The complete solution of a double inverse spectral problem for compact Hankel operators

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from joint works with Patrick Gérard (Université Paris-Sud)

Motivation

- Spectral theory of Hankel operators : a key tool in the study of some non dispersive Hamiltonian system :

  the cubic Szegő equation.
- A complete integrable system which admits two Lax pairs related to Hankel operators.
- Solve a double inverse spectral problem for compact Hankel operators.
- Apply it to obtain qualitative results on the dynamics of the cubic Szegő equation.

Part I : Classical Hankel operators (Hankel matrices).

Hankel operators in the real domain

A Hankel operator is an operator on $\ell^2_{\mathbb{R}}(\mathbb{Z}_+)$ of the form

$$(\Gamma_c(x))_n = \sum_{k=0}^{\infty} c_{n+k} x_k \ .$$

is selfadjoint and satisfies

$$\Gamma_c \Sigma = \Sigma^* \Gamma_c = \Gamma_{\Sigma^* c}$$

where Σ is the shift operator,

$$\Sigma : (x_0, x_1, \cdots) \mapsto (0, x_0, x_1, \cdots)$$

Nehari, 1957 : Γ_c is bounded iff

$$\exists f \in L^\infty(\mathbb{T}),\ \forall n \geq 0, c_n = \hat{f}(n) \ ,$$

or iff $u_c(e^{ix}) := \sum_{n=0}^{\infty} c_n \, e^{inx} \in BMO_+$ (C. Fefferman, 1971).

The compact case

Hartman, 1958 : Γ_c is compact iff

$$\exists f \in C(\mathbb{T}),\ \forall n \geq 0, c_n = \hat{f}(n)\ ,$$

or iff $u_c(\mathrm{e}^{ix}) = \sum_{n=0}^{\infty} c_n\,\mathrm{e}^{inx} \in VMO_+$.

In this case, Γ_c is compact and self-adjoint, hence
$\exists$ a sequence $(\lambda_j)_{j\geq 1}\ ,\ \lambda_j \in \mathbb{R}\ ,\ \lambda_j \to 0$, with

$$|\lambda_1| \geq |\lambda_2| \geq \dots$$

such that the eigenvalues of Γ_c are the λ_j's, repeated according to multiplicity, and possibly 0.

The Megretski–Peller–Treil theorem

What are the constraints on the λ_j's ?

Theorem (Megretski–Peller–Treil, 1995)

If $(\lambda_j)_{j\geq 1}$ is the sequence of eigenvalues of some selfadjoint compact Hankel operator, then, for every $\lambda \in \mathbb{R} \setminus \{0\}$,

$$|\#\{j : \lambda_j = \lambda\} - \#\{j : \lambda_j = -\lambda\}| \leq 1 \ .$$

Conversely, any sequence $(\lambda_j)_{j\geq 1}$ of real numbers satisfying the above condition and tending to 0 is the sequence of eigenvalues of some selfadjoint compact Hankel operator.

- Question : describe the isospectral classes.

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- Question : describe the isospectral classes.

No uniqueness expected : an example

Even in the rank one case, no uniqueness expected.
Indeed, Γ_c is a selfadjoint rank one operator if and only if

$$c_n = \alpha p^n \ , \ \alpha \in \mathbb{R}^* \ , \ p \in (-1, 1) \ .$$

The only nonzero eigenvalue is

$$\lambda_1 = \frac{\alpha}{1 - p^2} \ .$$

Isospectral sets are therefore manifolds diffeomorphic to $\mathbb{R}$.
Hence, we need to introduce additional parameters.

The shifted Hankel operator

Given a Hankel operator Γ_c, define $\tilde\Gamma_c$ as

$$\tilde\Gamma_c = \Sigma^*\Gamma_c = \Gamma_c\Sigma = \Gamma_{\Sigma^* c}\ .$$

Notice that

$$\tilde\Gamma_c^2 = \Gamma_c\Sigma\Sigma^*\Gamma_c = \Gamma_c^2 - (\,.\,|c)c\ .$$

If Γ_c is selfadjoint compact, so is $\tilde\Gamma_c$, and its eigenvalues $(\mu_j)_{j\geq 1}$ satisfy

$$|\lambda_1| \geq |\mu_1| \geq |\lambda_2| \geq |\mu_2| \geq \dots$$

The case with strict inequalities corresponds to a dense G_δ subset of $VMO_{+,\mathbb{R}}$, for which the inverse spectral problem has a particularly simple solution.

The generic case

Theorem (PG-S. Grellier, 2012)

Given two sequences $(\lambda_j)_{j\geq 1}, (\mu_j)_{j\geq 1}$ of real numbers such that

$$|\lambda_1| > |\mu_1| > |\lambda_2| > \cdots \to 0 \ ,$$

there exists a unique sequence $(c_n)_{n\geq 0}$ of real numbers such that Γ_c is compact and

- *The non zero eigenvalues of Γ_c are the λ_j's.*
- *The non zero eigenvalues of $\tilde{\Gamma}_c$ are the μ_j's.*

Back to the example

If $c_n = \alpha p^n$, $\alpha \in \mathbb{R}^*$, $p \in (-1, 1)$. The only nonzero eigenvalue of Γ_c is

$$\lambda_1 = \frac{\alpha}{1 - p^2} .$$

The only nonzero eigenvalue of $\tilde{\Gamma}_c$ is

$$\mu_1 = \frac{\alpha p}{1 - p^2} .$$

The knowledge of λ_1 and μ_1 characterizes α and p, hence c.

Catching the multiplicities

In the general case, consider the — finite or infinite tending to 0 — sequence of non zero eigenvalues of Γ_c and $\tilde{\Gamma}_c$, listed so that

$$|\lambda_1| \geq |\mu_1| \geq |\lambda_2| \geq |\mu_2| \geq \dots$$

Lemma (P. Gérard-S.G.)

$\forall \lambda \neq 0$ *such that* $\ker(\tilde{\Gamma}_c^2 - \lambda^2 I) + \ker(\Gamma_c^2 - \lambda^2 I) \neq \{0\}$,

$$|\dim \ker(\tilde{\Gamma}_c^2 - \lambda^2 I) - \dim \ker(\Gamma_c^2 - \lambda^2 I)| = 1 \ .$$

Consequently, in the series $|\lambda_1| \geq |\mu_1| \geq |\lambda_2| \geq |\mu_2| \geq \dots$, the length of a maximal string with consecutive equal terms is odd.

Theorem (P. Gérard-S.G, 2013)

Let $(\lambda_j), (\mu_j)$ be two — finite or infinite tending to 0— sequences of non zero real numbers satisfying

- $|\lambda_1| \geq |\mu_1| \geq |\lambda_2| \geq |\mu_2| \geq \dots$
- *In the above sequence, the lengths of maximal strings with consecutive equal terms are odd. Denote them by* $(2n_r - 1)_r$.
- $\forall \lambda \neq 0, |\#\{j : \lambda_j = \lambda\} - \#\{j : \lambda_j = -\lambda\}| \leq 1$.
- $\forall \mu \neq 0, |\#\{j : \mu_j = \mu\} - \#\{j : \mu_j = -\mu\}| \leq 1$.

Then there exists a sequence $(c_n)_{n\geq 0}$ of real numbers such that Γ_c is compact and

- *The non zero eigenvalues of Γ_c are the λ_j's.*
- *The non zero eigenvalues of $\tilde{\Gamma}_c$ are the μ_j's.*

Moreover, if $M = \sum_r (n_r - 1)$, the isospectral set is a manifold diffeomorphic to $\mathbb{R}^M$ if $M < \infty$, homeomorphic to $\mathbb{R}^\infty$ if $M = \infty$.

An example

In the case of a finite sequence of nonzero eigenvalues, explicit formulae for u_c. For instance, given four real numbers such that

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > 0 \ ,$$

we get

$$u_c(\mathrm{e}^{ix}) = \frac{\frac{\lambda_1 - \mu_1 \mathrm{e}^{ix}}{\lambda_1^2 - \mu_1^2} + \frac{\lambda_2 - \mu_2 \mathrm{e}^{ix}}{\lambda_2^2 - \mu_2^2} - \frac{\lambda_1 - \mu_2 \mathrm{e}^{ix}}{\lambda_1^2 - \mu_2^2} - \frac{\lambda_2 - \mu_1 \mathrm{e}^{ix}}{\lambda_2^2 - \mu_1^2}}{\begin{vmatrix} \frac{\lambda_1 - \mu_1 \mathrm{e}^{ix}}{\lambda_1^2 - \mu_1^2} & \frac{\lambda_2 - \mu_1 \mathrm{e}^{ix}}{\lambda_2^2 - \mu_1^2} \\ \frac{\lambda_1 - \mu_2 \mathrm{e}^{ix}}{\lambda_1^2 - \mu_2^2} & \frac{\lambda_2 - \mu_2 \mathrm{e}^{ix}}{\lambda_2^2 - \mu_2^2} \end{vmatrix}}$$

If $|\lambda_1| > |\lambda_2| > 0$ and $\mu_1 = \lambda_2, \mu_2 = -\lambda_2$, then, there exists $p \in (-1, 1)$ such that

$$u_c(\mathrm{e}^{ix}) = (\lambda_1^2 - \lambda_2^2) \frac{1 - p\,\mathrm{e}^{ix}}{\lambda_1 - p\,\mathrm{e}^{ix}(\lambda_1 - \lambda_2) - \lambda_2\,\mathrm{e}^{2ix}} \ .$$

Remarks

Hence, if λ_1, λ_2 are given such that $|\lambda_1| > |\lambda_2| > 0$, the corresponding isospectral set consists of sequences c given by the above two formulae.
Notice that the second expression is obtained from the first one by making $\mu_1 \to \lambda_2$, $\mu_2 \to -\lambda_2$, and

$$\frac{2\lambda_2 + \mu_2 - \mu_1}{\mu_1 + \mu_2} \to p \, .$$

Part II : Complexified version.

The Hardy space representation

$$L^2_+ = \{u : u(\mathrm{e}^{ix}) = \sum_{n=0}^{\infty} c_n \mathrm{e}^{inx}, \sum_{n=0}^{\infty} |c_n|^2 < \infty\} ,$$

$$\Pi : L^2(\mathbb{T}) \longrightarrow L^2_+ \text{ the Szegö projector ,}$$

Given $u \in VMO_+$, define H_u on L^2_+ by

$$H_u(h) = \Pi\left(u\overline{h}\right) .$$

H_u is a compact antilinear operator, non selfadjoint, and

$$\widehat{H_u(h)} = \Gamma_{\hat{u}}\left(\overline{\hat{h}}\right) , \widehat{K_u(h)} = \tilde{\Gamma}_{\hat{u}}\left(\overline{\hat{h}}\right)$$

$$K_u := S^* H_u = H_u S = H_{S^* u} , \; Sh(\mathrm{e}^{ix}) := \mathrm{e}^{ix} h(\mathrm{e}^{ix})$$

$$K_u^2 = H_u^2 - (\cdot|u)u .$$

Eigenspaces of H_u^2, K_u^2, $u \in VMO_+$

$$E_u(s) := \ker(H_u^2 - s^2 I)\ ,\ F_u(s) := \ker(K_u^2 - s^2 I)\ .$$

Lemma (P. Gérard-S.G., 2013)

Let $s > 0$ such that $E_u(s) + F_u(s) \neq \{0\}$.

$$|\dim E_u(s) - \dim F_u(s)| = 1\ .$$

Let $(s_j^2)_j$ – finite or infinite tending to 0 – the sequence of distinct eigenvalues of H_u^2 and K_u^2.

The s_{2j-1}'s are the singular values of H_u such that

$$\dim E_u(s_{2j-1}) = \dim F_u(s_{2j-1}) + 1\ .$$

The s_{2k}'s are the singular values of K_u such that

$$\dim F_u(s_{2k}) = \dim E_u(s_{2k}) + 1\ .$$

Finite Blaschke products

A finite Blaschke product is an inner function of the form

$$\Psi(z) = \mathrm{e}^{i\psi} \prod_{j=1}^{k} \frac{z - p_j}{1 - \overline{p}_j z} \ , \ \psi \in \mathbb{T} \ , \ p_j \in \mathbb{D} \ .$$

The integer k is called the degree of Ψ. Alternatively, Ψ can be written as

$$\Psi(z) = \mathrm{e}^{i\psi} \frac{z^k \overline{D}\left(\frac{1}{z}\right)}{D(z)} \ ,$$

where D is a polynomial of degree k, $D(0) = 1$, with all its roots outside $\overline{\mathbb{D}}$. We denote by $\mathcal{B}_k$ the set of Blaschke product of degree k. It is a classical result that $\mathcal{B}_k$ is diffeomorphic to $\mathbb{T} \times \mathbb{R}^{2k}$.

Action of H_u and K_u on the eigenspaces

Proposition (P. Gérard-S.G., 2013)

Let $s > 0$ *and* $u \in VMO_+(\mathbb{T})$. *Assume* $m := \dim E_u(s) = \dim F_u(s) + 1$. *Denote by* u_s *the orthogonal projection of* u *onto* $E_u(s)$. *There exists* Ψ_s, *a finite Blaschke product, of degree* $m-1$, *such that* $su_s = \Psi_s H_u(u_s)$ *and, if* $\Psi_s(z) = \mathrm{e}^{-i\psi_s} \frac{z^{m-1}\overline{D}_s\left(\frac{1}{z}\right)}{D_s(z)}$,

$$E_u(s) = \frac{H_u(u_s)}{D_s}\mathbb{C}_{m-1}[z]\ ,\ F_u(s) = \frac{H_u(u_s)}{D_s}\mathbb{C}_{m-2}[z],$$

$$H_u\left(\frac{z^a}{D_s}H_u(u_s)\right) = s\mathrm{e}^{-i\psi_s}\frac{z^{m-a-1}}{D_s}H_u(u_s)\ ,\ 0 \le a \le m-1$$

$$K_u\left(\frac{z^b}{D_s}H_u(u_s)\right) = s\mathrm{e}^{-i\psi_s}\frac{z^{m-b-2}}{D_s}H_u(u_s)\ ,\ 0 \le b \le m-2\ .$$

Action of H_u and K_u – continued

Assume $\ell := \dim F_u(s) = \dim E_u(s) + 1$. Denote by u_s' the orthogonal projection of u onto $F_u(s)$. There exists a finite Blaschke product Ψ_s of degree $\ell - 1$, such that $K_u(u_s') = s\Psi_s u_s'$ and, if $\Psi_s(z) = \mathrm{e}^{-i\psi_s}\frac{z^{\ell-1}\overline{D}_s\left(\frac{1}{z}\right)}{D_s(z)}$,

$$F_u(s) = \frac{u_s'}{D_s}\mathbb{C}_{\ell-1}[z]\ ,\ E_u(s) = \frac{zu_s'}{D_s}\mathbb{C}_{\ell-2}[z],$$

$$K_u\left(\frac{z^a}{D_s}u_s'\right) = s\mathrm{e}^{-i\psi_s}\frac{z^{\ell-a-1}}{D_s}u_s'\ ,\ 0 \le a \le \ell - 1$$

$$H_u\left(\frac{z^{b+1}}{D_s}u_s'\right) = s\mathrm{e}^{-i\psi_s}\frac{z^{\ell-b-1}}{D_s}u_s'\ ,\ 0 \le b \le \ell - 2\ .$$

Coming back to selfadjoint operators

Remark that the preceding identities provide very simple matrices for the action of H_u and K_u on $E_u(s)$ and $F_u(s)$.
Selfadjoint Hankel operators correspond to symbols u with real Fourier coefficients, hence the angles ψ_s belong to $\{0, \pi\}$.
In this case, one can easily check that the dimensions of the eigenspaces of these matrices associated to the eigenvalues $\pm s$ differ of at most 1 : the Megretskii–Peller–Treil condition.

Notation

- $\Omega_n := \{s_1 > s_2 > \cdots > s_n > 0\} \subset \mathbb{R}^n$.
- $\Omega_\infty = \{(s_n)_{n\geq 1}\ , s_1 > s_2 > \cdots > s_n \to 0\}$.

Given $u \in VMO_+(\mathbb{T}) \setminus \{0\}$, define a finite or infinite sequence $s = (s_1 > s_2 > \dots) \in \cup_{n=1}^{\infty} \Omega_n \cup \Omega_\infty$ such that

1. The s_{2j-1}'s are the singular values of H_u such that
$$\dim E_u(s_{2j-1}) = \dim F_u(s_{2j-1}) + 1 \ .$$
2. The s_{2k}'s are the singular values of K_u such that
$$\dim F_u(s_{2k}) = \dim E_u(s_{2k}) + 1 \ .$$

For every n, associate to each s_n an inner function Ψ_n.

The statement

Let

$$\mathcal{B} := \cup_{k=0}^{\infty} \mathcal{B}_k$$

and the mapping

$$\Phi : \begin{array}{ccc} VMO_+(\mathbb{T}) \setminus \{0\} & \longrightarrow & \cup_{n=1}^{\infty} \Omega_n \times \mathcal{B}^n \cup \Omega_\infty \times \mathcal{B}^\infty \\ u & \longmapsto & ((s_j), (\Psi_j)) \,. \end{array}$$

Theorem

The map Φ is bijective.
Moreover, explicit formula for Φ^{-1} on $\Omega_n \times \mathcal{B}^n$.

Topological features

Theorem

The following restriction maps of Φ,

$$\Phi_n : \Phi^{-1}(\Omega_n \times \mathcal{B}^n) \to \Omega_n \times \mathcal{B}^n \;,\; \Phi_\infty : \Phi^{-1}(\Omega_\infty \times \mathcal{B}^\infty) \to \Omega_\infty \times \mathcal{B}^\infty$$

are homeomorphisms. Moreover, given a positive integer n, and a sequence $(d_1, \dots, d_n)$ of nonnegative integers, the map

$$\Phi^{-1} : \Omega_n \times \prod_{r=1}^{n} \mathcal{B}_{d_r} \longrightarrow VMO_+(\mathbb{T})$$

is a smooth embedding.

Manifolds

As a consequence, the set

$$\mathcal{V}_{(d_1,\dots,d_n)} := \Phi^{-1}\left(\Omega_n \times \prod_{r=1}^{n} \mathcal{B}_{d_r}\right)$$

is a submanifold of $VMO_+(\mathbb{T})$ of dimension $n + \sum_{r=1}^{n} d_r$:
$\mathcal{V}_{(d_1,\dots,d_n)}$ is the set of symbols u such that

1. The singular values s_{2j-1} of H_u such that $\dim E_u(s_{2j-1}) = \dim F_u(s_{2j-1}) + 1$, ordered decreasingly, have respective multiplicities
$$d_1 + 1, d_3 + 1, \dots .$$
2. The singular values s_{2j} of K_u such that $\dim F_u(s_{2j}) = \dim E_u(s_{2j}) + 1$, ordered decreasingly, have respective multiplicities
$$d_2 + 1, d_4 + 1, \dots .$$

Back to the generic case

The generic finite rank case corresponds to $(d_1, \dots, d_n) = (0, \dots, 0)$. Denote by

$$\mathcal{V}(n) := \left\{ u;\ \mathrm{rk} H_u = \left[\frac{n+1}{2}\right],\ \mathrm{rk} K_u = \left[\frac{n}{2}\right] \right\} .$$

$\mathcal{V}(n)$ is a Kähler submanifold of L^2_+ of complex dimension n. Let $\mathcal{V}(n)_{\mathrm{gen}} := \mathcal{V}_{(0,\dots,0)}$ its open subset made of generic states u so that H_u and K_u have simple singular values. Through Φ,

$$\mathcal{V}(n)_{\mathrm{gen}} \simeq \Omega_n \times \mathcal{B}_0^n \simeq \Omega_n \times \mathbb{T}^n .$$

Main steps of the proof

- Reduce to finite rank case by a compactness argument .
- Φ_n is continuous and the degree of the Ψ_r's is locally constant.
- Prove that $\Phi_n : \mathcal{V}_{(d_1,\dots,d_n)} \mapsto \Omega_n \times \prod_{r=1}^n \mathcal{B}_{d_r}$ is a homeomorphism.
    - Injectivity : explicit formula for u in terms of its spectral data.
    - Surjectivity :
        - The mapping Φ_n is proper : compactness argument.
        - The mapping Φ_n is open : explicit calculation with the formulae giving u_s, u'_s.
        - Prove $\mathcal{V}_{(d_1,\dots,d_n)}$ is non empty.
    - Conclude by the connectedness of the target space $\Omega_n \times \prod_{r=1}^n \mathcal{B}_{d_r}$.
- Prove that Φ_n^{-1} is a smooth embedding of $\Omega_n \times \prod_{r=1}^n \mathcal{B}_{d_r}$ so that $\mathcal{V}_{(d_1,\dots,d_n)}$ is a smooth manifold.

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    - The mapping Φ_n is proper : compactness argument.
    - The mapping Φ_n is open : explicit
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- Prove that Φ_n^{-1} is a smooth embedding of $\Omega_n \times \prod_{r=1}^n \mathcal{B}_{d_r}$ so that $\mathcal{V}_{(d_1,\dots,d_n)}$ is a smooth manifold.

Main steps of the proof

- Reduce to finite rank case by a compactness argument .
- Φ_n is continuous and the degree of the Ψ_r's is locally constant.
- Prove that $\Phi_n : \mathcal{V}_{(d_1,\dots,d_n)} \mapsto \Omega_n \times \prod_{r=1}^n \mathcal{B}_{d_r}$ is a homeomorphism.
  - Injectivity : explicit formula for u in terms of its spectral data.
  - Surjectivity :
    - The mapping Φ_n is proper : compactness argument.
    - The mapping Φ_n is open : explicit calculation with the formulae giving u_s , u'_s.
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- $\mathcal{V}(n)_{\mathrm{gen}}$ is non empty :

$$u(z) = z^{q-1} + z^{q-2} \in \mathcal{V}(2q-1)_{\mathrm{gen}},$$

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A key lemma about Hankel operators

Key Lemma

Let N be a positive integer. Let $Q(z) := 1 - c_1 z - c_2 z^2 - \cdots - c_N z^N$ be a complex valued polynomial with no roots in the closed unit disc. Let H be an anti-linear operator on $\frac{\mathbb{C}_{N-1}[z]}{Q(z)}$ satisfying

$$S^* H S^* = H - (1|\cdot)u \, .$$

Then *H coïncides with the Hankel operator H_u* on $\frac{\mathbb{C}_{N-1}[z]}{Q(z)}$.

Link with the cubic Szegö equation

The simultaneous consideration of operators H_u and K_u was suggested by the study of the equation on L^2_+ endowed with the symplectic structure $\omega(u, v) := \mathrm{Im}\,(u|v)$.

$$i\dot{u} = \Pi(|u|^2 u)\ .$$

A Hamiltonian system for

$$E(u) = \frac{1}{4}\int_{\mathbb{T}} |u|^4 \,\frac{dx}{2\pi},$$

wellposed on $H^s_+(\mathbb{T})$, $s \geq \frac{1}{2}$.
This system enjoys a double Lax pair structure,

$$\frac{dH_u}{dt} = [B_u, H_u]\ ,\ \frac{dK_u}{dt} = [C_u, K_u]\ .$$

Generalized action angle coordinates

Given $u \in H^{1/2}_+(\mathbb{T})$, write $\Phi(u) = \left((s_r), (\Psi_r := \mathrm{e}^{-i\psi_r}\chi_r)\right)$.

Theorem

The evolution of the cubic Szegö equation on $H^{1/2}_+$ *reads*

$$\frac{ds_r}{dt} = 0\ ,\ \frac{d\psi_r}{dt} = (-1)^{r-1} s_r^2\ ,\ \frac{d\chi_r}{dt} = 0\ .$$

Moreover, on $\mathcal{V}_{(d_1,\dots,d_n)}$,

$$\omega_{|\mathcal{V}(d_1,\dots,d_n)} = \sum_{r=1}^n d\left(\frac{s_r^2}{2}\right) \wedge d\psi_r\ ,\ E = \frac{1}{4}\sum_{r=1}^n (-1)^{r-1} s_r^4\ .$$

In particular, $\mathcal{V}_{(d_1,\dots,d_n)}$ *is a an involutive submanifold of the Kähler manifold* $\mathcal{V}(d)$ *with* $d = n + 2\sum_{r=1}^n d_r$.

Perspectives

- Qualify the rational approximation it provides.
- Contrary to the $H^{1/2}(\mathbb{T})$ regularity, the $H^s(\mathbb{T})$ regularity is not easily described by the mapping Φ. One can even show that the conservation laws of the previous Hamiltonian system do not control this regularity. It is an open problem to find a criterion leading to high regularity of u in terms of $\Phi(u)$.