The complete solution of a double inverse spectral problem for compact Hankel operators

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Lyon – GdR AFHP – 22 octobre 2013

from joint works with Patrick Gérard (Université Paris-Sud)
Motivation

- **Spectral theory** of Hankel operators: a key tool in the study of some non-dispersive Hamiltonian system: the cubic Szegő equation.

- A complete integrable system which admits two Lax pairs related to Hankel operators.

- Solve a double inverse spectral problem for compact Hankel operators.

- Apply it to obtain qualitative results on the dynamics of the cubic Szegő equation.
PART I: CLASSICAL HANKEL OPERATORS (HANKEL MATRICES).
Hankel operators in the real domain

A Hankel operator is an operator on \( \ell^2_{\mathbb{R}}(\mathbb{Z}_+) \) of the form

\[
(\Gamma_c(x))_n = \sum_{k=0}^{\infty} c_{n+k} x_k .
\]

is selfadjoint and satisfies

\[
\Gamma_c \Sigma = \Sigma^* \Gamma_c = \Gamma \Sigma^* c
\]

where \( \Sigma \) is the shift operator,

\[
\Sigma : (x_0, x_1, \cdots) \mapsto (0, x_0, x_1, \cdots)
\]

Nehari, 1957 : \( \Gamma_c \) is bounded iff

\[
\exists f \in L^\infty(\mathbb{T}), \, \forall n \geq 0, \, c_n = \hat{f}(n) ,
\]

or iff \( u_c(e^{ix}) := \sum_{n=0}^{\infty} c_n e^{inx} \in BMO_+ \) (C. Fefferman, 1971).
Hartman, 1958: $\Gamma_c$ is compact iff

$$\exists f \in C(\mathbb{T}), \forall n \geq 0, c_n = \hat{f}(n),$$

or iff $u_c(e^{inx}) = \sum_{n=0}^{\infty} c_n e^{inx} \in VMO_+.$

In this case, $\Gamma_c$ is compact and self-adjoint, hence

$\exists$ a sequence $(\lambda_j)_{j \geq 1}, \lambda_j \in \mathbb{R}, \lambda_j \to 0,$ with

$$|\lambda_1| \geq |\lambda_2| \geq \ldots$$

such that the eigenvalues of $\Gamma_c$ are the $\lambda_j$'s, repeated according to multiplicity, and possibly 0.
Motivation

Real setting

Complex setting

Main result

The cubic Szegö equation

The Megretski–Peller–Treil theorem

What are the constraints on the $\lambda_j$’s?

**Theorem (Megretski–Peller–Treil, 1995)**

If $(\lambda_j)_{j \geq 1}$ is the sequence of eigenvalues of some selfadjoint compact Hankel operator, then, for every $\lambda \in \mathbb{R} \setminus \{0\}$,

$$\left| \# \{ j : \lambda_j = \lambda \} - \# \{ j : \lambda_j = -\lambda \} \right| \leq 1.$$  

Conversely, any sequence $(\lambda_j)_{j \geq 1}$ of real numbers satisfying the above condition and tending to 0 is the sequence of eigenvalues of some selfadjoint compact Hankel operator.

**Question:** describe the isospectral classes.
The Megretski–Peller–Treil theorem

What are the constraints on the $\lambda_j$’s?

**Theorem (Megretski–Peller–Treil, 1995)**

If $(\lambda_j)_{j \geq 1}$ is the sequence of eigenvalues of some selfadjoint compact Hankel operator, then, for every $\lambda \in \mathbb{R} \setminus \{0\}$,

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Conversely, any sequence $(\lambda_j)_{j \geq 1}$ of real numbers satisfying the above condition and tending to $0$ is the sequence of eigenvalues of some selfadjoint compact Hankel operator.

Question: describe the isospectral classes.
No uniqueness expected: an example

Even in the rank one case, no uniqueness expected. Indeed, $\Gamma_c$ is a selfadjoint rank one operator if and only if

$$c_n = \alpha p^n, \quad \alpha \in \mathbb{R}^*, \quad p \in (-1, 1).$$

The only nonzero eigenvalue is

$$\lambda_1 = \frac{\alpha}{1 - p^2}.$$

Isospectral sets are therefore manifolds diffeomorphic to $\mathbb{R}$. Hence, we need to introduce additional parameters.
The shifted Hankel operator

Given a Hankel operator $\Gamma_c$, define $\tilde{\Gamma}_c$ as

$$\tilde{\Gamma}_c = \Sigma^* \Gamma_c = \Gamma_c \Sigma = \Gamma \Sigma^* c.$$  

Notice that

$$\tilde{\Gamma}_c^2 = \Gamma_c \Sigma \Sigma^* \Gamma_c = \Gamma_c^2 - (c, c).$$

If $\Gamma_c$ is selfadjoint compact, so is $\tilde{\Gamma}_c$, and its eigenvalues $(\mu_j)_{j \geq 1}$ satisfy

$$|\lambda_1| \geq |\mu_1| \geq |\lambda_2| \geq |\mu_2| \geq \ldots$$

The case with strict inequalities corresponds to a dense $G_\delta$ subset of $VMO_{+,\mathbb{R}}$, for which the inverse spectral problem has a particularly simple solution.
The generic case

Theorem (PG-S. Grellier, 2012)

Given two sequences \((\lambda_j)_{j \geq 1}, (\mu_j)_{j \geq 1}\) of real numbers such that

\[
|\lambda_1| > |\mu_1| > |\lambda_2| > \cdots \to 0,
\]

there exists a unique sequence \((c_n)_{n \geq 0}\) of real numbers such that \(\Gamma_c\) is compact and

- The non zero eigenvalues of \(\Gamma_c\) are the \(\lambda_j\)’s.
- The non zero eigenvalues of \(\tilde{\Gamma}_c\) are the \(\mu_j\)’s.
If \( c_n = \alpha p^n, \alpha \in \mathbb{R}^*, p \in (-1, 1) \). The only nonzero eigenvalue of \( \Gamma_c \) is
\[
\lambda_1 = \frac{\alpha}{1 - p^2}.
\]
The only nonzero eigenvalue of \( \tilde{\Gamma}_c \) is
\[
\mu_1 = \frac{\alpha p}{1 - p^2}.
\]
The knowledge of \( \lambda_1 \) and \( \mu_1 \) characterizes \( \alpha \) and \( p \), hence \( c \).
Catching the multiplicities

In the general case, consider the — finite or infinite tending to 0 — sequence of non zero eigenvalues of $\Gamma_c$ and $\tilde{\Gamma}_c$, listed so that

$$|\lambda_1| \geq |\mu_1| \geq |\lambda_2| \geq |\mu_2| \geq \ldots$$

Lemma (P. Gérard-S.G.)

$\forall \lambda \neq 0$ such that $\ker(\tilde{\Gamma}_c^2 - \lambda^2 I) + \ker(\Gamma_c^2 - \lambda^2 I) \neq \{0\}$,

$$|\dim \ker(\tilde{\Gamma}_c^2 - \lambda^2 I) - \dim \ker(\Gamma_c^2 - \lambda^2 I)| = 1.$$

Consequently, in the series $|\lambda_1| \geq |\mu_1| \geq |\lambda_2| \geq |\mu_2| \geq \ldots$, the length of a maximal string with consecutive equal terms is odd.
Theorem (P. Gérard-S.G, 2013)

Let $(\lambda_j), (\mu_j)$ be two — finite or infinite tending to 0—sequences of non zero real numbers satisfying

- $|\lambda_1| \geq |\mu_1| \geq |\lambda_2| \geq |\mu_2| \geq \ldots$
- In the above sequence, the lengths of maximal strings with consecutive equal terms are odd. Denote them by $(2n_r - 1)_r$.

- $\forall \lambda \neq 0, |\#\{j : \lambda_j = \lambda\} - \#\{j : \lambda_j = -\lambda\}| \leq 1$.
- $\forall \mu \neq 0, |\#\{j : \mu_j = \mu\} - \#\{j : \mu_j = -\mu\}| \leq 1$.

Then there exists a sequence $(c_n)_{n \geq 0}$ of real numbers such that $\Gamma_c$ is compact and

- The non zero eigenvalues of $\Gamma_c$ are the $\lambda_j$’s.
- The non zero eigenvalues of $\tilde{\Gamma}_c$ are the $\mu_j$’s.

Moreover, if $M = \sum_r (n_r - 1)$, the isospectral set is a manifold diffeomorphic to $\mathbb{R}^M$ if $M < \infty$, homeomorphic to $\mathbb{R}^\infty$ if $M = \infty$. 
An example

In the case of a finite sequence of nonzero eigenvalues, explicit formulae for $u_c$. For instance, given four real numbers such that

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > 0,$$

we get

$$u_c(e^{ix}) = \frac{\lambda_1 - \mu_1 e^{ix}}{\lambda_1^2 - \mu_1^2} + \frac{\lambda_2 - \mu_2 e^{ix}}{\lambda_2^2 - \mu_2^2} - \frac{\lambda_1 - \mu_2 e^{ix}}{\lambda_1^2 - \mu_2^2} - \frac{\lambda_2 - \mu_1 e^{ix}}{\lambda_2^2 - \mu_1^2} + \frac{\lambda_1 - \mu_1 e^{ix}}{\lambda_1^2 - \mu_1^2} + \frac{\lambda_2 - \mu_2 e^{ix}}{\lambda_2^2 - \mu_2^2}.$$

If $|\lambda_1| > |\lambda_2| > 0$ and $\mu_1 = \lambda_2, \mu_2 = -\lambda_2$, then, there exists $p \in (-1, 1)$ such that

$$u_c(e^{ix}) = (\lambda_1^2 - \lambda_2^2) \frac{1 - p e^{ix}}{\lambda_1 - p e^{ix}(\lambda_1 - \lambda_2) - \lambda_2 e^{2ix}}.$$
Remarks

Hence, if $\lambda_1, \lambda_2$ are given such that $|\lambda_1| > |\lambda_2| > 0$, the corresponding isospectral set consists of sequences $c$ given by the above two formulae.

Notice that the second expression is obtained from the first one by making $\mu_1 \rightarrow \lambda_2$, $\mu_2 \rightarrow -\lambda_2$, and

$$\frac{2\lambda_2 + \mu_2 - \mu_1}{\mu_1 + \mu_2} \rightarrow p.$$
Part II: Complexified version.
The Hardy space representation

\[ L^2_+ = \left\{ u : u(e^{ix}) = \sum_{n=0}^{\infty} c_n e^{inx}, \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\}, \]

\[ \Pi : L^2(\mathbb{T}) \longrightarrow L^2_+ \text{ the Szegö projector}, \]

Given \( u \in VMO_+ \), define \( H_u \) on \( L^2_+ \) by

\[ H_u(h) = \Pi \left( u \hat{h} \right). \]

\( H_u \) is a compact antilinear operator, non selfadjoint, and

\[ \hat{H_u}(\hat{h}) = \Gamma \hat{u} \left( \hat{h} \right), \quad \hat{K_u}(\hat{h}) = \tilde{\Gamma} \hat{u} \left( \hat{h} \right) \]

\[ K_u := S^* H_u = H_u S = H_{S^*u}, \quad Sh(e^{ix}) := e^{ix} h(e^{ix}), \]

\[ K_u^2 = H_u^2 - (\cdot | u) u. \]
Eigenspaces of $H_u^2, K_u^2, u \in VMO_+$

$E_u(s) := \ker(H_u^2 - s^2 I)$, $F_u(s) := \ker(K_u^2 - s^2 I)$.

**Lemma (P. Gérard-S.G., 2013)**

Let $s > 0$ such that $E_u(s) + F_u(s) \not= \{0\}$.

$|\dim E_u(s) - \dim F_u(s)| = 1$.

Let $(s_j^2)_j$ – finite or infinite tending to $0$ – the sequence of distinct eigenvalues of $H_u^2$ and $K_u^2$.

The $s_{2j-1}$’s are the singular values of $H_u$ such that

$$\dim E_u(s_{2j-1}) = \dim F_u(s_{2j-1}) + 1.$$

The $s_{2k}$’s are the singular values of $K_u$ such that

$$\dim F_u(s_{2k}) = \dim E_u(s_{2k}) + 1.$$
A finite Blaschke product is an inner function of the form

$$\psi(z) = e^{i\psi} \prod_{j=1}^{k} \frac{z - p_j}{1 - \overline{p}_j z}, \quad \psi \in \mathbb{T}, \ p_j \in \mathbb{D}.$$  

The integer $k$ is called the degree of $\psi$. Alternatively, $\psi$ can be written as

$$\psi(z) = e^{i\psi} \frac{z^k D\left(\frac{1}{z}\right)}{D(z)},$$

where $D$ is a polynomial of degree $k$, $D(0) = 1$, with all its roots outside $\mathbb{D}$. We denote by $\mathcal{B}_k$ the set of Blaschke product of degree $k$. It is a classical result that $\mathcal{B}_k$ is diffeomorphic to $\mathbb{T} \times \mathbb{R}^{2k}$. 
Action of $H_u$ and $K_u$ on the eigenspaces

Proposition (P. Gérard-S.G., 2013)

Let $s > 0$ and $u \in \text{VMO}_+ (\mathbb{T})$. Assume $m := \dim E_u(s) = \dim F_u(s) + 1$. Denote by $u_s$ the orthogonal projection of $u$ onto $E_u(s)$. There exists $\psi_s$, a finite Blaschke product, of degree $m - 1$, such that $su_s = \psi_s H_u(u_s)$ and, if $\psi_s(z) = e^{-i\psi_s} \frac{z^{m-1}D_s(\frac{1}{z})}{D_s(z)}$,

$$E_u(s) = \frac{H_u(u_s)}{D_s}C_{m-1}[z], \quad F_u(s) = \frac{H_u(u_s)}{D_s}C_{m-2}[z],$$

$$H_u \left( \frac{z^a}{D_s} H_u(u_s) \right) = se^{-i\psi_s} \frac{z^{m-a-1}}{D_s} H_u(u_s), \quad 0 \leq a \leq m - 1$$

$$K_u \left( \frac{z^b}{D_s} H_u(u_s) \right) = se^{-i\psi_s} \frac{z^{m-b-2}}{D_s} H_u(u_s), \quad 0 \leq b \leq m - 2.$$
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Action of $H_u$ and $K_u$ – continued

Assume $\ell := \dim F_u(s) = \dim E_u(s) + 1$. Denote by $u'_s$ the orthogonal projection of $u$ onto $F_u(s)$. There exists a finite Blaschke product $\psi_s$ of degree $\ell - 1$, such that $K_u(u'_s) = s\psi_s u'_s$ and, if $\psi_s(z) = e^{-i\psi_s} \frac{z^{\ell-1} D_s(1/z)}{D_s(z)}$,

$$F_u(s) = \frac{u'_s}{D_s} C_{\ell-1}[z], \quad E_u(s) = \frac{zu'_s}{D_s} C_{\ell-2}[z],$$

$$K_u \left( \frac{z^a}{D_s} u'_s \right) = s e^{-i\psi_s} \frac{z^{\ell-a-1}}{D_s} u'_s, \quad 0 \leq a \leq \ell - 1$$

$$H_u \left( \frac{z^{b+1}}{D_s} u'_s \right) = s e^{-i\psi_s} \frac{z^{\ell-b-1}}{D_s} u'_s, \quad 0 \leq b \leq \ell - 2.$$
Coming back to selfadjoint operators

Remark that the preceding identities provide very simple matrices for the action of $H_u$ and $K_u$ on $E_u(s)$ and $F_u(s)$. Selfadjoint Hankel operators correspond to symbols $u$ with real Fourier coefficients, hence the angles $\psi_s$ belong to $\{0, \pi\}$. In this case, one can easily check that the dimensions of the eigenspaces of these matrices associated to the eigenvalues $\pm s$ differ of at most 1: the Megretskii–Peller–Treil condition.
Notation

- $\Omega_n := \{s_1 > s_2 > \cdots > s_n > 0\} \subset \mathbb{R}^n$.
- $\Omega_\infty = \{(s_n)_{n \geq 1}, s_1 > s_2 > \cdots > s_n \to 0\}$.

Given $u \in VMO_+ (\mathbb{T}) \setminus \{0\}$, define a finite or infinite sequence $s = (s_1 > s_2 > \cdots) \in \bigcup_{n=1}^\infty \Omega_n \cup \Omega_\infty$ such that

1. The $s_{2j-1}$’s are the singular values of $H_u$ such that
   \[ \dim E_u(s_{2j-1}) = \dim F_u(s_{2j-1}) + 1. \]

2. The $s_{2k}$’s are the singular values of $K_u$ such that
   \[ \dim F_u(s_{2k}) = \dim E_u(s_{2k}) + 1. \]

For every $n$, associate to each $s_n$ an inner function $\Psi_n$. 
The statement

Let

\[ \mathcal{B} := \bigcup_{k=0}^{\infty} \mathcal{B}_k \]

and the mapping

\[ \Phi : \overset{\text{VMO}_+(\mathbb{T}) \setminus \{0\}}{u} \rightarrow \bigcup_{n=1}^{\infty} \Omega_n \times \mathcal{B}^n \cup \Omega_\infty \times \mathcal{B}_\infty \]

\[ u \mapsto -((-s_j), (\Psi_j)) . \]

Theorem

*The map \( \Phi \) is bijective.*

*Moreover, explicit formula for \( \Phi^{-1} \) on \( \Omega_n \times \mathcal{B}^n \).*
The following restriction maps of $\Phi$,

$$
\Phi_n : \Phi^{-1}(\Omega_n \times \mathcal{B}^n) \to \Omega_n \times \mathcal{B}^n, \quad \Phi_\infty : \Phi^{-1}(\Omega_\infty \times \mathcal{B}^\infty) \to \Omega_\infty \times \mathcal{B}^\infty
$$

are homeomorphisms. Moreover, given a positive integer $n$, and a sequence $(d_1, \ldots, d_n)$ of nonnegative integers, the map

$$
\Phi^{-1} : \Omega_n \times \prod_{r=1}^n \mathcal{B}_{d_r} \longrightarrow \text{VMO}_+(\mathbb{T})
$$

is a smooth embedding.
Manifolds

As a consequence, the set

\[ V_{(d_1,\ldots,d_n)} := \Phi^{-1} \left( \Omega_n \times \prod_{r=1}^{n} B_{d_r} \right) \]

is a submanifold of $VMO_+(\mathbb{T})$ of dimension $n + \sum_{r=1}^{n} d_r$:

$V_{(d_1,\ldots,d_n)}$ is the set of symbols $u$ such that

1. The singular values $s_{2j-1}$ of $H_u$ such that $\dim E_u(s_{2j-1}) = \dim F_u(s_{2j-1}) + 1$, ordered decreasingly, have respective multiplicities $d_1 + 1, d_3 + 1, \ldots$.

2. The singular values $s_{2j}$ of $K_u$ such that $\dim F_u(s_{2j}) = \dim E_u(s_{2j}) + 1$, ordered decreasingly, have respective multiplicities $d_2 + 1, d_4 + 1, \ldots$. 
Back to the generic case

The generic finite rank case corresponds to $(d_1, \ldots, d_n) = (0, \ldots, 0)$. Denote by

$$V(n) := \left\{ u; \rk H_u = \left[ \frac{n+1}{2} \right], \rk K_u = \left[ \frac{n}{2} \right] \right\}.$$

$V(n)$ is a Kähler submanifold of $L^2_+$ of complex dimension $n$. Let $V(n)_{\text{gen}} := V(0, \ldots, 0)$ its open subset made of generic states $u$ so that $H_u$ and $K_u$ have simple singular values. Through $\Phi$,

$$V(n)_{\text{gen}} \simeq \Omega_n \times B_0^n \simeq \Omega_n \times \mathbb{T}^n.$$
Main steps of the proof

- Reduce to finite rank case by a compactness argument.
- $\Phi_n$ is continuous and the degree of the $\Psi_r$’s is locally constant.
- Prove that $\Phi_n : \mathcal{V}(d_1,...,d_n) \mapsto \Omega_n \times \prod_{r=1}^n B_{d_r}$ is a homeomorphism.
  - Injectivity: explicit formula for $u$ in terms of its spectral data.
  - Surjectivity:
    - The mapping $\Phi_n$ is proper: compactness argument.
    - The mapping $\Phi_n$ is open: explicit calculation with the formulae giving $u_s, u'_s$.
    - Prove $\mathcal{V}(d_1,...,d_n)$ non empty.
  - Conclude by the connectedness of the target space $\Omega_n \times \prod_{r=1}^n B_{d_r}$.
- Prove that $\Phi_n^{-1}$ is a smooth embedding of $\Omega_n \times \prod_{r=1}^n B_{d_r}$ so that $\mathcal{V}(d_1,...,d_n)$ is a smooth manifold.
Main steps of the proof

- Reduce to finite rank case by a compactness argument.
- \(\Phi_n\) is continuous and the degree of the \(\Psi_r\)'s is locally constant.
- Prove that \(\Phi_n : \mathcal{V}(d_1, \ldots, d_n) \mapsto \Omega_n \times \prod_{r=1}^{n} B_{d_r}\) is a homeomorphism.
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  - Surjectivity:
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    - The mapping \(\Phi_n\) is open: explicit calculation with the formulae giving \(u, u'\).
    - Prove \(\mathcal{V}(d_1, \ldots, d_n)\) is non-empty.
  - Conclude by the connectedness of the target space \(\Omega_n \times \prod_{r=1}^{n} B_{d_r}\).
- Prove that \(\Phi_n^{-1}\) is a smooth embedding of \(\Omega_n \times \prod_{r=1}^{n} B_{d_r}\) so that \(\mathcal{V}(d_1, \ldots, d_n)\) is a smooth manifold.
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  - Surjectivity:
    - The mapping $\Phi_n$ is proper: compactness argument.
    - The mapping $\Phi_n$ is open: explicit calculation with the formulae giving $\nu_s$, $\nu'_s$.
  - Prove $\text{Dom}(\Phi_n)$ is non-empty.
- Conclude by the connectedness of the target space $\Omega_n \times \prod_{r=1}^{n} B_{d_r}$.
- Prove that $\Phi_n^{-1}$ is a smooth embedding of $\Omega_n \times \prod_{r=1}^{n} B_{d_r}$ so that $\mathcal{V}_{(d_1,\ldots,d_n)}$ is a smooth manifold.
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- Prove that $\Phi_n^{-1}$ is a smooth embedding of $\Omega_n \times \prod_{r=1}^n B_{d_r}$ so that $\mathcal{V}(d_1,\ldots,d_n)$ is a smooth manifold.
Main steps of the proof

- Reduce to finite rank case by a compactness argument.
- \( \Phi_n \) is continuous and the degree of the \( \Psi_r \)'s is locally constant.
- Prove that \( \Phi_n : V(d_1, \ldots, d_n) \mapsto \Omega_n \times \prod_{r=1}^{n} B_{d_r} \) is a homeomorphism.
  - Injectivity: explicit formula for \( u \) in terms of its spectral data.
  - Surjectivity:
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    - The mapping \( \Phi_n \) is open: explicit calculation with the formulae giving \( u_s, u'_s \).
    - Prove \( V(d_1, \ldots, d_n) \) is non empty.
- Conclude by the connectedness of the target space \( \Omega_n \times \prod_{r=1}^{n} B_{d_r} \).
- Prove that \( \Phi_n^{-1} \) is a smooth embedding of \( \Omega_n \times \prod_{r=1}^{n} B_{d_r} \) so that \( V(d_1, \ldots, d_n) \) is a smooth manifold.
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    - The mapping $\Phi_n$ is proper: compactness argument.
    - The mapping $\Phi_n$ is open: explicit calculation with the formulae giving $u_s$, $u'_s$.
    - Prove $\mathcal{V}(d_1,...,d_n)$ is non empty.
- Conclude by the connectedness of the target space $\Omega_n \times \prod_{r=1}^{n} B_{d_r}$.
- Prove that $\Phi_n^{-1}$ is a smooth embedding of $\Omega_n \times \prod_{r=1}^{n} B_{d_r}$ so that $\mathcal{V}(d_1,...,d_n)$ is a smooth manifold.
Main steps of the proof

- Reduce to finite rank case by a compactness argument.
- $\Phi_n$ is continuous and the degree of the $\Psi_r$’s is locally constant.
- Prove that $\Phi_n : \mathcal{V}(d_1,\ldots,d_n) \mapsto \Omega_n \times \prod_{r=1}^{n} B_{d_r}$ is a homeomorphism.
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\( \mathcal{V}(d_1, \ldots, d_n) \) is non empty

- \( \mathcal{V}(n)_{\text{gen}} \) is non empty:

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 u(z) = z^{q-1} + z^{q-2} \in \mathcal{V}(2q - 1)_{\text{gen}},
\]

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 u(z) = \frac{z^{q-1} + z^{q-2}}{1 + \varepsilon z^q} \in \mathcal{V}(2q)_{\text{gen}}.
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- Prove that \( \mathcal{V}(d_1, \ldots, d_n) \) is non empty by induction on the \( d_j \)'s. At each step, we use the preceding homeomorphism. Induction starting from the generic case, by making \( s_{2r+1} - s_{2r-1} \) or \( s_{2k+2} - s_{2k} \) go to zero in the explicit formula.
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A key lemma about Hankel operators

Key Lemma

Let $N$ be a positive integer. Let

$Q(z) := 1 - c_1 z - c_2 z^2 - \cdots - c_N z^N$ be a complex valued polynomial with no roots in the closed unit disc. Let $H$ be an anti-linear operator on $\mathbb{C}_{N-1}[z]/Q(z)$ satisfying

$$S^* H S^* = H - (1|\cdot)u.$$ 

Then $H$ coincides with the Hankel operator $H_u$ on $\mathbb{C}_{N-1}[z]/Q(z)$. 
Link with the cubic Szegő equation

The simultaneous consideration of operators $H_u$ and $K_u$ was suggested by the study of the equation on $L^2_+$ endowed with the symplectic structure $\omega(u, v) := \text{Im} (u|v)$.

\[ i\dot{u} = \Pi(|u|^2u) \, . \]

A Hamiltonian system for

\[ E(u) = \frac{1}{4} \int_{\mathbb{T}} |u|^4 \frac{dx}{2\pi}, \]

wellposed on $H^s_+(\mathbb{T})$, $s \geq \frac{1}{2}$. This system enjoys a double Lax pair structure,

\[ \frac{dH_u}{dt} = [B_u, H_u] \, , \, \frac{dK_u}{dt} = [C_u, K_u] \, . \]
Generalized action angle coordinates

Given \( u \in H^{1/2}(\mathbb{T}) \), write \( \Phi(u) = ((s_r), (\Psi_r := e^{-i\psi_r} \chi_r)) \).

**Theorem**

The evolution of the cubic Szegő equation on \( H^{1/2}_+ \) reads

\[
\frac{ds_r}{dt} = 0, \quad \frac{d\psi_r}{dt} = (-1)^{r-1} s_r^2, \quad \frac{d\chi_r}{dt} = 0.
\]

Moreover, on \( \mathcal{V}(d_1, \ldots, d_n) \),

\[
\omega|_{\mathcal{V}(d_1, \ldots, d_n)} = \sum_{r=1}^{n} d \left( \frac{s_r^2}{2} \right) \wedge d\psi_r, \quad E = \frac{1}{4} \sum_{r=1}^{n} (-1)^{r-1} s_r^4.
\]

In particular, \( \mathcal{V}(d_1, \ldots, d_n) \) is a an involutive submanifold of the Kähler manifold \( \mathcal{V}(d) \) with \( d = n + 2 \sum_{r=1}^{n} d_r \).
Qualify the rational approximation it provides.

Contrary to the $H^{1/2}(\mathbb{T})$ regularity, the $H^s(\mathbb{T})$ regularity is not easily described by the mapping $\Phi$. One can even show that the conservation laws of the previous Hamiltonian system do not control this regularity. It is an open problem to find a criterion leading to high regularity of $u$ in terms of $\Phi(u)$. 