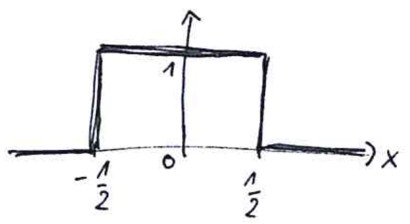
# $\boxed{\text{TD 3}}$ - Produits de convolution

**Ex 1 :** 1. $I = [-\frac{1}{2} ; \frac{1}{2}]$ , $\chi_I$ :

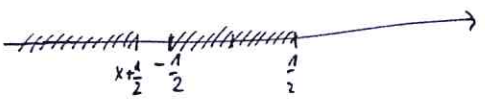



Soit $x \in \mathbb{R}$.

$$\chi_I * \chi_I(x) = \int_{\mathbb{R}} \chi_I(x-y)\chi_I(y)dy = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \chi_I(y)dy \quad \text{car} \quad -\frac{1}{2} \leqslant x-y \leqslant \frac{1}{2}$$

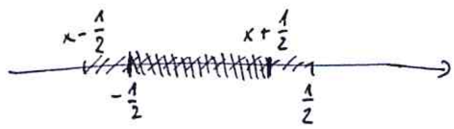
$$\iff x-\frac{1}{2} \leqslant y \leqslant x+\frac{1}{2}$$

Alors, * si $x \leqslant -1$,



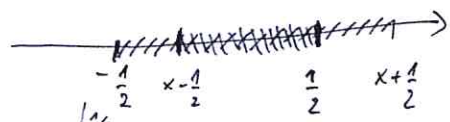

$$\chi_I * \chi_I(x) = 0.$$

* Si $x \in [-1 ; 0]$,




$$\chi_I * \chi_I(x) = \int_{-\frac{1}{2}}^{x+\frac{1}{2}} dy = x+1$$

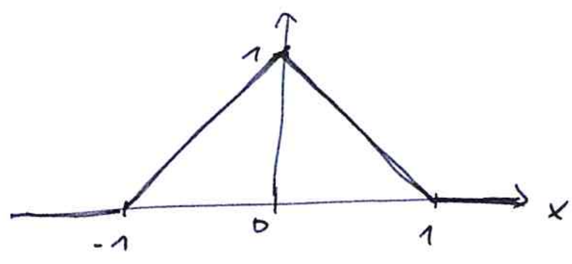
* Si $x \in [0 ; 1]$




$$\chi_I * \chi_I(x) = \int_{x-\frac{1}{2}}^{1/2} dy = 1-x$$

* Si $x \geqslant 1$, $\chi_I * \chi_I(x) = 0$. Ainsi,

$$\chi_I * \chi_I(x) = \begin{cases} 0 & \text{sur } ]-\infty ; -1] \\ 1+x & \text{sur } ]-1 ; 0] \\ 1-x & \text{sur } ]0 ; 1] \\ 0 & \text{sur } ]1 ; +\infty[ \end{cases}$$




$\chi_I * \chi_I$ est continue, ~~par~~ et linéaire par morceaux.

2. On fait la même étude que dans 1. Soit $x \in \mathbb{R}$.

$$\chi_I * \chi_I * \chi_I(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \chi_I * \chi_I(y)dy \, , \quad \text{donc}$$

on peut déjà affirmer que si: $|x| \geqslant \frac{3}{2}$, $\chi_I * \chi_I * \chi_I(x) = 0$.

$*$ Si: $x \in ]-\frac{3}{2}; -\frac{1}{2}]$, $\quad \chi_I * \chi_I * \chi_I(x) = \displaystyle\int_{-1}^{x+\frac{1}{2}} (1+y)\,dy = \left[\frac{1}{2}(1+y)^2\right]_{-1}^{x+\frac{1}{2}}$

$$= \frac{1}{2}\left(x+\frac{3}{2}\right)^2$$

$*$ Si: $x \in ]-\frac{1}{2}; \frac{1}{2}]$,

$$\chi_I * \chi_I * \chi_I(x) = \int_{x-\frac{1}{2}}^{0} (1+y)\,dy + \int_{0}^{x+\frac{1}{2}} (1-y)\,dy = \left[\frac{1}{2}(1+y)\right]_{x-\frac{1}{2}}^{0} + \left[-\frac{1}{2}(1-y)^2\right]_{0}^{x+\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{1}{2}\left(x+\frac{1}{2}\right)^2 - \frac{1}{2}\left(x-\frac{1}{2}\right)^2 + \frac{1}{2}$$

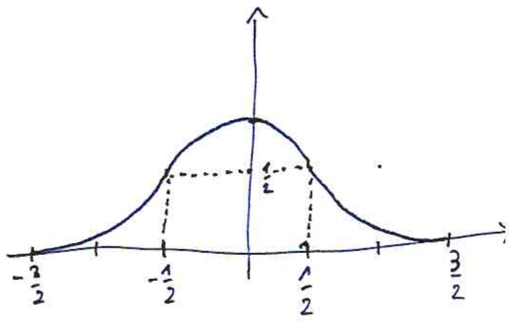
$$= -x^2 + \frac{3}{4}$$

$*$ Si: $x \in ]\frac{1}{2}; \frac{3}{2}]$,

$$\chi_I * \chi_I * \chi_I(x) = \int_{x-\frac{1}{2}}^{1} (1-y)\,dy = \frac{1}{2}\left(x-\frac{3}{2}\right)^2$$

Alors $\chi_I * \chi_I * \chi_I(x) = \begin{cases} 0 & \text{si: } x \in ]-\infty; -\frac{3}{2}] \\ \frac{1}{2}(x+\frac{3}{2})^2 & \text{si: } x \in ]-\frac{3}{2}; -\frac{1}{2}] \\ -x^2+\frac{3}{4} & \text{si: } x \in ]-\frac{1}{2}; \frac{1}{2}] \\ \frac{1}{2}(x-\frac{3}{2})^2 & \text{si: } x \in ]\frac{1}{2}; \frac{3}{2}] \\ 0 & \text{si: } x \in ]\frac{3}{2}; +\infty[ \end{cases}$

et donc $(\chi_I * \chi_I * \chi_I)'(x) = \begin{cases} 0 & \text{si: } \ldots \\ x+\frac{3}{2} & \ldots \\ -2x & \ldots \\ x-\frac{3}{2} & \ldots \\ 0 & \ldots \end{cases}$




la fonction est bien continûment dérivable et quadratique par morceaux

3. $h(x) = \begin{cases} \frac{1}{\sqrt{|x|}} & \text{si } x \neq 0 \\ 0 & \text{si: } x = 0. \end{cases}$

$\cancel{\chi_I * h(x) = \int^x}$ $\quad \chi_{[-1;1]} * h(x) = \displaystyle\int_{x-1}^{x+1} h(y)\,dy$

Si $x < -1$, $\displaystyle\int_{x-1}^{x+1} h(y)dy = \int_{x-1}^{x+1} \frac{1}{\sqrt{-y}} dy = \int_{x-1}^{x+1} (-y)^{-\frac{1}{2}} dy$

$$= \left[-2(-y)^{\frac{1}{2}}\right]_{x-1}^{x+1}$$

$$= 2\left((1-x)^{\frac{1}{2}} - (-1-x)^{\frac{1}{2}}\right)$$

Si $x \in ]-1;1]$, $\displaystyle\int_{x-1}^{x+1} h(y)dy = \int_{x-1}^{0} \frac{1}{\sqrt{-y}} dy + \int_{0}^{x+1} \frac{1}{\sqrt{y}} dy$

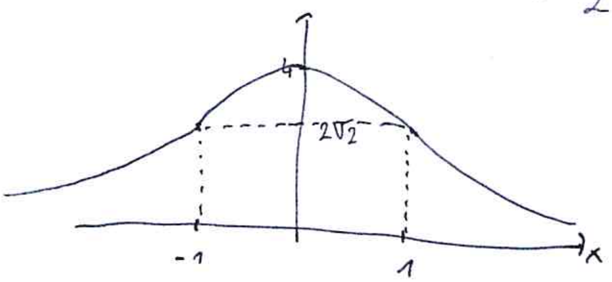
$$= -2\left[(-y)^{\frac{1}{2}}\right]_{x-1}^{0} + 2\left[y^{\frac{1}{2}}\right]_{0}^{x+1}$$

$$= 2\left((1-x)^{\frac{1}{2}} + (1+x)^{\frac{1}{2}}\right)$$

Si $x > 1$ $\displaystyle\int_{x-1}^{x+1} h(y)dy = \int_{x-1}^{x+1} \frac{1}{\sqrt{y}} dy = \left[2y^{\frac{1}{2}}\right]_{x-1}^{x+1}$

$$= 2\left((x+1)^{\frac{1}{2}} - (x-1)^{\frac{1}{2}}\right)$$

$\chi_{[-1;1]} * h$ :




$\chi_{[-1;1]} * h(-1^-) = \chi_{[-1;1]} * h(-1^+) = 2\sqrt{2} = \chi_{[-1;1]} * h(1^-) = \chi_{[-1;1]} * h(1^+)$

donc cette fonction est continue.

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Ex 2 : $f_a(x) = e^{ax}$

(a) Soient $a, b \in \mathbb{R}$, $x \in \mathbb{R}$.

$$g := (f_a \chi_{\mathbb{R}_+}) * (f_b \chi_{\mathbb{R}_+})(x) = \int_{\mathbb{R}} e^{ay} \chi_{\mathbb{R}_+}(y)\, e^{b(x-y)} \chi_{\mathbb{R}_+}(x-y)\, dy$$

$$= \int_0^x e^{ay} e^{b(x-y)} dy$$

ces fonctions sont continues sur $[0;x]$, donc l'intégrale converge. Le produit de convolution est donc bien défini. On peut le calculer :

$$g(x) = e^{bx} \int_0^x e^{y(a-b)} dy = \begin{cases} e^{bx}\left(\dfrac{e^{x(a-b)} - 1}{a-b}\right) = \dfrac{e^{ax} - e^{bx}}{a-b} & \text{si } a \neq b \\ x e^{bx} & \text{si } a = b \end{cases}$$

(b) $g := \chi_{[-a;a]} * \left(f_b\, \chi_{[0,+\infty[}\right)(x) = \int_0^{+\infty} e^{by}\, \chi_{[-a;a]}(x-y)\,dy$

* Si: $x < -a$, $g(x) = 0$
* Si: $x \in [-a;a]$, $g(x) = \int_0^{x+a} e^{by}\,dy = \begin{cases} \frac{1}{b}\left(e^{b(x+a)} - 1\right) & \text{si } b \neq 0 \\ x+a & \text{si } b = 0 \end{cases}$
* Si: $x > a$, $g(x) = \begin{cases} \frac{1}{b}\left(e^{b(x+a)} - e^{b(x-a)}\right) & \text{si } b \neq 0 \\ 2a & \text{si } b = 0. \end{cases}$

(c) $g(x) := \left(\chi_{[-a;a]} * f_b\right)(x)\,\chi_{[0,+\infty[}(x)$

$$= \left(\int_{x-a}^{x+a} e^{by}\,dy\right)\chi_{[0;+\infty[}(x) = \begin{cases} \dfrac{e^{b(x+a)} - e^{b(x-a)}}{b} & \text{si } b \neq 0,\ x \geqslant 0 \\ 2a & \text{si } b = 0,\ x \geqslant 0 \\ 0 & \text{si } x < 0 \end{cases}$$

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Ex 3 : $(m,\sigma) \in \mathbb{R} \times \mathbb{R}_+^*$, $g_{m,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$

1. $$\int_{-\infty}^{+\infty} g_{m,\sigma}(x)\,dx = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx \underset{\substack{y = \frac{x-m}{\sigma} \\ dy = \frac{1}{\sigma}dx}}{=} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy = 1$$

2. $$g_{p,\sigma} * g_{q,\tau}(x) = \int_{-\infty}^{+\infty} \left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-p)^2}{2\sigma^2}}\right)\left(\frac{1}{\tau\sqrt{2\pi}} e^{-\frac{(x-y-q)^2}{2\tau^2}}\right) dy$$

$$= \int_{-\infty}^{+\infty} g_{p,\sigma}(y)\, g_{q,\tau}(x-y)\,dy$$

$$= \int_{-\infty}^{+\infty} g_{0,\sigma}(y-p)\, g_{0,\tau}(x-y-q)\,dy$$

donc, après chgt de variables:

$$(g_{p,\sigma} * g_{q,\tau})(x) = \int_{-\infty}^{+\infty} g_{0,\sigma}(y)\, g_{0,\tau}(x-y-p-q)\,dy$$

$$= (g_{0,\sigma} * g_{0,\tau})(x-p-y)$$

Il suffit de calculer $g_{0,\sigma} * g_{0,\tau}$.

Alors :

$$(g_{0,\sigma} * g_{0,\tau})(x) = \int_{\mathbb{R}} \frac{1}{\sigma\tau(2\pi)} e^{-\frac{y^2}{2\sigma^2}} e^{-\frac{(x-y)^2}{2\tau^2}}\,dy$$

$$= \int_{\mathbb{R}} \frac{1}{2\pi\sigma\tau} \exp\left(\frac{-y^2}{2\sigma^2} - \frac{(x-y)^2}{2\tau^2}\right) dy$$

$$-\frac{y^2}{2\sigma^2} - \frac{(x-y)^2}{2\tau^2} = -y^2\left(\frac{1}{2\sigma^2} + \frac{1}{2\tau^2}\right) + \frac{xy}{\tau^2} - \frac{x^2}{2\tau^2}$$

$$= \frac{-1}{2\sigma^2\tau^2}\left(y^2(\sigma^2+\tau^2) - 2xy\sigma^2\right) - \frac{x^2}{2\tau^2}$$

$$= \frac{-1}{2\sigma^2\tau^2}\left(\left(y\sqrt{\sigma^2+\tau^2} - \frac{x\sigma^2}{\sqrt{\sigma^2+\tau^2}}\right)^2 - \frac{x^2\sigma^4}{\sigma^2+\tau^2}\right) - \frac{x^2}{2\tau^2}$$

$$= -\frac{(\sigma^2+\tau^2)}{2\sigma^2\tau^2}(y-x_0)^2 + \underbrace{\frac{x^2\sigma^2}{2\tau^2(\sigma^2+\tau^2)} - \frac{x^2}{2\tau^2}}$$

où $x_0 = x\dfrac{\sigma^2}{\sigma^2+\tau^2}$

$$= -\frac{(\sigma^2+\tau^2)}{2\sigma^2\tau^2}(y-x_0)^2 - \frac{x^2}{2(\sigma^2+\tau^2)}$$

donc $$(g_{0,\sigma} * g_{0,\tau})(x) = e^{-\frac{x^2}{2(\sigma^2+\tau^2)}} \int_{\mathbb{R}} \frac{1}{2\pi\sigma\tau} e^{-\frac{(\sigma^2+\tau^2)}{2\sigma^2\tau^2}(y-x_0)^2}\,dx$$

$$= \underbrace{\frac{1}{\sqrt{\sigma^2+\tau^2}\sqrt{2\pi}} e^{-\frac{x^2}{2(\sigma^2+\tau^2)}}} \int_{\mathbb{R}} \underbrace{\frac{\sqrt{\sigma^2+\tau^2}}{\sqrt{2\pi}\,\sigma\tau} e^{-\frac{(\sigma^2+\tau^2)}{2\sigma^2\tau^2}(y-x_0)^2}}\,dx$$

$$= g_{0,\sqrt{\sigma^2+\tau^2}}(x) \int_{\mathbb{R}} g_{x_0,\sigma\tau/\sqrt{\sigma^2+\tau^2}}(y)\,dy$$

donc, d'après la question 1,

$$g_{0,\sigma} * g_{0,\tau} = g_{0,\sqrt{\sigma^2+\tau^2}}$$

et finalement, $g_{p,\sigma} * g_{q,\tau} = g_{p+q,\sqrt{\sigma^2+\tau^2}}$, d'après la remarque de la page précédente.