On Hardy's and Caffarelli, Kohn, Nirenberg's inequalites.

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Outline

1. Known results related to Hardy's and Caffarelli, Kohn, Nirenberg's (CKN's) inequalities
2. CKN's inequalities for fractional Sobolev spaces
3. New perspectives of Hardy's and CKN's inequalities in Sobolev spaces.

Section 1: Known results related to Hardy's and CKN's inequalities

Hardy's inequalities

1. For $1 \leqslant p < d$,

$$\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p}\,dx \leqslant C \int_{\mathbb{R}^d} |\nabla u|^p\,dx \quad \forall u \in C_c^1(\mathbb{R}^d).$$

2. For $p > d$,

$$\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p}\,dx \leqslant C \int_{\mathbb{R}^d} |\nabla u|^p\,dx \quad \forall u \in C_c^1(\mathbb{R}^d \setminus \{0\}).$$

3. For $p = d > 1$, ...

Standard proof is based on integration by parts.

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Caffarelli, Kohn, Nirenberg's inequalities, CM 84

Let $p \geqslant 1$, $q \geqslant 1$, $\tau > 0$, $0 < a \leqslant 1$, α, β, $\gamma \in \mathbb{R}$. One has

$$\||x|^\gamma u\|_{L^\tau(\mathbb{R}^d)} \leqslant C \||x|^\alpha \nabla u\|^a_{L^p(\mathbb{R}^d)} \||x|^\beta u\|^{(1-a)}_{L^q(\mathbb{R}^d)} \quad \forall u \in C^1_c(\mathbb{R}^d)$$

under the following conditions

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\left(\frac{1}{p} + \frac{\alpha - 1}{d}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{d}\right),$$

with $\gamma = a\sigma + (1-a)\beta$,

$$0 \leqslant \alpha - \sigma \quad \text{and} \quad \left(\alpha - \sigma \leqslant 1 \quad \text{if} \quad \frac{1}{\tau} + \frac{\gamma}{d} = \frac{1}{p} + \frac{\alpha - 1}{d}\right),$$

and

$$\frac{1}{\tau} + \frac{\gamma}{d}, \quad \frac{1}{p} + \frac{\alpha}{d}, \quad \frac{1}{q} + \frac{\beta}{d} > 0.$$

Comments on the CKN inequality

• This inequality is related to Gagliardo-Nirenberg's inequality when $\alpha = \beta = \gamma = 0$, Gagliardo RM 59, Nirenberg ASNSP 59.

• A full story of Gagliardo-Nirenberg's inequality for fractional Sobolev spaces is due to Brezis and Mironescu AIHP 18.

• The proof of CKN's inequality is based on

- Integration by parts and symmetrization in the case $0 \leqslant \alpha - \sigma \leqslant 1$.
- Interpolation & the application of the previous case when $\alpha - \sigma > 1$ and $\frac{1}{\tau} + \frac{\gamma}{d} \neq \frac{1}{p} + \frac{\alpha - 1}{d}$

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Section 2: CKN's inequality for fractional Sobolev spaces

CKN's inequality for fractional Sobolev spaces

- Goals: extending CKN's inequality for fractional Sobolev spaces and searching for variants of Hardy's inequality when $p > d$. Recall

$$|u|^p_{W^{s,p}} := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} \, dx \, dy \text{ for } u \in L^p(\mathbb{R}^d).$$

- Known results: (Hardy's type-inequalities):
  - Mazya & Shaposhnikova JFA 02 (harmonic analysis, extension technique), Frank & Seiringer JFA 08 (ground state representation formula): $a = 1$, $\tau = p$, $\alpha = 0$ and $\gamma = -s$.
  - Abdellaoui & Bentifour JFA 17 (Picone's inequality): $a = 1$, $\tau = pd/(d - sp)$, $-(d - sp)/p < \alpha = \gamma < 0$, and $1 < p < d/s$.
  - Sharp constant and the remainder are considered by Frank & Seiringer and Abdellaoui & Bentifour.
- Notation:

$$|u|^p_{W^{s,p,\alpha}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|x|^{\frac{\alpha p}{2}} |y|^{\frac{\alpha p}{2}} |u(x) - u(y)|^p}{|x-y|^{d+sp}} \, dx \, dy.$$

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Statement of the results

Let $p > 1$, $q \geqslant 1$, $\tau > 0$, $0 < s < 1$, $0 < a \leqslant 1$, $\alpha, \beta, \gamma \in \mathbb{R}$ be s.t.

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\left(\frac{1}{p} + \frac{\alpha - s}{d}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{d}\right), \qquad (2.1)$$

and, with $\gamma = a\sigma + (1-a)\beta$,

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Theorem (Ng. & Squassina JFA 17)

Assume (2.1) *and* (2.2). *If* $1/\tau + \gamma/d > 0$ *then*

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- Starting point: Sobolev's and Poincare's inequalities; NO integration by parts.
- Inspiration: harmonic analysis; however, instead of localizing frequency, we localize space variables.
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A new proof of Hardy's inequality

Recall Hardy's inequality, for $1 \leqslant p < d$,

$$\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p}\,dx \leqslant C \int_{\mathbb{R}^d} |\nabla u|^p\,dx \quad \text{for } u \in C_c^1(\mathbb{R}^d).$$

Here is the proof. Set

$$\mathbb{C}_k := \{x \in \mathbb{R}^d : 2^k \leqslant |x| < 2^{k+1}\}.$$

By Poincare's inequality, we have

$$⨍_{\mathbb{C}_k} \left| u - ⨍_{\mathbb{C}_k} u \right|^p dx \leqslant C2^{kp} ⨍_{\mathbb{C}_k} |\nabla u|^p\,dx,$$

which yields

$$\int_{\mathbb{C}_k} \frac{|u|^p}{|x|^p}\,dx \sim 2^{-kp} \int_{\mathbb{C}_k} |u|^p \leqslant C \int_{\mathbb{C}_k} |\nabla u|^p\,dx + C2^{k(d-p)} \left| ⨍_{\mathbb{C}_k} u \right|^p.$$

By Poincare's inequality, we also have

$$\left| ⨍_{\mathbb{C}_{k+1}} u - ⨍_{\mathbb{C}_k} u \right|^p dx \leqslant C2^{kp} ⨍_{\mathbb{C}_k \cup \mathbb{C}_{k+1}} |\nabla u|^p\,dx.$$

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A new proof of Hardy's inequality

Recall Hardy's inequality, for $1 \leqslant p < d$,

$$\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p}\, dx \leqslant C \int_{\mathbb{R}^d} |\nabla u|^p \, dx \quad \text{for } u \in C_c^1(\mathbb{R}^d).$$

Here is the proof. Set

$$\mathbb{C}_k := \left\{ x \in \mathbb{R}^d : 2^k \leqslant |x| < 2^{k+1} \right\}.$$

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for $c > 2^{p-d}$, in particular for $c = (2^{p-d}+1)/2 < 1$. This implies

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The proof is complete. □

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The proof is complete. $\square$

Proof of CKN's inequality for the main case

Let $p > 1$, $q \geqslant 1$, $\tau > 0$, $0 < s < 1$, $0 < a \leqslant 1$, α, β, $\gamma \in \mathbb{R}$ be s.t.

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\left(\frac{1}{p} + \frac{\alpha - s}{d}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{d}\right), \tag{2.3}$$

and, with $\gamma = a\sigma + (1-a)\beta$,

$$0 \leqslant \alpha - \sigma \leqslant s \tag{2.4}$$

Theorem

Assume (2.3) *and* (2.4). *We have, for* $1/\tau + \gamma/d > 0$,

$$\||x|^{\gamma} u\|_{L^{\tau}(\mathbb{R}^d)} \leqslant C |u|^{a}_{W^{s,p,\alpha}(\mathbb{R}^d)} \||x|^{\beta} u\|^{(1-a)}_{L^{q}(\mathbb{R}^d)} \quad \forall u \in C^1_c(\mathbb{R}^d),$$

$$|u|^p_{W^{s,p,\alpha}(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|x|^{\frac{\alpha p}{2}} |y|^{\frac{\alpha p}{2}} |u(x) - u(y)|^p}{|x-y|^{d+sp}} \, dx \, dy.$$

Lemma (Gagliardo-Nirenberg's inequality)

Let $d \geqslant 1$, $0 < s < 1$, $p > 1$, $q \geqslant 1$, $\tau > 0$, *and* $0 < a \leqslant 1$ *be s.t.*

$$\frac{1}{\tau} = a\left(\frac{1}{p} - \frac{s}{d}\right) + (1-a)\frac{1}{q}.$$

We have

$$\|u\|_{L^\tau(\mathbb{R}^d)} \leqslant C|u|^a_{W^{s,p}(\mathbb{R}^d)}\|u\|^{1-a}_{L^q(\mathbb{R}^d)} \qquad \text{for } u \in C^1_c(\mathbb{R}^d).$$

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Lemma

Let $d \geqslant 1$, $p > 1$, $0 < s < 1$, $q \geqslant 1$, $\tau > 0$, *and* $0 < a \leqslant 1$ *be s.t.*

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Let $\lambda > 0$, $0 < r < R$, *and set* $D := \left\{x \in \mathbb{R}^d : \lambda r < |x| < \lambda R\right\}$. *Then,* $\forall u \in C^1(\bar{D})$,

$$\left(⨍_D \left|u - ⨍_D u\right|^\tau dx\right)^{1/\tau} \leqslant C\left(\lambda^{sp-d}|u|^p_{W^{s,p}(D)}\right)^{a/p}\left(⨍_D |u|^q\,dx\right)^{(1-a)/q}.$$

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Proof. By scaling, one can assume that $\lambda = 1$. Let $0 < s' \leqslant s$ and $\tau' \geqslant \tau$ be such that

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From the previous lemma, we derive that

$$\Big\| u - \fint_D u \Big\|_{L^{\tau'}(D)} \leqslant C\, |u|^{a}_{W^{s',p}(D)}\, \|u\|^{1-a}_{L^q(D)}.$$

Since $|u|_{W^{s',p}(D)} \leqslant C\,|u|_{W^{s,p}(D)}$, $\|u\|_{L^{\tau}(D)} \leqslant C\|u\|_{L^{\tau'}(D)}$, the conclusion follows. □

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Recall $\mathbb{C}_k := \left\{x \in \mathbb{R}^d : 2^k \leqslant |x| < 2^{k+1}\right\}$. We have, $\Big(\alpha - \sigma \geqslant 0 + \text{balance law}\Big)$,

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It follows that

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Using the balance law, we derive that

$$\int_{\mathbb{C}_k} |x|^{\gamma\tau}|u|^{\tau} dx \leqslant C|u|^{a\tau}_{W^{s,p,\alpha}(\mathbb{C}_k)} \left\||x|^{\beta}u\right\|^{(1-a)\tau}_{L^q(\mathbb{C}_k)} + C2^{(\gamma\tau+d)k}\left|\fint_{\mathbb{C}_k} u\right|^{\tau}, \tag{2.5}$$

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we obtain, with $c = 2/(1+2^{\gamma\tau+d}) < 1$,

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since $a/p + (1-a)/q \geqslant 1/\tau$ thanks to the fact $\alpha - \sigma \leqslant s$. □

Combining (2.5) and (2.6) yields

$$\int_{\mathbb{R}^d} |x|^{\gamma\tau}|u|^\tau \, dx \leqslant C \sum_k |u|^{a\tau}_{W^{s,p,\alpha}(\mathbb{C}_k \cup \mathbb{C}_{k+1})} \| |x|^\beta u \|^{(1-a)\tau}_{L^q(\mathbb{C}_k \cup \mathbb{C}_{k+1})}.$$

One has, for $s \geqslant 0$, $t \geqslant 0$ with $s + t \geqslant 1$, and for $x_k \geqslant 0$ and $y_k \geqslant 0$,

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On the limiting case

Theorem (Ng. & Squassina JFA 17)

Let $d \geqslant 1$, $p > 1$, $0 < s < 1$, $q \geqslant 1$, $\tau > 1$, $0 < a \leqslant 1$, $\alpha, \beta, \gamma \in \mathbb{R}$ *be such that* (2.3) *holds and*

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Let $u \in C_c^1(\mathbb{R}^d)$, *and* $0 < r < R$. *We have*

i) *if* $1/\tau + \gamma/d = 0$ *and* $\operatorname{supp} u \subset B_R$, *then*

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Section 3: New perspectives of Hardy's and Caffarelli, Kohn, Nirenberg's inequalities

Motivations

Define, for $d \geqslant 1$ and $p \geqslant 1$,

$$I_\delta(u) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}_{|u(x)-u(y)|>\delta} \frac{\delta^p}{|x-y|^{d+p}}\,dx\,dy \quad \forall u \in L^p(\mathbb{R}^d).$$

1. I_δ is related to the semi-norm of $W^{s,q}(\mathbb{R}^d)$:

$$|u|^q_{W^{s,q}(\mathbb{R}^d)} := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^q}{|x-y|^{d+sq}}\,dx\,dy.$$

2. I_δ appears in an estimate for the topological degree due to Bourgain, Brezis, & Ng., CRAS 05, and Ng. JAM 07

$$|\deg u| \leqslant C_d \int_{\mathbb{S}^d}\int_{\mathbb{S}^d}_{|u(x)-u(y)|\geqslant \ell_d} \frac{1}{|x-y|^{2d}}\,dx\,dy, \quad \forall u \in C(\mathbb{S}^d, \mathbb{S}^d),$$

where $\ell_d = \sqrt{2 + \frac{2}{d+1}}$.

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Let $d \geqslant 1$, $1 < p < +\infty$ *and* $u \in L^p(\mathbb{R}^d)$. *Then*

1. $$I_\delta(u) \leqslant C_{d,p} \int_{\mathbb{R}^d} |\nabla u|^p \quad \forall u \in W^{1,p}(\mathbb{R}^d).$$

2. $$\lim_{\delta \to 0} I_\delta(u) = K_{d,p} \int_{\mathbb{R}^d} |\nabla u|^p \quad \forall u \in W^{1,p}(\mathbb{R}^d).$$

3. *If*
$$\liminf_{\delta \to 0} I_\delta(u) < +\infty,$$
then $u \in W^{1,p}(\mathbb{R}^d)$.

Related works: Bourgain, Brezis, & Mironescu 01, Davila 02.

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Theorem (Ng. CVPDE, 11)

Let $p \geqslant 1$, Q *be a cube or a ball of* $\mathbb{R}^d$. *Then* $\exists C > 0$ *s.t. for all* $\delta > 0$:

$$\iint_{Q^2} |u(x) - u(y)|^p \, dx \, dy \leqslant C\left(|Q|^{\frac{d+p}{d}} \iint_{\substack{Q^2 \\ |u(x)-u(y)|>\delta}} \frac{\delta^p}{|x-y|^{d+p}} \, dx \, dy + \delta^p |Q|^2\right).$$

A variant of Sobolev's inequality also holds for I_δ for $1 < p < d$.

Question: How's about Hardy's and Caffarelli, Kohn, Nirenberg's inequalities?

Variants of Hardy's inequalities

Theorem (Ng. & Squassina JAM, to appear)

Let $d \geqslant 1$, $p \geqslant 1$, $0 < r < R$, *and* $u \in L^p(\mathbb{R}^d)$. *We have*

i) *if* $1 \leqslant p < d$ *and* $\operatorname{supp} u \subset B_R$, *then*

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^p} dx \leqslant C \left(I_\delta(u) + R^{d-p} \delta^p \right),$$

ii) *if* $p > d$ *and* $\operatorname{supp} u \subset \mathbb{R}^d \setminus B_r$, *then*

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Similar results hold for the case $p = d$.

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- Variants for $p = d \geqslant 2$ and also for $p = d = 1$ hold.
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Thank you for your attention!