proof (see [9, p.52]). A natural question is then to seek the generating function proof that corresponds or is closer to Sylvester's bijection. In this paper we shall investigate the dual problem : what kind of q-identities can be derived from Sylvester's bijection ?

There are several ways to describe Sylvester's bijection [6, 9, 19, 17], his original version was actually in terms of *2-modular* diagrams, later used by Bessenrodt [7], other descriptions are based on *fish hooks* diagrams [19, p. 13, vol. II] or Frobenius representation of partitions [6]. Let us recall Sylvester's bijection $\varphi : \mathcal{P}_O(n) \longrightarrow \mathcal{P}_D(n)$ through an example. If $\lambda = (9^2\,5\,3\,1) \in \mathcal{P}_O(27)$, then draw the 2-modular diagram and read it through its hooks, where each node "2" should be read twice :

$$\mathcal{P}_O(27) \ni (9^2\,5\,3\,1) \mapsto \begin{array}{ccccccccc} 2 & = & 2 & = & 2 & = & 2 & - & 1 \\ \| & & & & & & & & \\ 2 & & 2 & = & 2 & = & 2 & - & 1 \\ \| & & \| & & & & & & \\ 2 & & 2 & & 1 & & & & \\ \| & & | & & & & & & \\ 2 & & 1 & & & & & & \\ | & & & & & & & & \\ 1 & & & & & & & & \end{array} \mapsto (9\,7\,6\,4\,1) \in \mathcal{P}_{\mathcal{D}}(27).$$

Given a partition $\lambda = (\lambda_1,\, \lambda_2,\, \ldots)$ we shall note λ_1 the largest part of λ and $l(\lambda)$ the *length* of λ, i.e. the number of its parts. The *rank* of λ is its largest part less its length, i.e. $\lambda_1 - l(\lambda)$. The *alternative length* of λ is $l_a(\lambda) = \lambda_1 - \lambda_2 + \lambda_3 - \cdots$. A *chain* of λ is a subsequence $\lambda_p,\, \lambda_{p+1},\, \ldots, \lambda_q$ such that $\lambda_i - \lambda_{i+1} \leq 1$ for $p \leq i \leq q$ so in a strict partition, a *chain* is a maximal sequence of parts that are consecutive integers. Let $n_d(\lambda)$ be the number of *distinct* parts of λ and $n_c(\lambda)$ the number of chains of λ. Finally let $d_2(\lambda)$ be the size of Durfee square in the 2-modular diagram of λ.

For example, if $\lambda = (12,\, 10,\, 9,\, 8,\, 5,\, 5)$, then λ has three chains : $\{12\}$, $\{10, 9, 8\}$, $\{5, 5\}$, so $n_c(\lambda) = 3$, $l(\lambda) = 6$, $\lambda_1 = 12$, the rank is $12 - 6 = 6$, $n_d(\lambda) = 5$, $l_a(\lambda) = 12 - 10 + 9 - 8 + 6 - 5 = 4$ and $d_2(\lambda) = 4$.

Let $\mathcal{P}_{\mathcal{O}}$ denote the set of odd partitions and $\mathcal{P}_{\mathcal{D}}$ the set of strict partitions. We now summarize the most remarkable properties of Sylvester's bijection as following (cf. [7, Proposition 2.2] and [9, p. 51-52]).

Theorem 1 (Sylvester-Bessenrodt) *Let $\varphi : \mathcal{P}_O \longrightarrow \mathcal{P}_D$ be Sylvester's bijection and $\mu = \varphi(\lambda)$, then $|\lambda| = |\mu|$, and*

- $d_2(\lambda) = [(l(\mu)+1)/2]$,
- $n_d(\lambda) = n_c(\mu)$,
- $l(\lambda) = l_a(\mu)$,
- $l(\lambda) + (\lambda_1 - 1)/2 = \mu_1$.

Consider the corresponding generating functions of odd and strict partitions :

$$\begin{aligned} A &= \sum_{\lambda\in\mathcal{P}_O} x^{n_d(\lambda)} y^{l(\lambda)} z^{l(\lambda)+(\lambda_1-1)/2} t^{d_2(\lambda)} q^{|\lambda|} && (2)\\ B &= \sum_{\mu\in\mathcal{P}_D} x^{n_c(\mu)} y^{l_a(\mu)} z^{\mu_1} t^{\lfloor (l(\mu)+1)/2\rfloor} q^{|\mu|}. && (3)\end{aligned}$$

Il follows from Theorem 1 that $A = B$. Therefore one may expect to derive some interesting q-identities by calculating A and B separately. Surprisingly, the two analytical expressions thus obtained coincide, so we will not get a true q-identity but, instead, an *independent* proof of the identity $A = B$.

Throughout this paper we shall use the standard notations of q-series :

$$(a)_n := (a;\, q)_n = \prod_{i=1}^{n}(1-aq^{i-1}), \qquad (a)_\infty := (a;\, q)_\infty = \prod_{i=1}^{\infty}(1-aq^{i-1}).$$

Theorem 2 *The generating functions (2) and (3) are both equal to*

$$\begin{aligned} x\sum_{n\geq 0}(ty)^{n+1} z^{2n+1} q^{(n+1)(2n+1)} &\left[1-(1-x)yz^2\, q^{4n+3}\right]\\ &\times \frac{((1-x)zq^2;\, q^2)_n((1-x)yzq;\, q^2)_n}{(zq^2;\, q^2)_{n+1}(yzq;\, q^2)_{n+1}}. \qquad (4)\end{aligned}$$

Interesting q-identities will appear when we specialize some of the parameters. For example, we shall derive from the above theorem the so-called Hence Sylvester's bijection turns out to provide a combinatorial proof

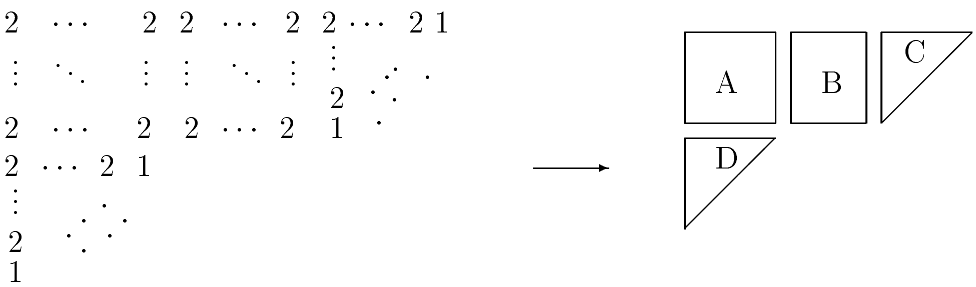

Figure 1: *Decomposition of a 2-modular diagram with 2 at the southeast corner of the Durfee square* $\boxed{A}$

of the Rogers-Fine identity. Note that Andrews [3] has given a different combinatorial proof of (5) using Durfee rectangles.

The rest of this paper is organized as follows. In section 2 we shall prove Theorem 2. In section 3 we will derive some special cases of Theorem 2, which include Rogers-Fine identity and an extension of Lebesgue's identity, which corresponds to Sylvester's original partition result. In section 4 we will review two partition theorems of Fine and show how a partition theorem of Fine [10], which does not follow from Sylvester's bijection, is related to Sylvester's bijection through generating function point of view. Finally, in section 5 we consider the extension of Sylvester's bijection when the odd partitions are replaced by partitions whose parts are in an arithmetical sequence $\{c+mk\}_{k\geq 0}$. In this context Theorem 4 generalizes the previous results of Bessenrodt [7] and Pak-Postnikov [20], while Theorem 5 is an extension of Theorem 2. However, as we will show in Proposition 1, the (m,c)-analogue of Rogers-Fine identity is actually equivalent to the ordinary one.

2 Proof of Theorem 2

Let's first consider the generating function (2). We shall apply the Durfee square dissection in the 2-modular diagram of an odd partition λ and distinguish two cases according to the element $c(\lambda)$ at the southeast corner of the Durfee square. Let $d_2(\lambda)=r$ with $r\geq 1$.

i) If $c(\lambda)=2$, then we decompose the 2-modular diagram of odd partitions as in Figure 1 and compute the generating functions of partitions corresponding to the diagrams A, B, C, D.

It is clear that the g.f. of "A" is $xt^ry^rz^{r+(r-1)}q^{2r^2}$, that of "B" is $\frac{1}{1-zq^{2r}}$, since the number of distinct parts of a partition is equal to that of its conjugate (see [1] for other applications of this invariant), the g.f.

of "C" can be obtained through the corresponding conjugate partitions as follows :

$$q^r \prod_{i=1}^{r-1} \left(1 + \frac{xzq^{2i}}{1 - zq^{2i}}\right) = q^r \frac{((1-x)zq^2; q^2)_{r-1}}{(zq^2; q^2)_{r-1}}$$

and that of "D" is

$$\prod_{i=0}^{r-1} \left(1 + \frac{xyzq^{2i+1}}{1 - yzq^{2i+1}}\right) = \frac{((1-x)zyq; q^2)_r}{(zyq; q^2)_r}.$$

Multiplying the four generating functions we get

$$xy^r z^{2r} t^r q^{2r^2+r} \frac{((1-x)zq^2; q^2)_{r-1}}{(zq^2; q^2)_r} \cdot \frac{((1-x)zyq; q^2)_r}{(zyq; q^2)_r}.$$

ii) If $c(\lambda) = 1$, then we decompose the 2-modular diagram of odd partitions as in Figure 2 where we move all the 1's at the end of r first rows to the last column of "A", which corresponds to the Durfee square, and compute the corresponding generating functions of partitions corresponding to "A", "B" and "C". It is easy to see that the g.f. of "A" is $xt^r y^r z^{r+(r-1)} q^{2r^2-r}$, by enumerating its conjugate partitions the g.f. of "B" is

$$\prod_{i=1}^{r-1} \left(1 + \frac{xzq^{2i}}{1 - zq^{2i}}\right) = \frac{((1-x)zq^2; q^2)_{r-1}}{(zq^2; q^2)_{r-1}},$$

that of "C" is

$$\frac{1}{1 - yzq^{2r-1}} \prod_{i=0}^{r-2} \left(1 + \frac{xyzq^{2i+1}}{1 - yzq^{2i+1}}\right) = \frac{((1-x)yzq; q^2)_{r-1}}{(yzq; q^2)_r}.$$

Multiplying the three generating functions we get

$$xy^r z^{2r-1} t^r q^{2r^2-r} \frac{((1-x)yzq; q^2)_{r-1}}{(yzq; q^2)_r} \cdot \frac{((1-x)zq^2; q^2)_{r-1}}{(zq^2; q^2)_{r-1}}$$

Summing up the two generating functions of i) and ii) and replacing r by $n+1$ we obtain the expression (4).

On the other hand, if μ is a strict partition, then each integer less than $l(\mu)$ appears as a part in its conjugate μ' and there is a gap in the parts of μ, i.e. $\mu_i - \mu_{i+1} \geq 2$, iff the corresponding part in μ' is

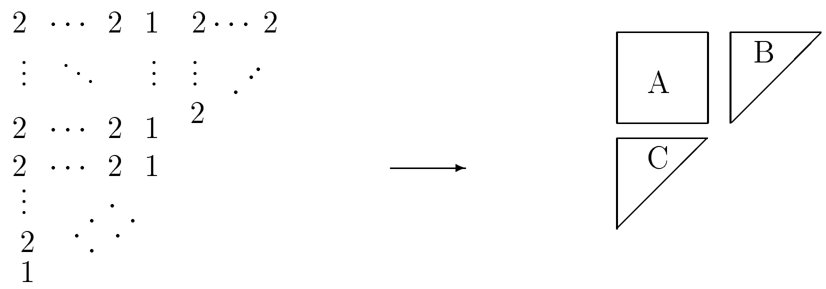

Figure 2: *Decomposition of a 2-modular diagram with 1 at the southeast corner of the Durfee square* $\boxed{A}$

repeated. Moreover, in view of the weight function (3) each odd (resp. even) part of μ' should be weighted by yz (resp. z) and each reapeted odd part should be weighted by x. Hence the relevant generating function of strict partitions of length $2n+2$ is

$$\begin{aligned} & \frac{xzt^{n+1}q^{2n+2}}{1-zq^{2n+2}} \prod_{i=0}^{n} yzq^{2i+1}\left(1+\frac{xyzq^{2i+1}}{1-yzq^{2i+1}}\right) \prod_{i=1}^{n} zq^{2i}\left(1+\frac{xzq^{2i}}{1-zq^{2i}}\right) \\ = \quad & x(qty)^{n+1}z^{2n+2}q^{2(n+1)^2}\frac{((1-x)yzq;q^2)_{n+1}((1-x)zq^2;q^2)_n}{(yzq;q^2)_{n+1}(zq^2;q^2)_{n+1}}; \end{aligned}$$

and that of strict partitions of length $2n+1$ is

$$\begin{aligned} & \frac{xyzt^{n+1}q^{2n+1}}{1-yzq^{2n+1}} \prod_{i=0}^{n-1} yzq^{2i+1}\left(1+\frac{xyzq^{2i+1}}{1-yzq^{2i+1}}\right) \prod_{i=1}^{n} zq^{2i}\left(1+\frac{xzq^{2i}}{1-zq^{2i}}\right) \\ = \quad & x(ty)^{n+1}z^{2n+1}q^{2n^2+3n+1}\frac{((1-x)yzq;q^2)_n((1-x)zq^2;q^2)_n}{(yzq;q^2)_{n+1}(zq^2;q^2)_n}. \end{aligned}$$

Summing up the above two cases we obtain the expression (4).

Remark: One can also derive Theorem 2 by combining the formula for A or B with Theorem 1. The above proof shows that the generating function of odd partitions obtained through Durfee square dissection in the 2-modular diagram corresponds to that of strict partitions obtained through Sylvester's bijection.

3 Some special cases

We first prove that the $t=1$ case of Theorem 2 is equivalent to the so-called Rogers-Fine identity [3] :

$$(1-t)\sum_{n\geq 0}\frac{(aq)_n}{(bq)_n}t^n = \sum_{n\geq 0}\frac{(aq)_n\,(atq/b)_n\,(1-atq^{2n+1})}{(bq)_n\,(tq)_n}(bt)^n q^{n^2}. \qquad (5)$$

Indeed, when $t=1$, the corresponding generating function of odd partitions (2) with largest part $2j+1$ ($j \geq 0$) is

$$\sum_{\substack{\lambda \in \mathcal{P}_O \\ \lambda_1 = 2j+1}} x^{n_d(\lambda)} y^{l(\lambda)} z^{l(\lambda)+(\lambda_1-1)/2} q^{|\lambda|} = \frac{xyz^{j+1}q^{2j+1}}{1-yzq^{2j+1}} \prod_{i=0}^{j-1} \left(1 + \frac{xyzq^{2i+1}}{1-yzq^{2i+1}}\right)$$

therefore,

$$\sum_{\lambda \in \mathcal{P}_O} x^{n_d(\lambda)} y^{l(\lambda)} z^{l(\lambda)+(\lambda_1-1)/2} q^{|\lambda|} = \sum_{j \geq 0} \frac{xyz^{j+1}q^{2j+1}((1-x)yzq;q^2)_j}{(yzq;q^2)_{j+1}}. \tag{6}$$

It follows from (6) and Theorem 2 with $t=1$ that :

$$\begin{aligned} \sum_{j \geq 0} \frac{xyz^{j+1}q^{2j+1}\,((1-x)yzq;\, q^2)_j}{(yzq;\, q^2)_{j+1}} &= x \sum_{n \geq 0} y^{n+1} z^{2n+1} q^{2n^2+3n+1} \qquad (7) \\ \times \left(1-(1-x)yz^2\, q^{4n+3}\right) & \frac{((1-x)zq^2;\, q^2)_n((1-x)yzq;\, q^2)_n}{(zq^2;\, q^2)_{n+1}(yzq;\, q^2)_{n+1}}. \end{aligned}$$

To see that (7) is equivalent to Rogers-Fine identity (5) we need to do some manipulations. In (7), substituting y by qy, q^2 by q and simplifying $xyzq$, we obtain :

$$\begin{aligned} &\sum_{j \geq 0} \frac{((1-x)yzq)_j}{(zyq)_{j+1}} (zq)^j \qquad (8) \\ &= \sum_{n \geq 0} \frac{((1-x)tq)_n\,((1-x)yzq)_n\,(1-(1-x)yz^2q^{2n+2})}{(zq)_{n+1}(yzq)_{n+1}} z^{2n} y^n q^{n^2+2n}. \end{aligned}$$

Finally, multiplying (8) by $(1-zq)(1-z\,y\,q)$ we get Rogers-Fine identity (5) with $a=(1-x)yz$, $b=yzq$ and $t=zq$.

Now, setting $z=1$ in (7), the left-hand side ofbecomes

$$\begin{aligned} &\sum_{j \geq 0} \frac{xyq^{2j+1}}{1-yq^{2j+1}} \frac{((1-x)yq;q^2)_j}{(yq;q^2)_j} \\ =\ &\sum_{j \geq 0} \left[\frac{((1-x)yq;q^2)_{j+1}}{(yq;q^2)_{j+1}} - \frac{((1-x)yq;q^2)_j}{(yq;q^2)_j}\right] \\ =\ &\frac{((1-x)yq;q^2)_\infty}{(yq;q^2)_\infty} - 1, \end{aligned} \tag{9}$$

while the right-hand side plus 1 reduces to

$$\begin{aligned}
&1+\sum_{n\geq 0} y^{n+1}q^{(n+1)(2n+1)}\frac{(1-x;q^2)_{n+1}((1-x)yq;q^2)_{n+1}}{(q^2;q^2)_n(yq;q^2)_{n+1}}\\
&+\sum_{n\geq 0} y^{n+1}q^{(n+1)(2n+3)}\frac{(1-x;q^2)_{n+1}((1-x)yq;q^2)_{n+1}}{(q^2;q^2)_{n+1}(yq;q^2)_{n+1}}\\
&=\sum_{n\geq 0}\frac{(1-x;q^2)_n((1-x)yq;q^2)_n y^n q^{n(2n+1)}}{(q^2;q^2)_n(yq;q^2)_{n+1}}\left[1-(1-x)yq^{4n+1}\right].
\end{aligned}$$

So, substituting x by $1-x$, (7) with $z=1$ reduces to

$$\sum_{n\geq 0} y^n q^{n(2n+1)}\frac{(x;q^2)_n(xyq;q^2)_n}{(q^2;q^2)_n(yq;q^2)_{n+1}}\left(1-xyq^{4n+1}\right)=\frac{(xyq;q^2)_\infty}{(yq;q^2)_\infty}. \tag{10}$$

Furthertmore, setting $y=1$ the left-hand side of (10) reduces to

$$\begin{aligned}
&\sum_{n\geq 0} q^{n(2n+1)}\left[1-q^{2n+1}+q^{2n+1}(1-xq^{2n})\right]\frac{(x)_{2n}}{(q)_{2n+1}}\\
=\;&\sum_{n\geq 0} q^{n(2n+1)}\frac{(x)_{2n}}{(q)_{2n}}+\sum_{n\geq 0} q^{(n+1)(2n+1)}\frac{(x)_{2n+1}}{(q)_{2n+1}}\\
=\;&\sum_{n\geq 1} q^{n(n+1)/2}\frac{(x)_n}{(q)_n}.
\end{aligned} \tag{11}$$

Hence, in view of Euler's formula (1), (10) reduces to Lebesgue's formula (12) :

$$\sum_{n\geq 0}\frac{(x)_n q^{n(n+1)/2}}{(q)_n}=\prod_{m=1}^{\infty}\left(1-xq^{2m-1}\right)(1+q^m). \tag{12}$$

Originally,Sylvester derived from his bijection the following partition theorem, which is known (see [4, p. 24-25]) to be equivalent to Lebesgue's identity.

Corollary 1 (Sylvester) *The number of partitions in $\mathcal{P}_{\mathcal{O}}(n)$ with k distinct odd parts is equal to the number of partitions in $\mathcal{P}_{\mathcal{D}}(n)$ with k runs.*

Finally, setting $x=0$ in (10) we get

$$\sum_{n=0}^{\infty}\frac{q^{n(2n+1)}y^n}{(q^2;q^2)_n(yq;q^2)_{n+1}}=\frac{1}{(yq;q^2)_\infty},$$

which is clearly equivalent to Cauchy's formula [4, (2.2.8), p. 20] :

$$\sum_{i=0}^{\infty} \frac{q^{i^2} y^i}{(q)_i (yq)_i} = \frac{1}{(yq)_\infty}. \tag{13}$$

The corresponding partition theorem is the following

Corollary 2 *The number of partitions of $\mathcal{P}_{\mathcal{O}}(n)$ with length l is equal to the number of partitions of $\mathcal{P}_{\mathcal{D}}(n)$ with alternative length l.*

Note that the above result follows also from Sylvester's bijection seems to be published for the first time in [7] (see also [8, 15]).

4 On Fine's two partition theorems

In [10, 11] Fine gave two refinements of Euler's result, that we call Fine's first and second partition theorems respectively. His first partition theorem reads as follows :

Corollary 3 (Fine 1) *The number of partitions of $\mathcal{P}_{\mathcal{D}}(n)$ with largest part k is equal to the number of partitions in $\mathcal{P}_{\mathcal{O}}(n)$ such that $2k+1$ equals the largest part plus twice the number of parts.*

This corresponds to the $x = y = 1$ case of (7), i.e.,

$$\sum_{j \geq 0} \frac{z^{j+1} q^{2j+1}}{(zq; q^2)_{j+1}} = \sum_{n \geq 0} \frac{(zq^{n+1})^{2n+1}}{(zq)_{2n+2}}. \tag{14}$$

A proof of (14), based on Heine's transformation, can be found in [4, p. 27]. In fact we can give a short proof of this result by paraphrasing his statement as follows.

Corollary 4 (Fine 1 bis) *The number of partitions of $\mathcal{P}_{\mathcal{D}}(n)$ with largest part $\leq k$ equals the number of partitions of $\mathcal{P}_{\mathcal{O}}(n)$ such that the largest part plus twice the number of parts is $\leq 2k+1$.*

Since the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q = (q)_n/(q)_k\,(q)_{n-k}$ is the generating function of partitions whose Ferrer diagrams are inscribed in the rectangle $k \times (n-k)$, the above corollary is equivalent to the identity :

$$\sum_{l=0}^{k} q^l \begin{bmatrix} k \\ l \end{bmatrix}_{q^2} = (1+q)(1+q^2)\cdots(1+q^k), \tag{15}$$

which is a special value of the Rogers-Szegö polynomials [4, Example 5, p. 49] and can be calculated as follows : Set $f_k(t;q) = \sum_{l=0}^{n} t^l \begin{bmatrix} n \\ l \end{bmatrix}$, then

$$\sum_{k\geq 0} \frac{f_k(t;q)}{(q;q)_n} x^k = \sum_{l\geq 0} \frac{t^l x^l}{(q;q)_l} \sum_{k\geq 0} \frac{x^k}{(q;q)_k} = \frac{1}{(tx;q)_\infty} \frac{1}{(x;q)_\infty}. \tag{16}$$

Substituting q by q^2 and t by q we get

$$\sum_{n\geq 0} \frac{f_n(q;q^2)}{(q^2;q^2)_n} x^n = \frac{1}{(x;q)_\infty} = \sum_{n\geq 0} \frac{x^n}{(q;q)_n}.$$

Equating the coefficients of x^n yields (15). Note that (15) can also be derived from Sylvester's bijection through the following graphical decomposition, where $k = 9$:

$$\le l\left\{\overbrace{\begin{array}{cccc}2&2&2&2\\2&2&2&2\\2&2&&\\2&&&\end{array}}^{\le k-l}\right. \quad \left.\begin{array}{c}1\\1\\1\\1\\1\end{array}\right\} l \quad \to l\left\{\begin{array}{l}\overbrace{2\;\;2\;\;2\;\;2}^{\le k-l}\;\;1\\ 2\;\;2\;\;2\;\;2\;\;1\\ 2\;\;2\;\;1\\ 2\;\;1\\ 1\end{array}\right. \quad \to (9,\ 7,\ 6,\ 4,\ 1).$$

Since $2(k-l)+1+2l = 2k+1$, this completes the proof.

Now, we come to Fine's second partition theorem.

Corollary 5 (Fine 2) *The number of partitions of n into odd parts with largest part equal to $2r+1$ ($r \geq 0$) is equal to the number of partitions of n into distinct parts with rank $2r$ or $2r+1$.*

This theorem of Fine does not follow from Sylvester's bijection directly. For example, for $r = 0$, the partition $n = 1+1+\cdots+1$ maps to n through Sylvester's bijection, but $rank(n) = n-1 \geq 2$ if $n \geq 3$. Actually no direct combinatorial proof of this result is known. The analytical proof was based on an "elegant and difficult" q-identity (see [2, 5, 11]). However, as we will see, Fine's second theorem follows from Lebesgue's identity, which is the generating function version of Sylvester's partition theorem. Clearly Fine's second theorem is equivalent to the following statement

Corollary 6 (Fine 2 bis) *The number of partitions of n into odd parts each $\leq 2r+1$ ($r \geq 0$) is equal to the number of partitions of n into distinct parts with rank $\leq 2r+1$.*

Now, the above corollary is clearly equivalent to the following q-series identity :

$$\sum_{n=0}^{\infty} \begin{bmatrix} n+2r+1 \\ n \end{bmatrix} q^{n(n+1)/2} = \frac{1}{(1-q)(1-q^3)\cdots(1-q^{2r+1})}, \qquad (17)$$

which is Lebesgues's identity (12) with $x = q^{2r+2}$.
Remark: It is interesting to note that (15) and (17) are two finite extensions of Euler's identity (1). Other finite extensions of (1) are given in [8] in the framework of *Lecture Hall partitions.*

We end this section with an even analogue of Fine's second theorem. Let $D(n, 2r)$ denote the number of partitions of n with rank less than or equal to $2r$, and $E(n, 2r)$ the number of partitions of n with even parts each of them is less than or equal to $2r$. Then we have the following related result.

Theorem 3 *We have*

$$D(n,\, 2r) = \sum_{m\geq 0} E(n - m(m+1)/2,\, 2r).$$

Proof. First of all, setting $x = q$ in Lebesgue's formula we obtain the following result of Gauss:

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \prod_{m=1}^{\infty} (1-q^{2m})(1+q^m).$$

Now, applying Lebesgue's formula with $x = q^{2r+1}$ $(r \geq 0)$ and the Gauss formula we have

$$\sum_{n=0}^{\infty} \begin{bmatrix} n+2r \\ n \end{bmatrix} q^{n(n+1)/2} = \frac{\prod_{m\geq 1}(1+q^m)(1-q^{2m})}{(q^2;q^2)_r} = \frac{\sum_{n=0}^{\infty} q^{n(n+1)/2}}{(q^2;q^2)_r}.$$

This is clearly equivalent to our result. □

Remark: Hirschhorn [14] obtained a similar result related to Sylvester's partition theorem.

5 On (m, c)-analogue of Sylvester's theorem

Let c and m be two integers such that $1 \leq c < m$. We can replace the sequence of odd integers $\{1, 3, 5, \ldots\}$ by any arithmetical sequence of form $\{c, m + c, \ldots, km + c, \ldots\}$. Instead of 2-modular diagram we can draw an m-modular diagram as follows: if $\lambda_i = mk_i + c$ then we draw a Ferrer diagram of the partition $(k_1 + 1, k_2 + 1, \ldots)$ and interpret each dot except the rightmost at each row as m dots and put c at the rightmost dot. If we read this diagram by "hook" then we obtain some special type of partitions $\mu = (l_1^c, l_2^{m-c}, l_3^c, l_4^{m-c}, \ldots)$. For example, if $\lambda = ((4m + c)^2,\ 2m + c,\ m + c,\ c)$ then we draw the m-modular diagram

$$
\begin{array}{ccccccccc}
m & - & m & - & m & - & m & - & c \\
| & & & & & & & & \\
m & & m & - & m & - & m & - & c \\
| & & | & & & & & & \\
m & & m & & \mathbf{c} & & & & \\
| & & | & & & & & & \\
m & & c & & & & & & \\
| & & & & & & & & \\
c & & & & & & & &
\end{array}
$$

reading the above diagram by "hooks" we obtain

$$\mu = (9^c,\ 7^{m-c},\ 6^c, 4^{m-c},\ 1^c).$$

Denote by $\varphi_{m,c}$ the (m, c)-analog of Sylvester's bijection :$\lambda \mapsto \mu$. In what follows we shall state the similar results for this bijection and leave the proofs to the interested reader.

A partition λ is said to be of type $(h_1, h_2, \ldots)$ if λ has exactly $h_1 \geq 1$ parts of maximal length, $h_2 \geq 1$ second by length parts, etc. Let $\mathcal{A}(n)$ be the set of partitions of n with parts $\equiv c \pmod m$ and $\mathcal{B}(n)$ the set of partitions of n of type $(c, m - c, c, m - c, \ldots)$. Set $\mathcal{A} = \cup_{n\geq 0}\mathcal{A}(n)$ and $\mathcal{B} = \cup_{n\geq 0}\mathcal{B}(n)$.

In [7] Bessenrodt briefly alluded to this generalisation at the end of her paper. Later on, using Fronenius's representation of partitions, Pak and Postnikov [20] independently worked out some more details of this generalisation, but they overlooked the parameter of μ corresponding to the length of λ.

For any partition $\lambda \in \mathcal{A}$ let $d_m(\lambda)$ be the size of Durfee square in the m-modular diagram of λ, i.e., $d_m(\lambda) = \#\{i : \lambda_i \geq m(i-1)+c\}$. The following is the (m,c)-analog of Theorem 1 and merges the results of [7] and [20] into one single whole.

Theorem 4 *Let $\varphi_{m,c} : \mathcal{A} \longrightarrow \mathcal{B}$ be (m,c)-analog of Sylvester's bijection and $\mu = \varphi(\lambda)$, then $|\lambda| = |\mu|$, and*

- $d_m(\lambda) = [(l(\mu)+m-c)/m]$,
- $n_d(\lambda) = n_c(\mu)$,
- $l(\lambda) = l_a(\mu)$,
- $l(\lambda) + (\lambda_1 - c)/m = \mu_1$.

Remark: Bessenrodt [7] pointed out the equation $l(\lambda) = l_a(\mu)$, while the three other equations were proved by Pak & Postnikov [20].

Clearly Theorem 4 implies immediately the following identity :

$$\begin{aligned}\sum_{\lambda\in\mathcal{A}} & x^{n_d(\lambda)} y^{l(\lambda)} z^{l(\lambda)+(\lambda_1-c)/m} t^{d_m(\lambda)} q^{|\lambda|} \\ &= \sum_{\mu\in\mathcal{B}} x^{n_c(\mu)} y^{l_a(\mu)} z^{\mu_1} t^{\lfloor (l(\mu)+m-c)/m \rfloor} q^{|\mu|} \end{aligned} \tag{18}$$

Similarly we have the following result.

Theorem 5 *The generating functions of (18) have the following explicit formula :*

$$\begin{aligned} x \sum_{n\geq 0} (ty)^{n+1} z^{2n+1} q^{(n+1)(mn+c)} & \left[1-(1-x)yz^2\, q^{m(2n+1)+c}\right] \\ & \times \frac{((1-x)zq^m;\, q^m)_n ((1-x)yzq^c;\, q^m)_n}{(zq^m;\, q^m)_{n+1} (yzq^c;\, q^m)_{n+1}}. \end{aligned} \tag{19}$$

In what follows we shall derive some interesting q-identities from the above theorem by specializing the parameters.

Corollary 7 ((m,c)-analog of Rogers-Fine identity) *The following identity holds true*

$$\begin{aligned}
&\sum_{j=0}^{\infty} xyz^{j+1}q^{mj+c}\frac{((1-x)yzq^c;q^m)_j}{(yzq^c;q^m)_{j+1}}\\
&= x\sum_{n\geq 0} y^{n+1}z^{2n+1}q^{(n+1)(mn+c)}\left[1-(1-x)yz^2q^{m(2n+1)+c}\right]\\
&\times\frac{((1-x)zq^m;q^m)_n((1-x)yzq^c;q^m)_n}{(zq^m;q^m)_{n+1}(yzq^c;q^m)_{n+1}}. \qquad (20)
\end{aligned}$$

Proof. Since the g.f. of partitions in $\mathcal{A}$ with one distinct part $mj+c$ is

$$1+\frac{xyz^{j+1}q^{mj+c}}{1-yzq^{mj+c}}, \qquad j\geq 0,$$

so the g.f. of partitions in $\mathcal{A}$ with largest part $mj+c$ ($j\geq 0$) is

$$\begin{aligned}
\sum_{\lambda\in\mathcal{A}} x^{n_d(\lambda)}y^{l(\lambda)}z^{l(\lambda)+\frac{\lambda_1-c}{m}}q^{|\lambda|} &= \sum_{j\geq 0}\frac{xyz^{j+1}q^{mj+c}}{1-yzq^{mj+c}}\prod_{k=0}^{j-1}\left(1+\frac{xyzq^{mk+c}}{1-yzq^{mk+c}}\right)\\
&= \sum_{j=0}^{\infty}\frac{xyz^{j+1}q^{mj+c}((1-x)yzq^c;q^m)_j}{(yzq^c;q^m)_{j+1}}.
\end{aligned}$$

The identity (20) follows then from Theorem 5 with $t=1$. □

However the following result shows that the (m,c)-analog of Rogers-Fine identity (20) is not more general than the classical Rogers-Fine identity.

Proposition 1 *The (m,c)-analog of Rogers-Fine identity (20) is equivalent to the classical Rogers-Fine identity, i.e., the case where $m=2$ and $c=1$.*

Proof. We shall prove that (20) is equivalent to (7). Indeed, setting $m=2$ and $c=1$ in (20) yields (7). Inversely, substituting y by $yq^{2c/m-1}$ and then q by $q^{m/2}$ in (7) we recover (20). □

Note that setting $z=1$ in (20) the left-hand side becomes

$$\sum_{j\geq 0}\frac{xyq^{mj+c}}{1-yq^{mj+c}}\frac{((1-x)yq^c;q^m)_j}{(yq^c;q^m)_j}$$

$$
\begin{aligned}
&= \sum_{j\geq 0}\left[\frac{((1-x)yq^c;q^m)_{j+1}}{(yq^c;q^m)_{j+1}}-\frac{((1-x)yq^c;q^m)_j}{(yq^c;q^m)_j}\right]\\
&= \frac{((1-x)yq^c;q^m)_\infty}{(yq^c;q^m)_\infty}-1. \qquad (21)
\end{aligned}
$$

On the other hand, since $1-wyq^{m(2n+1)+c}=1-q^{m(n+1)}+q^{m(n+1)}(1-wyq^{mn+c})$, if $z=1$ we can rewrite the right-hand side of (20) plus 1 as follows :

$$
\begin{aligned}
&1+\sum_{n\geq 0} y^{n+1}q^{(n+1)(mn+c)}\frac{(1-x;q^m)_{n+1}((1-x)yq;q^m)_n}{(q^m;q^m)_n(yq;q^m)_{n+1}}\\
&+\sum_{n\geq 0} y^{n+1}q^{(n+1)(mn+m+c)}\frac{(1-x;q^m)_{n+1}((1-x)yq;q^m)_{n+1}}{(q^m;q^m)_{n+1}(yq;q^m)_{n+1}}\\
&=\sum_{n\geq 0}\frac{(1-x;q^m)_n((1-x)yq;q^m)_n}{(q^m;q^m)_n(yq^c;q^m)_{n+1}}y^nq^{n(mn+c)}\left[1-(1-x)yq^{2mn+c}\right].
\end{aligned}
$$

Therefore substituting x by $1-x$ we get the following

Corollary 8 *There holds*

$$
\sum_{n\geq 0}\frac{(x;q^m)_n(xyq;q^m)_ny^nq^{n(mn+c)}}{(q^m;q^m)_n(yq^c;q^m)_{n+1}}\left(1-xyq^{2mn+c}\right)=\frac{(xyq^c;q^m)_\infty}{(yq^c;q^m)_\infty}. \qquad (22)
$$

Furthermore, for $y=1$ (22) reduces to

Corollary 9 ((m,c)-analogue of Lebesgue's identity) *:*

$$
\sum_{n\geq 0}\frac{(x;q^m)_n(xq^c;q^m)_n\,q^{n(c+mn)}}{(q^m;q^m)_n(q^c;q^m)_{n+1}}(1-x\,q^{c+2mn})=\frac{(xq^c;q^m)_\infty}{(q^c;q^m)_\infty}; \qquad (23)
$$

while for $x=0$ (22) reduces to

Corollary 10 ((m,c)-analogue of Cauchy's identity) *:*

$$
\sum_{n\geq 0}\frac{q^{n(m(n-1)+c)}y^n}{(q^m;q^m)_n(yq^c;q^m)_n}=\frac{1}{(yq^c;q^m)_\infty}. \qquad (24)
$$

Evidently, when $m=2$ and $c=1$, (23) and (24) reduce to identities of Lebesgue and Cauchy, respecticely.

As for the Cauchy formula, we can prove the (m,c)-Cauchy formula (24) using m-module Ferrers diagrams. Clearly, the right-hand side of (24) is the g.f. of partitions in $\mathcal{A}$ with y enumerating the length of each partition. Now consider the m-module Ferrers diagrams of partitions in $\mathcal{A}$. Let $n \geq 0$ be the size of Durfee square. There are three cases :

(1) if the diagram is empty, the g.f. is 1;

(2) if m is at the southeast corner, the g.f. of such partitions is

$$\frac{q^{mn^2}}{(q^m;q^m)_n} \cdot \frac{y^n q^{nc}}{(yq^c;q^m)_n} \qquad n \geq 1;$$

(3) if c is at the southeast corner, the g.f. of such partitions is

$$\frac{q^{mn(n-1)}}{(q^m;q^m)_{n-1}} \cdot \frac{y^n q^{nc}}{(yq^c;q^m)_n} \qquad n \geq 1.$$

Summing up the above three cases we get the generating function for the partitions having a Durfee square of size $n \geq 0$:

$$\frac{q^{mn(n-1)+nc}y^n}{(q^m;q^m)_n(yq^c;q^m)_n}.$$

This establishes (24).

$$\begin{array}{cccccc} m & m & m & m & m & c \\ m & m & m & m & c & \\ m & m & \mathbf{m} & m & c & \\ m & m & c & & & \\ m & c & & & & \\ c & & & & & \end{array} \qquad \begin{array}{ccccc} m & m & m & m & c \\ m & m & m & m & c \\ m & m & \mathbf{c} & & \\ m & c & & & \\ c & & & & \end{array}$$

For instance,

$$\begin{aligned} \lambda &= (5m+c,\, (4m+c)^2,\, 2m+c,\, m+c,\, c) \\ &\mapsto \lambda' = (11^c,\, 9^{m-c},\, 7^c,\, 5^{m-c}, 4^c,\, 2^{m-c}). \end{aligned}$$

Note that $rank\lambda' = 11 - 3m$, which is negative if $m \geq 4$.
Remark: It's known [3, p. 571] that Rogers-Fine identity (5) is a special case of Watson's transformation formula for a terminating very-well-poised ${}_8\phi_7$ series as a multiple of a terminating balanced ${}_4\phi_3$ series [12, p. 35]. In view of Proposition 1, all the specializations of (20) or (5) would be limiting cases of Watson's formula. For example, as pointed out by Krattenthaler [16], setting $a = xyq^{c/m}$ and $b = x$, (22) can be written in standard basic hypergeometric function notation as follows :

$$ {}_4\phi_5 \left[\begin{matrix} a, & b, & q\sqrt{a}, & -q\sqrt{a} \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & 0,\ 0 \end{matrix} ; q,\ aq/b \right] = \frac{(aq;q)_\infty}{(aq/b;q)_\infty}, $$

which is a limiting case of Bailey's ${}_6\phi_5$ summation [12, Appendix (II.20)].

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