Some *smooth* applications of *non-smooth* Ricci curvature lower bounds

Andrea Mondino (University of Warwick)

Curvature-Dimension in Lyon
Université de Lyon,
15^{th} March 2017

Plan of the talk

GOAL: discuss some recent geometric applications to *smooth* Riemannian manifolds of *non-smooth* synthetic Ricci curvature lower bounds

1. Almost rigidity in the Levy-Gromov isoperimetric inequality,
2. Almost Euclidean isoperimetric inequality in a small ball in a space with Ricci curvature bounded below & application to Ricci flow,

Part 1. Almost rigidity in the Levy-Gromov isoperimetric inequality

Isoperimetric problem

One of oldest problems in mathematics, roots in myths of 2000 years ago (Queen Dido's problem). Roughly 3 questions:

- Q1 Given a space X what is the minimal amount of area needed to enclose a fixed volume $v > 0$?
- Q2 Is there an optimal shape?
- Q3 Describe/characterize the optimal shapes.

Examples

Not many examples of spaces X where we can fully answer Q1,Q2,Q3:

- $X = \mathbb{R}^n \rightsquigarrow$ only optimal shapes are round balls: $|\partial E| \geq |\partial B|$ where B is a round ball s.t. $|B| = |E|$.
- $X = S^n$ or $X = H^n$ analogous: only optimal shapes are metric balls: $|\partial E| \geq |\partial B|$ where B is a metric ball s.t. $|B| = |E|$
- Not many other examples (e.g. $\mathbb{R}P^3$ by Ritoré-Ros): in general the spaces for which we can fully answer Q1,Q2,Q3 are either very symmetric or perturbations of very symmetric spaces.
- Results in presence of mild singularities but still very symmetric (conical manifolds: Morgan-Ritoré '02, Milman-Rotem '14. Polytopes: Morgan '07).

Levy-Gromov inequality

Besides the euclidean one, probably the most famous isoperimetric inequality is the Levy-Gromov isoperimetric inequality:

Levy-Gromov Isoperimetric inequality

Let (M^n, g) Riemannian manifold with $Ric_g \geq Kg$, $K > 0$, and $E \subset M$ domain with smooth boundary ∂E. Then

$$\frac{|\partial E|}{|M|} \geq \frac{|\partial B|}{|S|} \quad (LGI)$$

where $S = S^n_K$ round sphere with $Ric \equiv K$, and $B \subset S$ is a spherical cap s.t. $\frac{|E|}{|M|} = \frac{|B|}{|S|}$.

Rigidity: If for some $E \subset M$ equality holds in (LGI) then $(M^n, g) \simeq S^n_K$ isometric.

Question: Almost rigidity? i.e. If (LGI) is almost attained, can we say that (M^n, g) is close to S^n_K in an appropriate sense?

In order to study the almost rigidity it is convenient to enlarge the class of spaces to allow *non-smooth spaces with Ricci curvature bounded below*.

Non smooth spaces with Ricci bounded below. 1

Notations:

- $(X, \mathrm{d}, \mathfrak{m})$ compact metric space (for simplicity, but everything holds for complete and separable, with appropriate changes) with a finite non-negative Borel measure $\mathfrak{m}$ (σ-finite would be enough)
- $(\mathcal{P}(X), W_2)$: metric space of probability measures on X endowed with quadratic transportation distance (Wasserstein)
- Entropy functional $\mathcal{U}_{N,\mathfrak{m}}(\mu)$ if $\mu = \rho\mathfrak{m} \ll \mathfrak{m}$

$$
\begin{aligned}
\mathcal{U}_{N,\mathfrak{m}}(\rho\mathfrak{m}) &:= -N \int \rho^{1-\frac{1}{N}} d\mathfrak{m} \quad \text{if } 1 \le N < \infty \\
&= \int \rho \log \rho \, d\mathfrak{m} \quad \text{if } N = \infty
\end{aligned}
$$

(if μ is not a.c. then the singular part does not contribute in case $N < \infty$, in case $N = \infty$ we set $\mathcal{U}_{\infty,\mathfrak{m}}(\mu) = \infty$).

Non smooth spaces with Ricci bounded below. 2

- Crucial observation If $(X, \mathrm{d}, \mathfrak{m})$ is a smooth Riemannian manifold (M, g), then $Ric_g \geq 0$ and dim $M \leq N$ iff the entropy functional $\mathcal{U}_{N,\mathfrak{m}}$ is convex along geodesics in $(\mathcal{P}(X), W_2)$.
- But the notion of convexity of the Entropy is purely of metric-measure nature, i.e. it makes sense in a general metric measure space $(X, \mathrm{d}, \mathfrak{m})$.
- DEF of $CD(K, N)$ condition [Lott-Sturm-Villani]: fixed $N \in [1, \infty]$, $(X, \mathrm{d}, \mathfrak{m})$ is a $CD(0, N)$-space if the Entropy $\mathcal{U}_{N,\mathfrak{m}}$ is convex along geodesics in $(\mathcal{P}(X), W_2)$.
  For $K \in \mathbb{R}$, $(X, \mathrm{d}, \mathfrak{m})$ is a $CD(K, N)$-space if the Entropy $\mathcal{U}_{N,\mathfrak{m}}$ is (K, N)-convex along geodesics in $(\mathcal{P}(X), W_2)$ (more complicated non linear condition).

Non smooth spaces with Ricci bounded below. 3

Keep in mind:
- $CD(K,N) spaces$ $\rightsquigarrow$ weak objects with Ricci curvature $\geq K$ and dimension $\leq N$
- the more convex is $\mathcal{U}_{N,\mathfrak{m}}$ along geodesics in $(\mathcal{P}(X), W_2)$, the more the space is positively Ricci curved.

Good properties:

- CONSISTENT: (M,g) satisfies $CD(K,N)$ iff $Ric \geq K$ and $dim(M) \leq N$
- GEOMETRIC PROPERTIES: Bishop-Gromov volume growth, Bonnet-Myers diameter bound, Lichnerowictz spectral gap, etc.
- STABLE under measured-Gromov Hausdorff convergence $\Longrightarrow$ all Ricci limit spaces are $CD(K,N)$ no matter if collapsing or not.
- Finsler manifolds with lower Ricci bounds are $CD(K,N)$.

$CD(K, N)$ and Ricci limits

- If our goal is to isolate a (synthetic) class of spaces which is closed under mGH convergence and which contains smooth manifolds with Ric $\geq K$ and dim$\leq N$, the class of $CD(K, N)$ spaces is TOO LARGE:
- compact Finsler manifolds satisfy $CD(K, N)$ for some $K \in \mathbb{R}$ and $N \geq 1$ but if a smooth Finsler manifold M is a Ricci-limit space then M is Riemannian (Cheeger-Colding '00).
- $\rightsquigarrow$ We would like to reinforce the $CD(K, N)$ condition in order to rule out Finsler structures, but in a sufficiently weak way in order to still get a STABLE notion.

Cheeger energy and $RCD(K,N)$-spaces

Given a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ and $f \in L^2(X, \mathfrak{m})$, define the Cheeger energy

$$Ch_{\mathfrak{m}}(f) := \frac{1}{2}\int_X |\nabla f|_w^2 \, d\mathfrak{m} = \liminf_{u \to f \ in L^2} \frac{1}{2}\int_X (\mathrm{lip} u)^2 d\mathfrak{m}$$

where $|\nabla f|_w$ is the minimal weak upper gradient.

Crucial observation: On a Finsler manifold M, the Cheeger energy is quadratic (i.e. parallelogram identity holds) iff M is Riemannian.

Idea[Ambrosio-Gigli-Savaré] reinforce the CD condition with the requirement that $Ch_{\mathfrak{m}}$ is quadratic (or, equivalently, the heat flow is linear).

Definition Given $K \in \mathbb{R}$ and $N \in [1, \infty]$, $(X, \mathrm{d}, \mathfrak{m})$ is an $RCD(K,N)$ space if it is a $CD(K,N)$ space & the Cheeger energy is quadratic.

Good properties of $RCD(K, N)$

- Stability under mGH convergence (Ambrosio-Gigli-Savaré and Gigli-M.-Savaré)
- Equivalent to Bochner inequality (for $N = \infty$ Gigli-Kuwada-Ohta + Ambrosio-Gigli-Savaré, for $N \in [1, \infty)$ Erbar-Kuwada-Sturm Vs Ambrosio-M.-Savaré)
- Implies Li-Yau inequalities (Garofalo-M. and Jiang)
- Implies Cheeger-Gromoll Splitting Theorem (Gigli)
- Local structure: euclidean tangent cones (Gigli-M.-Rajala and M.-Naber) and rectifiability (M.-Naber)
- Implies that Isometries are a Lie Group (Guijarro-Rodriguez and Sosa)
- Implies existence of a universal cover + classical Theorems on the (revised) fundamental group (M.-Wei)
- Local to Global property (Ambrosio-M.-Savaré, Cavalletti-E. Milman)
- ...

Examples of *RCD*-spaces

- Ricci limits, no matter if collapsed or not and no matter if the dimension is bounded above or not (in the first case get $RCD(K, N)$, in the latter get $RCD(K, \infty)$)
- Finite dimensional Alexandrov spaces with curvature bounded below (Perelman 90'ies and Otsu-Shioya '94: *Ch* is quadratic, Petrunin '12: CD is satisfied)
- Weighted Riemannian manifolds with Bakry-Émery $N - Ricci \geq K$: i.e. (M^n, g) Riemannian manifold, let $\mathfrak{m} := \Psi\, vol_g$ for some smooth function $\Psi \geq 0$, then $Ric_{g,\Psi,N} := Ric_g - (N-n)\frac{\nabla^2 \Psi^{1/N-n}}{\Psi^{1/N-n}} \geq Kg$ iff $(M, \mathrm{d}_g, \mathfrak{m})$ is $RCD(K, N)$.
- Cones or spherical suspensions over *RCD* spaces (Ketterer '13)
- ...

Levy-Gromov isoperimetric inequality in $RCD(K, N)$-spaces

DEF: Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. with $\mathfrak{m}(X) = 1$ and let $E \subset X$ be a Borel set. Define the outer Minkowski content

$$\mathfrak{m}^+(E) := \liminf_{\varepsilon \to 0^+} \frac{\mathfrak{m}(E^\varepsilon) - \mathfrak{m}(E)}{\varepsilon}$$

where $E^\varepsilon := \{x \in X \, : \, \mathsf{d}(x, E) < \varepsilon\}$

THM [Cavalletti-M. '15]: If $(X, \mathsf{d}, \mathfrak{m})$, with $\mathfrak{m}(X) = 1$, is an $RCD(K, N)$ space for some $K > 0$ and $2 \leq N \in \mathbb{N}$ then (LGI) holds, i.e. for every Borel subset $E \subset X$

$$\mathfrak{m}^+(E) \geq \frac{|\partial B|}{|S|}$$

where $S = S^N_K$ round sphere with $Ric \equiv K$, and $B \subset S$ is a spherical cap s.t. $\mathfrak{m}(E) = \frac{|B|}{|S|}$.

Method of proof and its roots

Main idea of proof: reduce the inequality to a family of 1-dimensional problems, via L^1-optimal transport.

Such a 1-dimensional reduction is called 1-D localization:

- In $\mathbb{R}^n$ or S^n, using the high symmetry of the space, 1-D localizations can be usually obtained via iterative bisections
  - Roots in a paper by Payne-Weinberger '60 about sharp estimate of 1^{st} eigenvalue of Neumann Laplacian in compact convex sets of $\mathbb{R}^n$
  - Formalized by Gromov-V. Milman '87, Kannan - Lovász - Simonovits '95
- Extended by B. Klartag '14 to Riemannian manifolds via L^1-optimal trasport: no symmetry but still heavily using the smoothness of the space (estimates on 2^{nd} fundamental form of level sets of the Kantorovich potential φ)
- Our contribute: simplify the approach via L^1-transport and extend it to non-smooth framework getting new applications.

Rigidity

Fact: we cannot hope to have the same rigidity as in the smooth setting, since (non-smooth) spherical suspensions have the same isoperimetric profile function of the round sphere.
Q: Are these the only cases of equality in (LGI)?

THM[Rigidity] (Cavalletti-M. '15) Let $(X, \mathrm{d}, \mathfrak{m})$ be an $RCD(N-1, N)$ space with $\mathfrak{m}(X) = 1$. Assume there exists $E \subset X$ with $\mathfrak{m}(E) \in (0, 1)$ such that

$$\mathfrak{m}^+(E) = \frac{|\partial B|}{|S|}$$

where $S = S^N$ round sphere of unit radius, and $B \subset S$ is a spherical cap s.t. $\mathfrak{m}(E) = \frac{|B|}{|S|}$.
Then $(X, \mathrm{d}, \mathfrak{m})$ is a spherical suspension: $X \simeq [0, \pi] \times_{\sin}^{N-1} Y$ as m.m.s. for some $RCD(N-2, N-1)$ space $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$

Almost Rigidity

Q: If (LGI) has almost equality, must the space be close to a spherical suspension?

THM[Almost Rigidity] (Cavalletti-M. '15) Let $(X, \mathsf{d}, \mathfrak{m})$ be an $RCD(N-1-\delta, N+\delta)$ space with $\mathfrak{m}(X)=1$. Assume there exists $E \subset X$ with $\mathfrak{m}(E)=v \in (0,1)$ such that

$$\mathfrak{m}^+(E) \leq \frac{|\partial B|}{|S|} + \delta.$$

Then $(X, \mathsf{d}, \mathfrak{m})$ is $\varepsilon = \varepsilon(\delta|N, v)$-mGH close to a spherical suspension:

$$\mathsf{d}_{mGH}(X, [0,\pi] \times_{\sin}^{N-1} Y) \leq \varepsilon$$

for some $RCD(N-2, N-1)$ space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$.

RK The almost rigidity seems to be new even in case $(X, \mathsf{d}, \mathfrak{m})$ is a smooth Riemannian manifold of dimension N and Ricci$\geq N-1-\delta$.

Part 2. Ricci flow, Perelman's Pseudo Locality Theorem and Almost euclidean isoperimetric inequalities.

Perelman's Pseudo-locality Theorem

THM[Theorem 10.1, Perelman's first Ricci flow paper 2002] For every $\alpha > 0$ there exists $\delta > 0, \varepsilon > 0$ with the following property. Suppose we have a smooth solution to the Ricci flow $(g_{ij})_t = -2R_{ij}$, $0 \le t \le \varepsilon^2$, and assume that at $t = 0$ we have

$$R_{g_0}(x) \ge -1 \quad \& \quad |\partial\Omega|_{g_0} \ge (1-\delta)\, c_n |\Omega|_{g_0}^{(n-1)/n},\ \forall x, \Omega \subset B_1(x_0),$$

where c_n is the euclidean isoperimetric constant.
Then we have an estimate $|Rm|(x,t) \le \alpha t^{-1} + \varepsilon^{-2}$ whenever $0 < t \le \varepsilon^2$, $\mathrm{d}_{g_t}(x, x_0) < \varepsilon$.

RK: fundamental difference from Ricci flow and heat flow. Heat flow has infinite speed of propagation, Ricci flow not. The non-linearity of Ricci flow here helps: if we have good geometric control on ball, and no assumtions outside, the Ricci flow for small times improves the geometric control in the ball regardless how bad the manifold is outside.

Perelman's Pseudo-locality Theorem revisited by Tian&Wang

THM[Tian-Wang JAMS 2015] For every $\alpha > 0$ there exists $\delta > 0, \varepsilon > 0$ with the following property. Suppose we have a smooth solution to the Ricci flow $(g_{ij})_t = -2R_{ij}$, $0 \leq t \leq \varepsilon^2$, and assume that at $t = 0$ we have

$$Ric_{g_0}(x) \geq -\delta^2 g_0 \text{ on } B_1(x_0) \quad \& \quad |B_1(x_0)|_{g_0} \geq (1 - \delta)\, \omega_n.$$

Then $|Rm|(x, t) \leq \alpha t^{-1} + \varepsilon^{-2}$ for $0 < t \leq \varepsilon^2$, $\mathrm{d}_{g_t}(x, x_0) < \varepsilon$.

RK: - From Bishop Gromov we have $|B|_{g_0} \leq (1 + C\,\delta)\, \omega_n$, so the condition $|B|_{g_0} \geq (1 - \delta)\, \omega_n$ is an almost maximal volume assumption.
-The proof by Tian-Wang is highly technical and it does not easily reduce to Perelman's Theorem.

Almost euclidean isoperimetric inequality

Q: do the assumptions of Tian-Wang's Pseudo-locality imply the assumptions of Perelman's Pseudo-Locality? I.E.

$$Ric_{g_0}(x) \geq -\delta^2 g_0 \ \& \ |B|_{g_0} \geq (1-\delta)\,\omega_n \stackrel{?}{\Rightarrow} |\partial\Omega|^n_{g_0} \geq (1-\varepsilon)\, c_n |\Omega|^{n-1}_{g_0}$$

for all $\Omega \subset B_\varepsilon(x_0)$.

THM[Cavalletti-M. '17] For every $N \in [2,\infty) \cap \mathbb{N}$ there exist $\bar{\varepsilon}_N, \bar{\delta}_N, C_N > 0$ such that the next holds.
Let (M, g) be a smooth N-dim. Riem. manifold and let $\bar{x} \in M$.
Assume that $B_1(\bar{x})$ is rel. compact and for some $\delta \in [0, \bar{\delta}_N]$

$$|B_1(\bar{x})| \geq (1-\delta)\omega_N \quad \& \quad Ric_g \geq -\delta^2 g \ \text{ on } B_1(\bar{x}).$$

Then for every subset $E \subset B_{\varepsilon_N}(\bar{x})$:

$$|\partial E|_g \geq N\omega_N^{1/N}(1 - C_N\delta)\,|E|_g^{\frac{N-1}{N}}.$$

RK Actually we prove the corresponding statement more generally for a m. m. space $(X, \mathsf{d}, \mathfrak{m})$ which is essentially non-branching, $CD_{loc}(-\delta, N)$ on a ball $B_1(\bar{x})$ and $\mathfrak{m}(B_1(\bar{x})) \geq (1-\delta)\omega_N$.

Combining Colding's volume convergence Theorem (Annals of Math. '97) with the above result we get:

COR[Cavalletti-M. '17] For every $N \in [2, \infty) \cap \mathbb{N}$ there exist $\bar{\varepsilon}_N, \bar{\delta}_N, C_N > 0$ such that the next holds.
Let (M, g) be a smooth N-dim. Riem. manifold and let $\bar{x} \in M$. Assume that $B_1(\bar{x})$ is rel. compact and for some $\delta \in [0, \bar{\delta}_N]$, it holds:

$$Ric_g \geq -\delta^2 g \text{ on } B_1(\bar{x}) \quad \& \quad \mathsf{d}_{GH}(B_1(\bar{x}), B_1^{\mathbb{R}^N}) \leq \delta.$$

Then for every subset $E \subset B_{\varepsilon_N}(\bar{x})$:

$$|\partial E|_g \geq N\omega_N^{1/N}(1 - C_N\delta)\, |E|_g^{\frac{N-1}{N}}.$$

RK closeness in GH-distance is a priori a very weak assumption (a manifold is δ-GH close to a δ-net which is a discrete space); so it is remarkable that GH-close + lower Ricci bound $\Rightarrow$ almost euclidean isoperimetric inequality.

Some comments, 1.

Q: Why the almost euclidean isoperimetric inequality remained an open problem since Perelman's work?

Classical method for proving Levy-Gromov isoperimetric inequality in a nutshell:

1. in a compact manifold, for every fixed volume v there is a minimizer Ω of the perimeter having volume v.
2. $\partial\Omega$ is smooth (up to a singular set of large codimension) and the smooth part has constant mean curvature (the regularity is now classical but it is not trivial at all!).
3. Using the regularity of $\partial\Omega$ (crucial: regular part has CMC) perform computations $\rightsquigarrow$ get a lower bound on $|\partial\Omega|$ (so a fortiori get a lower bound of the perimeter of any set since Ω is a minimizer).

Some comments, 2.

DIFFICULTY If we want to prove an AE isoperimetric ineq on $B_1(\bar{x})$

- A minimizing sequence for the perimeter can approach $\partial B_1(\bar{x})$ and so the minimizer Ω will hit $\partial B_1(\bar{x})$.
- On the contact region we have an obstacle problem, regularity is more tricky (partial regularity by Caffarelli in 70ies); in any case $\partial\Omega \cap \partial B_1(\bar{x})$ may not have constant mean curvature (if $\partial B_1(\bar{x})$ has not)
- $\rightsquigarrow$ not in good shape to perform computations of Levy-Gromov on the minimizer.

What we do: Via 1-D localization, we prove the lower bound on the perimeter of EVERY subset, not just of the minimizers, without any regularity assumption.
$\rightsquigarrow$ Use of synthetic Ricci curvature lower bounds via optimal transport to prove an exquisitely new smooth statement.

!!THANK YOU FOR THE ATTENTION!!