Riesz transforms of the Hodge-De Rham Laplacian on Riemannian manifolds

Jocelyn Magniez

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We suppose that :
$\rightarrow$ M satisfies the **doubling volume property**, that is there exist constants $C, D > 0$ such that :

$$v(x, \lambda r) \leq C\lambda^D v(x, r), \ \forall x \in M, r \geq 0, \lambda \geq 1, \qquad \text{(D)}$$

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$\rightarrow$ $p_t(x, y)$ satisfies a **Gaussian upper bound**, that is there exist constants $c, C > 0$ such that :

$$p_t(x, y) \leq \frac{C}{v(x, \sqrt{t})} exp(-c\frac{\rho^2(x, y)}{t}), \forall t > 0, \forall x, y \in M. \qquad \text{(G)}$$

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Motivation

We consider the heat equation :

$$\frac{\partial}{\partial t}u + \Delta u = 0, \ \ \forall t > 0 \ \text{ and } \ u(0) = u_0 \in L^p(M). \tag{1}$$

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It is well known that $-\Delta$ generates an analytic semi-group of contractions $(e^{-t\Delta})_{t\geqslant 0}$ on $L^p(M)$ and that the solution of (1) is $u(t) = e^{-t\Delta}u_0$.

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Let $d(\Delta)^{-\frac{1}{2}}$ be the Riesz transform associated to Δ. We have : $d(\Delta)^{-\frac{1}{2}} \in \mathcal{L}(L^p) \Longleftrightarrow \forall f \in \mathcal{D}(\Delta),\ \|d\,f\|_p \leqslant C\|\Delta^{\frac{1}{2}}f\|_p$.

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In this case we obtain :

$$u(t) \in W^{1,p}(M).$$

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Some known results

Since we have by integration by parts :

$$\|d\,f\|_2 = \|\nabla f\|_2 = \|\Delta^{\frac{1}{2}} f\|_2, \forall f \in \mathcal{C}_0^\infty(M),$$

it is obvious that $d(\Delta)^{-\frac{1}{2}}$ extends to a bounded operator from $L^2(M)$ to $L^2(\Lambda^1 T^* M)$ where $\Lambda^1 T^* M$ denotes the space of 1-forms on M. An interesting question is to know if $d(\Delta)^{-\frac{1}{2}}$ can extend to a bounded operator from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for $p \neq 2$.

Some known results

→ In 1999, Coulhon and Duong proved that under the assumptions (D) and (G), the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ is of weak-type $(1,1)$ and then bounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for all $p \in (1,2]$. In addition, they gave a complete non-compact Riemannian manifold satisfying (D) and (G) on which $d(\Delta)^{-\frac{1}{2}}$ is unbounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for $p > 2$.

Some known results

→ In 2003, Coulhon and Duong showed that if the manifold M satisfies (D), (G) and the heat kernel $\vec{p_t}(x,y)$ associated with the Hodge-De Rham Laplacian $\vec{\Delta}$ acting on 1-forms satisfies a Gaussian upper bound :

$$\|\vec{p_t}(x,y)\| \leq \frac{C}{v(x,\sqrt{t})} exp(-c\frac{\rho^2(x,y)}{t}), \forall t>0, \forall x,y \in M,$$

then the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for all $p \in (1,\infty)$.

Some known results

→ In 1987, Bakry proved that if the Ricci curvature is non-negative, then the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for all $p \in (1, \infty)$. The proof uses probabilistic technics and is based on the domination :

$$|e^{-t\vec{\Delta}}\omega| \leq e^{-t\Delta}|\omega|, \forall t > 0, \omega \in C_0^\infty(\Lambda^1 T^* M).$$

Some known results

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Devyver assumed that R_- is ε-sub critical : for some $\varepsilon \in [0, 1)$,

$$0 \leq (R_-\omega, \omega) \leq \varepsilon\,(H\omega, \omega), \forall\omega \in \mathcal{C}_0^\infty(\Lambda^1 T^*M), \qquad \text{(S-C)}$$

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Besides, assuming $R_- \in L^{\frac{D}{2}-\eta} \cap L^\infty$, he obtained a Gaussian upper-bound for $\vec{p_t}(x,y)$ and then the boundedness of the Riesz transform $d(-\Delta)^{-\frac{1}{2}}$ from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for $p \in (1,\infty)$. In the same time, he proved that if $R_- \in L^{\frac{D}{2}}$, then R_- is ε-sub-critical if and only if there is no harmonic 1-form on M.

Some known results

$\rightarrow$ In 2010, Assaad and Ouhabaz studied the boundedness of Riesz transforms associated to Schrodinger operators $A = \Delta + V_+ - V_-$. They proved that if (D), (G) are satisfied and if V_- is ϵ-sub-critical, then $\nabla A^{-\frac{1}{2}}$ is bounded on $L^p(M)$ for all $p \in (p_0', 2]$ where $p_0 = \frac{2D}{(D-2)(1-\sqrt{1-\varepsilon})} > 2$.

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Main results

Theorem

Assume that $(D), (G)$ *and* $(S-C)$ *are satisfied, then :*

(i) *the Riesz transform* $d^*(\overrightarrow{\Delta})^{-\frac{1}{2}}$ *is bounded from* $L^p(\Lambda^1 T^* M)$ *to* $L^p(M)$ *for all* $p \in (p_0', 2]$ *where* $p_0 = \frac{2D}{(D-2)(1-\sqrt{1-\varepsilon})}$.

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Corollary

Under the assumptions $(D), (G)$ *and* $(S-C)$, *the Riesz transform* $d(\Delta)^{-\frac{1}{2}}$ *is bounded from* $L^p(M)$ *to* $L^p(\Lambda^1 T^* M)$ *for all* $p \in (1, p_0)$ *where* $p_0 = \frac{2D}{(D-2)(1-\sqrt{1-\varepsilon})} > 2$.

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$\vec{\Delta} = d^*d + dd^*$ is the Hodge-De Rham Laplacian acting on $L^2(\Lambda^1 T^*M)$. Here, according to the context, d denotes the exterior derivative on functions or 1-forms and d^* its L^2-adjoint.

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Böchner formula : $\vec{\Delta} = \nabla^*\nabla + R_+ - R_- = H - R_-$, where R_+ (resp. R_-) is the positive part (resp. negative part) of the Ricci curvature and ∇ denotes the Levi-Civita connection on M.

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Böchner formula : $\vec{\Delta} = \nabla^*\nabla + R_+ - R_- = H - R_-$, where R_+ (resp. R_-) is the positive part (resp. negative part) of the Ricci curvature and ∇ denotes the Levi-Civita connection on M.

We suppose that R_- is ε-sub-critical, that is for some $0 \leq \varepsilon < 1$:

$$0 \leq (R_-\omega, \omega) \leq \varepsilon\,(H\omega, \omega), \forall \omega \in \mathcal{C}_0^\infty(\Lambda^1 T^*M). \qquad \text{(S-C)}$$

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That is for all $\omega, \eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$:

$$\overrightarrow{\mathfrak{h}}(\omega,\eta) = \int_M < \nabla\omega(x), \nabla\eta(x) >_x \, dm + \int_M < R_+(x)\omega(x), \eta(x) >_x dm,$$

$$\text{and } \mathcal{D}(\overrightarrow{\mathfrak{h}}) = \overline{\mathcal{C}_0^\infty(\Lambda^1 T^*M)}^{\|.\|_{\overrightarrow{\mathfrak{h}}}},$$

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where $\|\omega\|_{\vec{\mathfrak{h}}} = \sqrt{\vec{\mathfrak{h}}(\omega,\omega) + \|\omega\|_2^2}$.

Since R_- is ε-sub-critical, we can define the auto-adjoint operator $\vec{\Delta} = \nabla^*\nabla + R_+ - R_-$ on $L^2(\Lambda^1 T^*M)$ with the form :

$$\vec{\mathfrak{a}}(\omega,\eta) = (H\omega,\eta) - \int_M < R_-(x)\omega(x), \eta(x) >_x dm,$$

$$\mathcal{D}(\vec{\mathfrak{a}}) = \mathcal{D}(\vec{\mathfrak{h}}).$$

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L^p theory of the heat semi-group on forms

Theorem

Suppose that (D) *and* (G) *are satisfied and that the negative part* R_- *of the Ricci curvature is* ε*-sub-critical. Then the operator* $\vec{\Delta} = \nabla^*\nabla + R_+ - R_-$ *generates a* $\mathcal{C}^0$*-semi-group of contractions on* $L^p(\Lambda^1 T^*M)$ *for all* $p \in (p_0', p_0)$ *where* $p_0 = \frac{2D}{(D-2)(1-\sqrt{1-\varepsilon})}$.

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Lemma

For any suitable $\omega \in \Lambda^1 T^* M$ and for every $x \in M$:

$$< \nabla(\omega|\omega|^{p-2})(x), \nabla\omega(x) >_x \geq \frac{4(p-1)}{p^2} |\nabla(\omega|\omega|^{\frac{p}{2}-1})(x)|_x^2$$

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Proof.

We consider $\eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$ and set $\omega_t = e^{-t\vec{\Delta}}\eta$ for all $t \geq 0$. The previous lemma and the ε-sub-criticality of R_- lead to :

$$-\frac{1}{p}\frac{d}{dt}\|\omega_t\|_p^p \geq \left(\frac{4(p-1)}{p^2} - \varepsilon\right)\|H^{\frac{1}{2}}(|\omega_t|^{\frac{p}{2}-1}\omega_t)\|_2^2.$$

Then for all $p \in [\frac{2}{1+\sqrt{1-\varepsilon}}, \frac{2}{1-\sqrt{1-\varepsilon}}]$:

$$-\frac{1}{p}\frac{d}{dt}\|\omega_t\|_p^p \geq 0.$$

Therefore $\|\omega_t\|_p \leq \|\omega_0\|_p$, that is :

$$\|e^{-t\vec{\Delta}}\eta\|_p \leq \|\eta\|_p, \forall \eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M),$$

and we conclude by density considerations. □

Theorem

Suppose that (D), (G) and (S − C) are satisfied. We consider $2 \leq p < p_1$ *and* q *such that* $1 \leq q \leq \infty$ *and* $\frac{q-1}{q} D < 2$*. Then for all* $x \in M$ *and* $t > 0$ *:*

$$\|\chi_{B(x,\sqrt{t})} e^{-s\vec{\Delta}}\|_{p-pq} \leq \frac{C}{v(x,\sqrt{t})^{\frac{1}{p}-\frac{1}{pq}}} \left(max\left(1, \sqrt{\frac{t}{s}}\right)\right)^{\frac{2}{p}}.$$

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Proposition (Davies-Gaffney estimate)

Let E, F *be two closed subsets of* M. *For any* $\eta \in L^2(\Lambda^1 T^* M)$ *with support in* E, *we have :*

$$\|e^{-t\vec{\Delta}} \eta\|_{L^2(F)} \leq e^{-\frac{\rho^2(E,F)}{2t}} \|\eta\|_2.$$

L^p theory of the heat semi-group on forms

Theorem

Suppose that the negative part R_- of the Ricci curvature is ε-sub-critical. Then the operator $\vec{\Delta} = \nabla^*\nabla + R_+ - R_-$ generates a $\mathcal{C}^0$-semi-group of contractions on $L^p(\Lambda^1 T^*M)$ for all $p \in (p_0', p_0)$ where $p_0 = \frac{2D}{(D-2)(1-\sqrt{1-\varepsilon})}$.

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Let E, F be two closed subsets of M. For any $\eta \in L^2(\Lambda^1 T^* M)$ with support in E we have :

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Lemma

For any suitable $\omega \in \Lambda^1 T^*M$ and for every $x \in M$:

(i) $|d\omega(x)|_x \leq 2|\nabla\omega(x)|_x$.

(ii) $|d^*\omega(x)|_x \leq \sqrt{N}|\nabla\omega(x)|_x$.

Corollary

Let E, F be two closed subsets of M. For any $\eta \in L^2(\Lambda^1 T^* M)$ with support in E we have :

(i) $\|de^{-t\vec{\Delta}}\eta\|_{L^2(F)} \leq \frac{C}{\sqrt{t}} e^{-c\frac{\rho^2(E,F)}{t}} \|\eta\|_2.$

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Theorem

Suppose that (D), (G) and $(S-C)$ are satisfied. Then for all $r, s > 0$, $x, y \in M$ and all $p \in (p_0', p_0)$, $q \in [p, p_0)$,

$$\|\chi_{C_j(x,r)} e^{-s\vec{\Delta}} \chi_{B(x,r)}\|_{p-q} \leq \frac{Ce^{-c\frac{4^j r^2}{s}}}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \left(max\left(\frac{2^{j+1} r}{\sqrt{s}}, \frac{\sqrt{s}}{2^{j+1} r}\right)\right)^{\beta}$$

Theorem

Suppose that (D), (G) and (S − C) are satisfied. Then for all $r, s > 0$, $x, y \in M$ and all $p \in (p_0', 2]$,

(i) $\|\chi_{C_j(x,r)} d\, e^{-s\vec{\Delta}} \chi_{B(x,r)}\|_{p-2} \leq \dfrac{Ce^{-c\frac{4^j r^2}{s}}}{\sqrt{s}\, v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \left(max(\frac{r}{\sqrt{s}}, \frac{\sqrt{s}}{r})\right)^{\beta} 2^{j\beta}$

(ii) $\|\chi_{C_j(x,r)} d^* e^{-s\vec{\Delta}} \chi_{B(x,r)}\|_{p-2} \leq \dfrac{Ce^{-c\frac{4^j r^2}{s}}}{\sqrt{s}\, v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \left(max(\frac{r}{\sqrt{s}}, \frac{\sqrt{s}}{r})\right)^{\beta} 2^{j\beta}$

Theorem (Blunck, Kunstmann)

Let $p \in (1,2]$. Suppose that T is a sublinear operator of strong type $(2,2)$, and let $(A_r)_{r>0}$ be a family of linear operators acting on L^2. Assume that for $j \geq 2$ and every ball $B = B(x,r)$,

$$\left(\frac{1}{v(x,2^{j+1}r)}\int_{C_j(x,r)}|T(I-A_r)f|^2\right)^{\frac{1}{2}} \leq g(j)\left(\frac{1}{v(x,r)}\int_B |f|^p\right)^{\frac{1}{p}}, \qquad (2)$$

and for $j \geq 1$,

$$\left(\frac{1}{v(x,2^{j+1}r)}\int_{C_j(x,r)}|A_r f|^2\right)^{\frac{1}{2}} \leq g(j)\left(\frac{1}{v(x,r)}\int_B |f|^p\right)^{\frac{1}{p}}, \qquad (3)$$

for all f supported in B. If $\Sigma := \sum_j g(j)2^{Dj} < \infty$, then T is of weak type (p,p), with a bound depending only on the strong type $(2,2)$ bound of T, p and Σ.

Main result

Theorem

Assume that $(D), (G)$ and $(S-C)$ are satisfied, then :

(i) *the Riesz transform $d^*(\overrightarrow{\Delta})^{-\frac{1}{2}}$ is bounded from $L^p(\Lambda^1 T^* M)$ to $L^p(M)$ for all $p \in (p_0', 2]$ where $p_0 = \frac{2D}{(D-2)(1-\sqrt{1-\varepsilon})}$.*

(ii) *the Riesz transform $d(\overrightarrow{\Delta})^{-\frac{1}{2}}$ is bounded from $L^p(\Lambda^1 T^* M)$ to $L^p(\Lambda^2 T^* M)$ for all $p \in (p_0', 2]$ where $p_0 = \frac{2D}{(D-2)(1-\sqrt{1-\varepsilon})}$.*

Sketch of proof.

Set $T = d^*(\overrightarrow{\Delta})^{-\frac{1}{2}}$ and consider the operators
$A_r = I - (I - e^{-r^2\overrightarrow{\Delta}})^m$ for some integer m large enough.

□

Sketch of proof.

Set $T = d^*(\overrightarrow{\Delta})^{-\frac{1}{2}}$ and consider the operators $A_r = I - (I - e^{-r^2\overrightarrow{\Delta}})^m$ for some integer m large enough.

We prove (3) by using the estimate :

$$\|\chi_{C_j(x,r)} e^{-s\overrightarrow{\Delta}} \chi_{B(x,r)}\|_{p-q} \leq \frac{Ce^{-c\frac{4^j r^2}{s}}}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \left(max(\frac{2^{j+1}r}{\sqrt{s}}, \frac{\sqrt{s}}{2^{j+1}r}) \right)^{\beta}$$

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We prove (2) by using the estimate :

$$\|\chi_{C_j(x,r)} d^* e^{-s\overrightarrow{\Delta}} \chi_{B(x,r)}\|_{p-2} \leq \frac{Ce^{-c\frac{4^j r^2}{s}}}{\sqrt{s}\, v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \left(max(\frac{r}{\sqrt{s}}, \frac{\sqrt{s}}{r}) \right)^{\beta} 2^{j\beta}.$$

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It is the same proof for $T = d(\overrightarrow{\Delta})^{-\frac{1}{2}}$.

□

1 Introduction

2 Preliminaries

3 L^p theory of the heat semi-group on forms

4 The Riesz transforms $d^*(\vec{\Delta})^{-\frac{1}{2}}$ and $d(\vec{\Delta})^{-\frac{1}{2}}$

5 Around the sub-critical assumption

Assaad and Ouhabaz introduced the following quantities :

$$\alpha_1 = \int_0^1 \left\| \frac{R_-^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_1}}} \right\|_{r_1} \frac{dt}{\sqrt{t}}, \quad \alpha_2 = \int_1^\infty \left\| \frac{R_-^{\frac{1}{2}}}{v(.,\sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \frac{dt}{\sqrt{t}}, \quad (4)$$

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We set :

$$\|R_-^{\frac{1}{2}}\|_v := \alpha_1 + \alpha_2.$$

It is an easy exercise to see that when the volume is polynomial, that is when $v(x,r) = r^N$, then $\|R_-^{\frac{1}{2}}\|_v < \infty$ if and only if $R_- \in L^{\frac{N}{2}-\eta} \cap L^{\frac{N}{2}+\eta}$ for some $\eta > 0$. This kind of assumption is classical when studying the boundedness of Riesz transform of Schrödinger operators on L^p for $p > 2$.

Theorem

Assume that the manifold M satisfies (D), (G), $\|R_-^{\frac{1}{2}}\|_v < \infty$. If $Ker_{L^2}(\overrightarrow{\Delta}) = \{0\}$, then R_- is ε-sub-critical.

Sketch of proof.

Using the assumptions (D) and (G), we obtain :

$$(R_{-}\omega,\omega) \leq C\|R_{-}^{\frac{1}{2}}\|_{v}^{2}(H\omega,\omega), \forall\omega \in \mathcal{D}(\overrightarrow{\mathfrak{h}}). \tag{5}$$

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We set $\Lambda := H^{-\frac{1}{2}}R_{-}H^{-\frac{1}{2}} = (R_{-}^{\frac{1}{2}}H^{-\frac{1}{2}})^{*}(R_{-}^{\frac{1}{2}}H^{-\frac{1}{2}})$ and show that the operator $H^{\frac{1}{2}}$ is an isomorphism from $Ker_{H^1}(\overrightarrow{\Delta})$ to $Ker_{L^2}(I-\Lambda)$ where
$Ker_{H^1}(\overrightarrow{\Delta}) = \{\omega \in \mathcal{D}(\overrightarrow{\mathfrak{a}}) : \forall \eta \in \mathcal{C}_0^{\infty}(\Lambda^1 T^* M), \overrightarrow{\mathfrak{a}}(\omega, \eta) = 0\}$.

Sketch of proof.

Using the assumptions (D) and (G), we obtain :

$$(R_-\omega,\omega) \leq C\|R_-^{\frac{1}{2}}\|_v^2(H\omega,\omega), \forall\omega \in \mathcal{D}(\vec{\mathfrak{h}}). \qquad (5)$$

We set $\Lambda := H^{-\frac{1}{2}}R_-H^{-\frac{1}{2}} = (R_-^{\frac{1}{2}}H^{-\frac{1}{2}})^*(R_-^{\frac{1}{2}}H^{-\frac{1}{2}})$ and show that the operator $H^{\frac{1}{2}}$ is an isomorphism from $Ker_{H^1}(\vec{\Delta})$ to $Ker_{L^2}(I-\Lambda)$ where
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Besides Λ is a compact operator on $L^2(\Lambda^1T^*M)$; it is a consequence of (5).

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Besides Λ is a compact operator on $L^2(\Lambda^1 T^* M)$; it is a consequence of (5).
Since $\vec{\Delta}$ is non-negative, we have : $\|\Lambda\|_{2-2} \leq 1$. Then the self-adjointness and the compactness of Λ ensure that :
$Ker_{H^1}(\vec{\Delta}) = \{0\} \iff \|\Lambda\|_{2,2} < 1$.

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To conclude it suffices to remark that :
R_{-} is sub-critical $\iff \exists\, 0\leq\epsilon<1, \|\Lambda\|_{2,2}\leq\epsilon$.

Thanks for your attention!