On the structure of semigroups on $L_p$ with a bounded $H^\infty$-calculus

Stephan Fackler

Institute of Applied Analysis, University of Ulm

Journées du GdR «Analyse Fonctionelle, Harmonique et Probabilités»
Notation: $\Sigma_\varphi := \{ z \in \mathbb{C} : |\arg(z)| < \varphi \}$.

**Definition (Sectorial Operator)**

$(A, D(A))$ densely defined operator with $\omega \in (0, \pi)$ such that

$(S_\omega)$ \quad $\sigma(A) \subset \overline{\Sigma_\omega}$ and $\sup_{\lambda \notin \Sigma_{\omega + \epsilon}} \| \lambda R(\lambda, A) \| < \infty$ $\forall \epsilon > 0$.

Then $\omega(A) := \inf \{ \omega : (S_\omega) \text{ holds} \}$.
Definition (Analytic $C_0$-semigroup)

Family of operators $(T(z))_{z \in \Sigma_\delta}$ $(\delta \in (0, \frac{\pi}{2}))$ satisfying

(i) $z \mapsto T(z)$ is analytic

(ii) $T(z_1 + z_2) = T(z_1)T(z_2)$ $\forall z_1, z_2 \in \Sigma_\delta$

(iii) $\lim_{z \to 0} \frac{T(z)x}{z} = x$ $\forall \delta' \in (0, \delta), \forall x \in X$

It is called bounded if $\sup_{z \in \Sigma_{\delta'}} \|T(z)\| < \infty$ for all $\delta' \in (0, \delta)$.

One has 1:1 correspondence

bounded analytic $C_0$-semigroups $\leftrightarrow$ A sectorial with $\omega(A) < \frac{\pi}{2}$

At least formally $T(z) = e^{-zA}$. 
Given $f \in H_0^\infty(\Sigma_\sigma) := \left\{ f : \Sigma_\sigma \to \mathbb{C} \text{ analytic} : |f(z)| \leq \frac{|\lambda|^\varepsilon}{(1+|\lambda|)^{2\varepsilon}} \right\}$ define

$$f(A) := \int_{\partial \Sigma_{\sigma'}} f(\lambda) R(\lambda, A) \, d\lambda \quad (\omega(A) < \sigma' < \sigma).$$

**Definition (Bounded $H^\infty$-calculus)**

$(A, D(A))$ sectorial has bounded $H^\infty(\Sigma_\sigma)$-calculus if for some $C \geq 0$

$$(H_{\sigma}) \quad \|f(A)\| \leq C \sup_{z \in \Sigma_\sigma} |f(z)| \quad \forall f \in H_0^\infty(\Sigma_\sigma).$$

Then $\omega_{H^\infty}(A) := \inf\{\sigma : (H_{\sigma}) \text{ holds}\}.$
Theorem (C. Le Merdy)

$-A \sim (T(z))_{z \in \Sigma}$ bounded analytic $C_0$-semigroup on Hilbert space $H$.

Equivalent:

(i) $A$ has a bounded $H^\infty$-calculus

(ii) there exists $S \in \mathcal{B}(H)$ invertible such that

$$\|S^{-1}T(t)S\| \leq 1 \quad \forall t \geq 0.$$ 

Put differently: contractive semigroups are generic for all semigroups with a bounded $H^\infty$-calculus.
Can this be generalized to $L_p$ ($1 < p < \infty$)? In one direction, one has

**Theorem (L. Weis)**

$-A \sim (T(z))_{z \in \Sigma}$ bounded analytic $C_0$-semigroup on $L_p$, positive and contractive on the real line. Then $A$ has a bounded $H^\infty$-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$.

Can all semigroups on $L_p$ with a bounded $H^\infty$-calculus be obtained from such semigroups?
Theorem (S.F.)

\(-A \sim (T(z))_{z \in \Sigma}\) bounded analytic \(C_0\)-semigroup on \(L_p(\Omega)\) \((1 < p < \infty)\).

Equivalent:

(i) \(A\) has a bounded \(H^\infty\)-calculus with \(\omega_{H^\infty}(A) < \frac{\pi}{2}\).
Theorem (S.F.)

\(-A \sim (T(z))_{z \in \Sigma}\) bounded analytic \(C_0\)-semigroup on \(L_p(\Omega)\) \((1 < p < \infty)\).

Equivalent:

(i) \(A\) has a bounded \(H^\infty\)-calculus with \(\omega_{H^\infty}(A) < \frac{\pi}{2}\).

(ii) There exists a bounded holomorphic \(C_0\)-semigroup \((R(z))_{z \in \tilde{\Sigma}}\) in some \(L_p(\tilde{\Omega})\), positive and contractive on the real line with

- \(N \subset M \subset L_p(\tilde{\Omega})\) closed subspaces invariant under \((R(z))\)
- \(S \in B(L_p(\Omega), M/N)\) isomorphism

such that

\[ T(z) = S^{-1}R_{M/N}(z)S \quad \forall z \in \tilde{\Sigma}. \]
Theorem (S.F. (Reminder))

$-A \sim (T(z))_{z \in \Sigma}$ bounded analytic $C_0$-semigroup on $L_p(\Omega)$. Equivalent:

(i) $A$ has a bounded $H^\infty$-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$.

(ii) $T(z) = S^{-1}R_{M/N}(z)S \quad \forall z \in \tilde{\Sigma}$.

- On Hilbert spaces $\omega_{H^\infty}(A) < \frac{\pi}{2}$ holds automatically.
- This seems to be open for $L_p$-spaces, but false for general subspaces of $L_p$-spaces (N.J. Kalton).
- $(i) \Rightarrow (ii)$ holds on every UMD-Banach lattice ($(R(z))$ lives on another UMD-Banach lattice).
| Problem | Does the result hold without factorizing through a subspace-quotient as in the Hilbert space case? |
Problem

Does the result hold without factorizing through a subspace-quotient as in the Hilbert space case?

Problem

Can the constructed semigroup \((R(z))\) be chosen to be contractive on a whole sector as in the Hilbert space case?
Does the result hold without factorizing through a subspace-quotient as in the Hilbert space case?

Can the constructed semigroup \((R(z))\) be chosen to be contractive on a whole sector as in the Hilbert space case?

Does every positive contractive \(C_0\)-semigroup on a UMD Banach lattice have a bounded \(H^\infty\)-calculus?
Main ideas of the proof (I):

- For $\alpha > 1$ give $H^\infty_0(\sum \frac{\pi}{2\alpha} +)$ a $p$-operator space structure as follows:

$$H^\infty(\sum \frac{\pi}{2\alpha} +) \hookrightarrow \mathcal{B}(L_p(\mathbb{R}; Y))$$

$$f \mapsto f(B^{\frac{1}{\alpha}}),$$

where $-B$ generates the shift semigroup $V(t)g(s) = g(s - t)$ on $L_p(\mathbb{R}; Y)$ for some vector-valued $L_p$-space $Y$. 
Main ideas of the proof (I):

- For $\alpha > 1$ give $H^\infty_0(\sum \frac{\pi}{2\alpha} +)$ a $p$-operator space structure as follows:

  $$H^\infty(\sum \frac{\pi}{2\alpha} +) \hookrightarrow B(L_p(\mathbb{R}; Y))$$

  $$f \mapsto f(B^{1/\alpha}),$$

  where $-B$ generates the shift semigroup $V(t)g(s) = g(s - t)$ on $L_p(\mathbb{R}; Y)$ for some vector-valued $L_p$-space $Y$.

- $p$-complete boundedness of the functional calculus, e.g. mappings

  $$B(\ell^n_p(H^\infty(\Sigma))) \supset M_n(H^\infty) \to M_n(B(L_p(\mathbb{R}; Y))) \simeq B(\ell^n_p(L_p(\mathbb{R}; Y)))$$

  $$[f_{ij}] \mapsto [f_{ij}(B)]$$

  are uniformly bounded in $n$. 
Main ideas of the proof (II):

- A factorization theorem of G. Pisier yields a semigroup as asserted, except for strong continuity (ultraproduct construction).
- Reduce to the strongly continuous part.
Conclusions

Every bounded analytic $C_0$-semigroup on $L_p(\Omega)$ with generator $-A$ satisfying $\omega_{H^\infty}(A) < \frac{\pi}{2}$ can be obtained

- from a bounded analytic $C_0$-semigroup on $L_p(\tilde{\Omega})$, positive and contractive on the real line
- after passing to invariant subspace-quotients and similarity transforms
Je vous remercie de votre attention.
Je vous remercie de votre attention.

Et bon appétit!