Renormalization Theory, II

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Ordinary ϕ_4^4

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A_G(z_1, \cdots, z_N) = \int \prod_{v=1}^{n} d^d x_v \prod_{\ell} C(x_\ell, x'_\ell)
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Higher and higher values of the scale index i probe shorter and shorter distances.

Scale attributions, high subgraphs

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At fixed scale attribution, some subgraphs play an essential role. They are the connected subgraphs whose internal lines all have higher scale index than all the external lines of the subgraph. Let's call them the "high" subgraphs. They form a single forest for the inclusion relation.

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In four dimension by the previous estimates of a single scale propagator C, power counting delivers a factor M^{2i} per line and M^{-4i} per vertex integration $\int d^4x$. There are $n-1$ "internal" integrations to perform to compare a high connected subgraph to a local vertex.

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For a connected ϕ_4^4 graph, the net factor is $2l(G) - 4(n(G) - 1) = 4 - N(G)$ (because $4n = 2l + N$). When this factor is strictly negative, the sum is geometrically convergent, otherwise it diverges.

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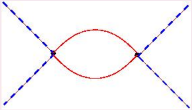

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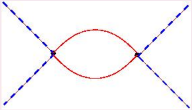

For instance for this graph the sum over the red scale i at fixed blue scale diverges (logarithmically) because there are two line factors M^{2i} and a single internal integration M^{-4i}.

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However (by the locality principle) this divergence can be absorbed into a change of the three parameters (coupling constant, mass and wave function) which appeared in the initial model.

This means physically that the parameters of the model do change with the observation scale but not the structure of the model itself. This is a kind of non-trivial self-similarity.

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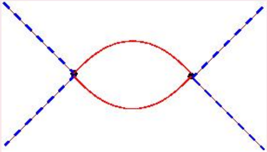

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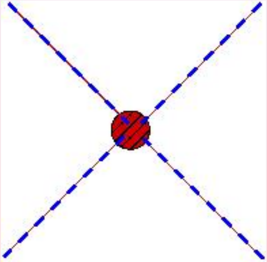

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- The renormalized expansion requires some complicated book-keeping (Zimmermann's forests, Connes-Kreimer Hopf algebra).
- It subtracts local pieces of divergent subgraphs irrespective of whether they are high or not.
- There is a price to pay, called renormalons.

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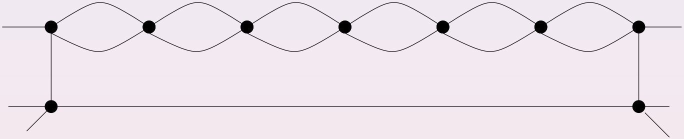
$$A_G^{ren}(k) = \int d^4p \frac{1}{(p^2+m^2)(p+k)^2+m^2)} - \int \frac{1}{(p^2+m^2)^2}$$

is finite but unbounded: $|A_G^{eff}(k)| \sim_{|k|\to\infty} c \log|k/m|$.

Renormalons, II



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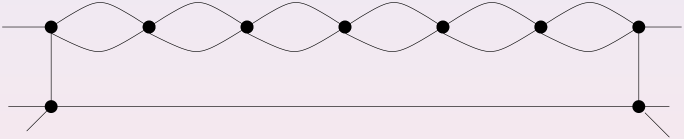

A chain of n such graphs as above behaves as $[\log|q|]^n$. Inserting them in a convergent loop leads to a total amplitude of P_n

$$\int [\log|q|]^n \frac{d^4q}{[q^2+m^2]^3} \simeq_{n\to\infty} c^n n!$$

which cannot be summed over n.

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- Expressing the theory in terms of all the running couplings leads to the effective expansion (which is not a power series in a single coupling).
- Because there is a single forest subtracted there is no book-keeping, (no need for Zimmermann's forests nor Connes-Kreimer Hopf algebras)
- There are also no renormalons, so the effective expansion is good for constructive purpose.

The Forest Formula or ”constructive swiss knife”

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Let F be a smooth function of $n(n-1)/2$ line variables x_ℓ, $\ell = (i,j)$, $1 \leq i < j \leq n$. The forest formula states

$$F(1,...,1) = \sum_{\mathcal{F}} \Big\{ \prod_{\ell \in \mathcal{F}} \Big[\int_0^1 dw_\ell \Big] \Big\} \Big\{ \prod_{\ell \in \mathcal{F}} \frac{\partial}{\partial x_\ell} F \Big\} \Big[x^{\mathcal{F}}(\{w\}) \Big], \text{ where}$$

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Given any series a_n, there is at most one such function f. When there is one, it is called the Borel sum, and it can be computed from the series to arbitrary accuracy.

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- ... your way here?

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\begin{aligned}
F &= 1 + H, \ H = \sum_{p \geq 1} a_p (-\lambda)^p, \ a_p = \frac{(4p)!!}{p!} \\
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G &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{H(\lambda)^n}{n} = \sum_{k \geq 1} b_k (-\lambda)^k, \\
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Borel summability is unclear. Even the sign of b_k is unclear.

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Borel summability unclear. $b_k \geq 0$ clear.

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Cluster expansion = Taylor-Lagrange expansion of the functional integral:

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Mayer expansion: define $H_i = -\lambda \int_0^1 dt \int_{-\infty}^{+\infty} x_i^4 e^{-\lambda t x_i^4 - x_i^2/2} \frac{dx_i}{\sqrt{2\pi}} \ = H \ \forall i$, $\varepsilon_{ij} = 0 \ \forall i, j$ and write

$$F = 1 + H = \sum_{n=0}^{\infty} \prod_{i=1}^{n} H_i(\lambda) \prod_{1 \le i < j \le n} \varepsilon_{ij}$$

Defining $\eta_{ij} = -1$, $\varepsilon_{ij} = 1 + \eta_{ij} = 1 + x_{ij}\eta_{ij}|_{x_{ij}=1}$ and apply swiss knife.

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$$G = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T}} \prod_{i=1}^{n} H_i(\lambda) \Big\{ \prod_{\ell \in \mathcal{T}} \big[\int_0^1 dw_\ell \big] \eta_\ell \Big\} \prod_{\ell \notin \mathcal{T}} \big[1 + \eta_\ell x_\ell^{\mathcal{T}}(\{w\}) \big)\big]$$

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- Convergence easy because each H_i contains a different "copy" $\int dx_i$ of functional integration.
- Borel summability now easy from the Borel summability of H. But this method does not extend to noncommutative theory.

Loop Vertices

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Intermediate field representation

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$$\begin{aligned} F &= \int_{-\infty}^{+\infty} e^{-\lambda x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i\sqrt{2\lambda}\sigma x^2 - x^2/2 - \sigma^2/2} \frac{dx}{\sqrt{2\pi}} \frac{d\sigma}{\sqrt{2\pi}} \\ &= \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\log[1+i\sqrt{8\lambda}\sigma] - \sigma^2/2} \frac{d\sigma}{\sqrt{2\pi}} \\ &= \int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} \frac{V^n}{n!} d\mu(\sigma) \end{aligned} \tag{2.2}$$

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\end{aligned}
$$

Apply swiss knife by making copies: $V^n(\sigma) \to \prod_{i=1}^n V_i(\sigma_i)$, $d\mu(\sigma) \to d\mu_C(\{\sigma_i\})$, $C_{ij} = 1 = x_{ij}|_{x_{ij}=1}$.

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One can picture the result as a sum over trees on loops, or "cacti". Since

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