L^2-asymptotic stability of mild solutions to Navier-Stokes system in $\mathbb{R}^3$

The classical theory of viscous incompressible fluid flow is governed by the celebrated Navier-Stokes equations

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = F, \quad (x,t) \in \mathbb{R}^3 \times (0,\infty),$$
$$\text{div } u = 0,$$
$$u(x,0) = u_0(x),$$

where $u = \big(u_1, u_2, u_3\big)$ is the velocity of the fluid and $p = p(x,t)$ is the scalar pressure.

There are two main approaches for the construction of solutions to the Navier-Stokes equations. In the 1934 pioneering paper by Leray, weak solutions are obtained for all divergence free initial data $u_0 \in L^2(\mathbb{R}^3)^3$ and $F = 0$. These solutions satisfy the above equations in the distributional sense and fulfill a suitable energy inequality. The second approach leads to mild solutions. These solutions are given by an integral formulation using the Duhamel principle and they are obtained by means of the Banach contraction principle.

I will describe a link between these two approaches. More precisely, assume that $V = V(x,t)$ is a global-in-time mild solution of the Navier-Stokes equations, which is small in a certain scaling invariant space. Usually, such solutions do not belong to L^2, however, we establish their asymptotic stability under arbitrarily large initial L^2-perturbations. That is we show first that the initial value problem with data $V(x,0)$ perturbed by an arbitrarily large divergence free L^2-vector field has a global-in-time weak solution in the sense of Leray. WE then show that this weak solution converges as $t \to \infty$ in the energy L^2-norm towards the mild solution $V = V(x,t)$. In other words, sufficiently small mild solutions of the Navier-Stokes equations are, in some sense, asymptotically stable weak solutions under all divergence free initial perturbations from $L^2(\mathbb{R}^3)^3$.