Lieb-Thirring vs Blaschke
for non-selfadjoint perturbations
of certain model operators

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Plan of the talk

Recent results by:
A. Borichev, L. Golinskii, SK,
M. Demuth, M. Hansmann, G. Katriel,
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1. Introduction: classical Lieb-Thirring inequalities.
2. Basic construction and results on zeros of holomorphic functions.
3. Applications to different models.
Let $H_0 = -\Delta$ considered on $L^2(\mathbb{R}^d)$. One has $\sigma(H_0) = \mathbb{R}_+$. 

Theorem (Lieb-Thirring’ 1975) Let $p > 0$, $d \geq 3$. Then

\[
\sum_{\lambda \in \sigma_d(H)} |\lambda|^p \leq C_p d \int_{\mathbb{R}^d} V^{p-\frac{d}{2}} dx = C_p d ||V||_{L^p}^{p-\frac{d}{2}},
\]

where $V^{-} = \max\{V, 0\}$. 

S. Kupin (U. Bordeaux 1)
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where $V \in L^p(\mathbb{R}^d), p > 0$, is a real-valued potential.
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One has $V(H_0 - \lambda)^{-1} \in S_\infty$, the class of compact operators, and, by Weyl’s theorem $\sigma_{ess}(H) = \sigma_{ess}(H_0) = \mathbb{R}_+$. 

Theorem (Lieb-Thirring' 1975)

Let $p > 0$, $d \geq 3$. Then

$$\sum_{\lambda \in \sigma_d(H)} |\lambda|^p \leq C_p^p,$$

$$\int \mathbb{R}^d |V(x) - \lambda|^{p+\frac{d}{2}} dx = C_p^{p+\frac{d}{2}},$$

where $V^\pm = \max\{-V, 0\}$. 

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Theorem (Lieb-Thirring’ 1975)

Let $p > 0$, $d \geq 3$. Then

$$\sum_{\lambda \in \sigma_d(H)} |\lambda|^p \leq C_{p,d} \int_{\mathbb{R}^d} V_-(x)^{p+d/2} \, dx = C_{p,d} \|V_-\|_{L^{p+d/2}}^{p+d/2}$$

where $V_- = \max \{-V, 0\}$. 
Consider now a complex-valued $V \in L^p(\mathbb{R}^d), p > 0$. 

**Theorem (Frank-Laptev-Lieb-Seiringer' 2007)**

Let $p \geq 1, d \geq 1$. Then

$$
\sum_{\lambda \in \sigma_d(H)} \text{Re} \lambda < 0 \quad \left| \text{Re} \lambda \right|^p \leq C_{p,d} \int_{\mathbb{R}^d} \left| \text{Re} V \right|^p + \frac{d}{2} \, dx = C_{p,d} \left\| \left| \text{Re} V \right|^p + \frac{d}{2} \right\|_{L^p}.
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$$\sum_{\lambda \in \sigma_d(H): \text{Re } \lambda < 0} |\text{Re } \lambda|^p \leq C_{p,d} \int_{\mathbb{R}^d} (\text{Re } V)^{p+d/2} \, dx$$

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For $\lambda \in \rho(A_0) = \bar{\mathbb{C}} \setminus \sigma(A_0)$, consider

$$f(\lambda) = \det_p(A - \lambda)(A_0 - \lambda)^{-1} = \det_p(I + (A - A_0)(A_0 - \lambda)^{-1}).$$
BGK : a basic method

One has:

- $\lambda \in \sigma_d(A) \iff \text{Ker}(A - \lambda I)$ is non-trivial and of finite dimension
- $\iff \lambda \in Z_f$ (i.e., $f(\lambda) = 0$) counting multiplicities,
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- $\lambda \in \sigma_d(A) \iff \text{Ker}(A - \lambda I)$ is non-trivial and of finite dimension
  $\iff \lambda \in Z_f$ (i.e., $f(\lambda) = 0$) counting multiplicities,
- $f \in Hol(\rho(A_0))$ and the following bound holds

$$|f(\lambda)| \leq \exp \left( \| (A - A_0)(A_0 - \lambda)^{-1} \|_{S_p}^p \right)$$
$$\leq \exp \left( \Gamma_p \| A - A_0 \|_{S_p}^p \right) \leq \exp \left( \frac{\Gamma_p \| A - A_0 \|_{S_p}^p}{d(\lambda, \sigma(A_0))^p} \right).$$
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On the next stage, one uses the (conformal) uniformization

$$\rho(A_0) = \mathbb{C} \setminus \sigma(A_0) \xrightarrow{\phi} \mathbb{D} = \{ |z| < 1 \}$$
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$$d(\lambda, \sigma(A_0)) \asymp F(z)d(z, \mathbb{T}) = F(z)(1 - |z|),$$
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Rather often, one sees that $F(z) = d(z, E)$, where $E \subset \mathbb{T}, \# E < \infty$. 
BGK: on zeros of holomorphic functions from certain classes

So one gets to the study of the zeros of classes of holomorphic functions appearing in the following theorems.

Theorem (Borichev-Golinskii-K’ 2010)

Let $f \in \text{Hol}(D)$, $|f(0)| = 1$, and $\log |f(z)| \leq D d(z, E)^q$, with $q \geq 0$. Then for any $\varepsilon > 0$,

$$\sum_{z \in \mathbb{Z}} f(1 - |z|) d(z, E)^{q - 1 + \varepsilon} \leq C(q, \varepsilon) D.$$
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where \( p, q \geq 0 \). Then for any \( \varepsilon > 0 \),

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\sum_{z \in Z_f} (1 - |z|)^{p+1+\varepsilon} d(z, E)^{(q-1+\varepsilon)_+} \leq C(p, q, \varepsilon) D.
\]
Applications and extensions

• (bounded) Jacobi matrices: Let $J - J_0 \in S_\infty$, where

$$J_0 = J(\{1\}, \{0\}, \{1\}) = \begin{bmatrix} 0 & 1 & 0 & \ldots \\ 1 & 0 & 1 & \ldots \\ 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

and

$$J = J(\{a_k\}, \{b_k\}, \{c_k\}) = \begin{bmatrix} b_1 & c_1 & 0 & \ldots \\ a_1 & b_2 & c_2 & \ldots \\ 0 & a_2 & b_3 & \ldots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

and $\{a_k\}, \{b_k\}, \{c_k\} \subset \mathbb{C}$.
Applications and extensions

It is clear that $\sigma(J_0) = [-2, 2]$. Then [BGK’ 2010]: for $p > 1$ and $\forall \varepsilon > 0$

$$\sum_{\lambda \in \sigma_d(J)} \frac{d(\lambda, [-2, 2])^{p+1+\varepsilon}}{|\lambda - 2||\lambda + 2|} \leq C(p, \varepsilon)\|J - J_0\|_{S_p}^p.$$
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- Similar results for $d$-dimensional Jacobi matrices.

- **Hansmann-Katriel’ 2011** : the above relation is improved to

$$\sum_{\lambda \in \sigma_d(J)} \frac{d(\lambda, [-2, 2])^{p+\varepsilon}}{(|\lambda - 2||\lambda + 2|)^{1/2}} \leq C(p, \varepsilon)\|J - J_0\|_{Sp}^p.$$
Applications and extensions

- Let $\tilde{J}_0$ be periodic (or finite-zone) Jacobi matrix. In particular,

$$\sigma(\tilde{J}_0) = \bigcup_{j=1}^{n} [\alpha_j, \beta_j].$$

Let $\tilde{E} = \{\alpha_j, \beta_j\}_{j=1,...,n}$, and $J - \tilde{J}_0 \in S_{\infty}$. 

- Hansmann' 2012: let $A = A_0 + B$, where $A_0 = A^*$ is bounded, and $B \in S_p$, $p > 1$.

Then

$$\sum_{\lambda \in \sigma} d(A) d(\lambda, \sigma(A_0)) \leq C_p ||B||_{S_p}. \quad (1)$$
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$$\sum_{\lambda \in \sigma_d(J)} \frac{d(\lambda, \sigma(\tilde{J}_0))^{p+1+\varepsilon}}{d(\lambda, \tilde{E})(1 + |\lambda|)} \leq C(p, \varepsilon)\|J - \tilde{J}_0\|_{S^p}. $$

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Extensions to unbounded operators

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Theorem (Dubuisson’ 2013)

Let $V \in L^p(\mathbb{R}^d)$, $p > d$.

- $(m > 0)$ For $0 < \tau$ small enough

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\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+\tau}}{|\lambda - m| (1 + |\lambda|)^{p+\max\{p/2,d\}+2\tau-1}} \leq C(p, d, \tau) \|V\|_{L^p}^p.
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\sum_{\lambda \in \sigma_d(K)} d(\lambda, \sigma(K_m))^{p+\tau} \frac{|\lambda - m| (1 + |\lambda|)^{p+\max\{p/2, d\}+2\tau-1}}{\min\{(p+\tau)/2, d\}(1 + |\lambda|)^{p/2+\max\{p,2d\}-d+2\tau}} \leq C(p, d, \tau) \|V\|_{L^p}^p.
$$

• ($m = 0$) Then, for $0 < \tau$ small enough

$$
\sum_{\lambda \in \sigma_d(K)} d(\lambda, \sigma(K_0))^{p+\tau} \frac{|\lambda|^{p+\max\{p,2d\}-d+2\tau}}{\min\{(p+\tau)/2, d\}(1 + |\lambda|)^{p/2+\max\{p,2d\}-d+2\tau}} \leq C(p, d, \tau) \|V\|_{L^p}^p.
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Open problems

- Non-selfadjoint perturbations of other operators of mathematical physics, e.g.

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- Geometry of eigen- (root-) subspaces associated to \( \sigma_d(H) \): a functional model? interpolation in certain spaces of holomorphic functions, etc.?
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