Estimation of conditional risk measures for elliptic distributions.

Véronique Maume-Deschamps
Joint work with Didier Rullière and Antoine Usseglio-Carleve.

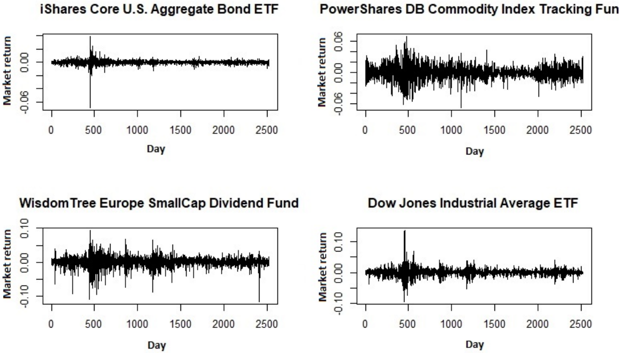
Lisbon, October 17th, 2018

Plan

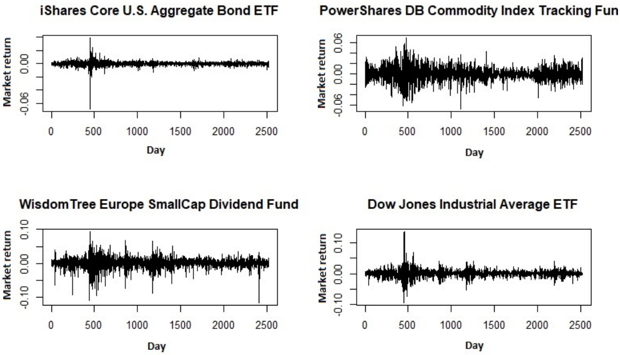
Plan

A financial example

Consider four assets: iShares Core U.S. Aggregate Bond ETF, PowerShares DB Commodity Index Tracking Fund, WisdomTree Europe SmallCap Dividend Fund and SPDR Dow Jones Industrial Average ETF.



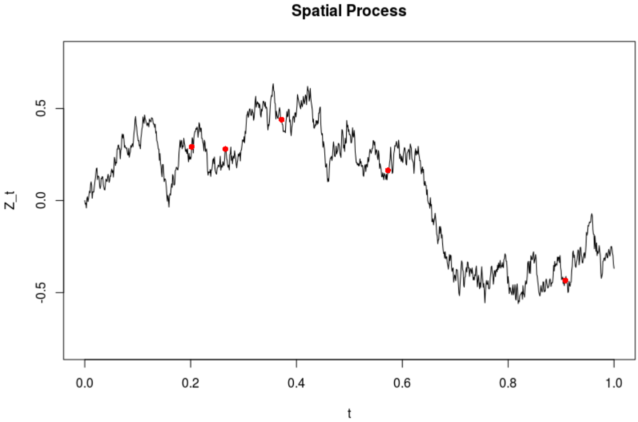
A financial example



How to use the knowledge of the 4 variables in order to estimate risk measures for WisdomTree Japan Hedged Equity Fund.

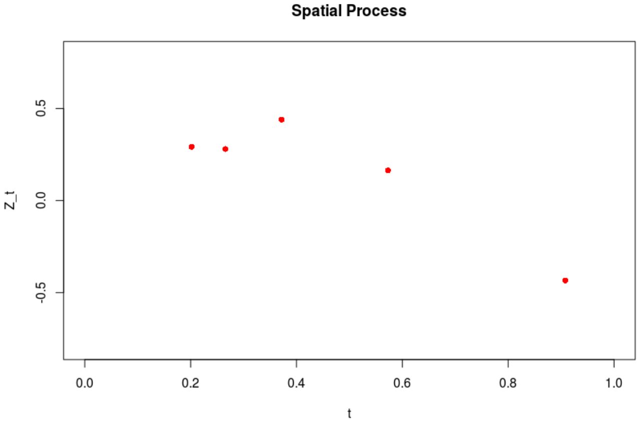
A general temporal problematic

Consider $(Z_t)_{t \in T}$ a random field observed at n points $t_1, \ldots, t_n$.



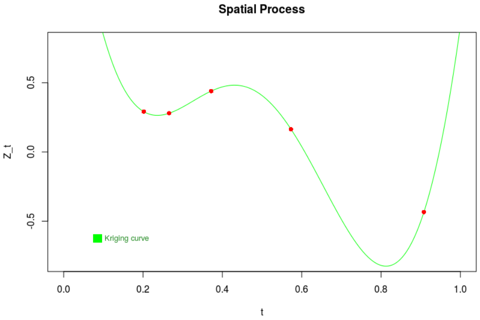
A general temporal problematic

Consider $(Z_t)_{t \in T}$ a random field observed at n points $t_1, \ldots, t_n$.



A general temporal problematic

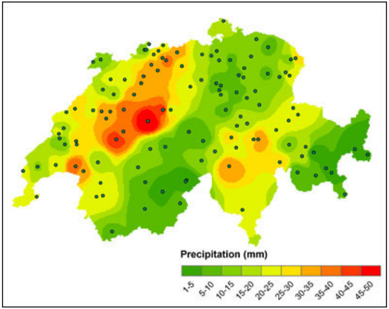
Consider $(Z_t)_{t \in T}$ a random field observed at n points $t_1, \ldots, t_n$. Using kriging ([Krige, 1951], [Matheron, 1963]) lead to estimations of the conditional mean, reliable for Gaussian processes.



How to estimate conditionnal risk measures (quantiles, expectiles, TVaR), for non necessarily gaussian random fields ?

A spatial example

Rainfall amounts measured at some locations $\mathbf{X} \in \mathbb{R}$, one unknown value $Y \in \mathbb{R}$.



Source: Geographic Information Technology Training Alliance.

How to estimate risk measures related to Y knowing $\mathbf{X}$?

Our purpose

- Focus on elliptic distributions,
- Consider quantile / expectile regression
- Propose an alternative estimation, for high level risk measures.

Plan

Definition

Definition (Representation Theorem [Cambanis et al., 1981])

A $\mathbb{R}^d$ random vector X is elliptical with parameters μ and Σ ($X \sim \mathcal{E}_d(\mu, \Sigma)$) if it writes:

$$X \overset{d}{=} \mu + R\Lambda U^{(d)}$$

where $\Lambda\Lambda^T = \Sigma$, $U^{(d)}$ is a $d-$dimensional random vector uniformly distributed on $\mathcal{S}^{d-1}$, R is a non-negative random variable, independent of $U^{(d)}$.

- R is called the radius of X.

Consistent Elliptic distributions

Definition (Kano, 1994 [Kano, 1994])

$X \sim \mathcal{E}_d(\mu, \Sigma, R)$ has the consistency property if:

$$R \stackrel{d}{=} \chi_d \epsilon$$

where ϵ is a positive random variable unrelated to d and independent of χ_d. We shall say that X is a consistent $(R, d)-$elliptical random vector with parameters μ and Σ.

Remark: the above definition rewrites as

$$X \stackrel{d}{=} \mu + \epsilon \mathcal{N}(0, \Sigma)$$

with ϵ independent of the normal vector. This means that finally, an elliptical distribution is a normal distribution with random variance $\epsilon^2 \Sigma$).

Properties of elliptic distributions

- Sub-vectors of elliptical vectors are elliptical, more precisely, Let $X = (X_1, X_2)$ be a consistent $(R, d)-$elliptical random vector with parameters μ and Σ. X_1 and X_2 are d_1 and d_2-dimensional subvectors of X. Let us write Σ :
$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$
Then X_1 and X_2 are respectively $(R, d_1)-$ and $(R, d_2)-$elliptical with parameters μ_1, Σ_{11} and μ_2, Σ_{22}, respectively.
- Conditional distributions of elliptical vectors are also elliptical.

Properties of elliptic distributions

- Sub-vectors of elliptical vectors are elliptical,
- Conditional distributions of elliptical vectors are also elliptical. More precisely,
  $X_2|(X_1 = x_1)$ is still elliptical, with radius R^* given by:

$$R^* \stackrel{d}{=} R\sqrt{1-\beta}\left| \left(R\sqrt{\beta}U^{(d)} = C_{11}^{-1}(x_1 - \mu_1) \right) \right.,$$

where C_{11} is the Cholesky root of Σ_{11}, and $\beta \sim Beta(\frac{d_1}{2}, \frac{d_2}{2})$.

Conditional measures.

Properties of elliptic distributions

- Sub-vectors of elliptical vectors are elliptical,
- Conditional distributions of elliptical vectors are also elliptical.

Definition

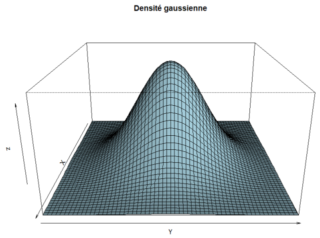
A random field $(Z_t)_{t \in T}$, $t \in \mathbb{R}$ is ϵ-elliptical if

$$Z_t = \mu + \epsilon X_t$$

where $(X_t)_{t \in T}$ is a gaussian field.

Examples

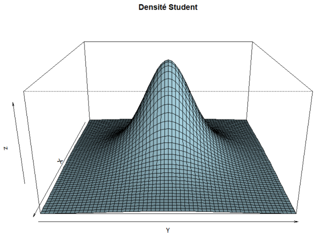
- Normal distributions: $\epsilon = 1$.



- Student distributions: with ν degrees of freedom: $\epsilon \overset{d}{=} \sqrt{\frac{\nu}{\chi^2_d}}$.
- Slash distributions: $\epsilon \overset{d}{=} \mathcal{P}(1, a)$.
- Laplace distibution: $\epsilon \overset{d}{=} \sqrt{\mathcal{E}(\lambda)}$.
- Many other.

Examples

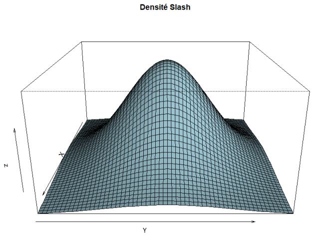
- Normal distributions: $\epsilon = 1$.
- Student distributions: with ν degrees of freedom: $\epsilon \overset{d}{=} \sqrt{\frac{\nu}{\chi^2_d}}$.



- Slash distributions: $\epsilon \overset{d}{=} \mathcal{P}(1, a)$.
- Laplace distibution: $\epsilon \overset{d}{=} \sqrt{\mathcal{E}(\lambda)}$.
- Many other.

Examples

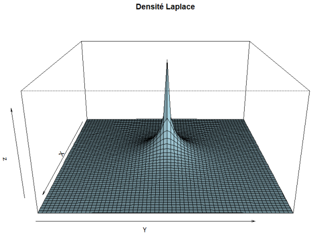
- Normal distributions: $\epsilon = 1$.
- Student distributions: with ν degrees of freedom: $\epsilon \stackrel{d}{=} \sqrt{\frac{\nu}{\chi^2_d}}$.
- Slash distributions: $\epsilon \stackrel{d}{=} \mathcal{P}(1, a)$.



- Laplace distibution: $\epsilon \stackrel{d}{=} \sqrt{\mathcal{E}(\lambda)}$.
- Many other.

Examples

- Normal distributions: $\epsilon = 1$.
- Student distributions: with ν degrees of freedom: $\epsilon \overset{d}{=} \sqrt{\frac{\nu}{\chi_d^2}}$.
- Slash distributions: $\epsilon \overset{d}{=} \mathcal{P}(1, a)$.
- Laplace distibution: $\epsilon \overset{d}{=} \sqrt{\mathcal{E}(\lambda)}$.



- Many other.

Quantiles and expectiles

Quantile is widely used as a risk measure (VaR). Recall: for X a random variable (a risk) with distribution function F_X,

- $q_\alpha(X) = \text{VaR}_\alpha(X) = \inf\{t \;/\; F_X(t) \geq \alpha\} = F_X^{-1}(\alpha)$,
- RiskMetrics popularized the use of VaR as a risk measure (1994).
- Basel Committee : Internal approach to capital management using VaR (1996),
- formalisation of coherent risk measures [Artzner et al., 1999], VaR is not coherent (because not sub-additive in general). VaR is a risk threshold but does not gives information on the risk above the threshold.

Quantiles and expectiles

TVaR / Expected shortfall: (for continuous distributions) - see e.g. [Artzner et al., 1999] -

$$\begin{aligned}\text{TVaR}_\alpha(X) &= \text{ES}_\alpha(X) = \mathbb{E}(X|X > \text{VaR}_\alpha(X)) \\ &= \frac{1}{1-\alpha}\int_\alpha^1 q_u(X)du.\end{aligned}$$

TVaR is a coherent risk measure.
The notion of ellicitability has taken some importance these last years [Bellini and Di Bernardino, 2017] in risk management.

Quantiles and expectiles

A statistic T is ellicitable ([Gneiting, 2011, Ziegel, 2016]) if it can be written as a minimizer of a scoring function denoted $s : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

$$T(X) = \underset{x \in \mathbb{R}}{\operatorname{argmin}} \, \mathbb{E}[s(X, x)].$$

Interest:

- Ability to compare different statistical methods
- Makes backtesting procedures easier
- Allow the use of stochastic approximation tools.

Quantiles and expectiles

VaR is ellicitable: take

$$s(x,X) = [\alpha(X-x)_+ + (1-\alpha)(X-x)_-] .$$

TVaR is not ellicitable.

Define the expectile as ([Newey and Powell, 1987]):

$$e_\alpha(X) = \underset{x\in\mathbb{R}}{\text{argmin}}\ \mathbb{E}\left[\alpha(X-x)_+^2 + (1-\alpha)(X-x)_-^2\right] .$$

Expectiles are:

- Ellicitable (of course),
- Coherent,
- the only risk measure which is both coherent and ellicitable ([Bellini and Bignozzi, 2015]).

Expectiles satisfy

$$\alpha\mathbb{E}\left[(X-e_\alpha(X))_+\right] = (1-\alpha)\mathbb{E}\left[(X-e_\alpha(X))_-\right] .$$

and thus take into account the risk over the threshold.

Conditional risk measures

We are interested in conditional risk measures for elliptical distributions. Focus on quantiles (results also for TVaR and expectiles).

Let $X = (X_1, X_2)$ be a $(R, N+1)-$elliptical random vector with parameters μ and Σ. $X_1 \in \mathbb{R}^N$ and $X_2 \in \mathbb{R}$.

Calculate / estimate

$$q_\alpha(X_2|X_1 = x_1), \; e_\alpha(X_2|X_1 = x_1).$$

Conditional risk measures

Conditional distribution.

Proposition

Let $X = (X_1, X_2)$ *a* $(R, N+1)-$*elliptical random vector with parameters* μ *and* Σ. *Write*

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

Then

$$q_\alpha (X_2 | X_1 = x_1) = \mu_{2|1} + \sqrt{\Sigma_{2|1}} \Phi_{R^*}^{-1}(\alpha)$$

with $\begin{cases} \mu_{2|1} = & \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1) \\ \Sigma_{2|1} = & \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{cases}$ *and for a random variable* Y, Φ_Y *is the distribution function of* $\frac{Y - \mathbb{E}(Y)}{\sigma_Y}$.

Conditional risk measures

$$q_\alpha\left(X_2|X_1 = x_1\right) = \mu_{2|1} + \sqrt{\Sigma_{2|1}}\Phi_{R^*}^{-1}(\alpha)$$

Problem: the distribution of R^* is hardly accessible.

- Explicit formulas for Gaussian and Student distributions,
- First idea: quantiles regression,
- Extreme estimations (ie $\alpha \to 1$).

Plan

Quantile Regression

Approximate the conditional quantile by:
([Koenker and Bassett, 1978])

$$\hat{q}_\alpha(X_2|X_1 = x_1) = {\beta^*}^T x_1 + \beta_0^*$$

where β^* and β_0^* are the solutions of the following minimization problem:

$$(\beta^*, \beta_0^*) = \underset{\beta \in \mathbb{R}^N, \beta_0 \in \mathbb{R}}{\arg\min} \; \mathbb{E}[s_\alpha(X_2 - \beta^T X_1 - \beta_0)]$$

with the scoring function s_α:

$$s_\alpha(x) = (\alpha - 1)x\mathbf{1}_{\{x<0\}} + \alpha x\mathbf{1}_{\{x>0\}}$$

Quantile Regression for elliptic distributions

Theorem

Let $X = (X_1, X_2)$ be an elliptical distribution, the optimal β^* is given by :

$$\beta^* = \Sigma_{11}^{-1}\Sigma_{12}$$

The Quantile Regression Predictor with level $\alpha \in [0,1]$ is given by:

$$\hat{q}_\alpha(X_2|X_1 = x_1) = \mu_{2|1} + \sqrt{\Sigma_{2|1}}\Phi_R^{-1}(\alpha)$$

It satisfies

$$\hat{q}_\alpha(X_2|X_1) \sim \mathcal{E}_1\left(\mu_2 + \sigma_{2|1}\Phi_R^{-1}(\alpha), \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}, R\right)$$

Recall the theoretical conditional quantile:

$$q_\alpha\left(X_2|X_1 = x_1\right) = \mu_{2|1} + \sqrt{\Sigma_{2|1}}\Phi_{R^*}^{-1}(\alpha)$$

How good is the quantile regression?

Gaussian case

$$\begin{cases} q_\alpha(X_2|X_1 = x_1) = & \mu_{2|1} + \sigma_{2|1}\Phi^{-1}(\alpha) \\ \hat{q}_\alpha(X_2|X_1 = x_1) = & \mu_{2|1} + \sigma_{2|1}\Phi^{-1}(\alpha) \end{cases}$$

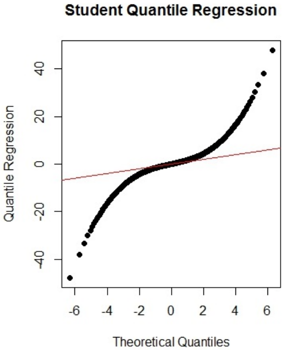
The Quantile Regression Predictor is exactly the conditional quantile.

How good is the quantile regression?

Student case

$$\begin{cases} q_\alpha(X_2|X_1 = x_1) = & \mu_{2|1} + \sigma_{2|1}\sqrt{\frac{\nu}{\nu+N}}\sqrt{1+\frac{1}{\nu}d_1}\Phi^{-1}_{\nu+N}(\alpha) \\ \hat{q}_\alpha(X_2|X_1 = x_1) = & \mu_{2|1} + \sigma_{2|1}\Phi^{-1}_{\nu}(\alpha) \end{cases}$$

The error may be huge, especially if the Mahalanobis distance $d_1 = (x_1 - \mu_1)^T \Sigma_{11}^{-1}(x_1 - \mu_1)$ is high. Below $N = 5$.



Conditional expectiles

Recall that the $\alpha-$expectile of X is defined as:

$$\arg\min_{x\in\mathbb{R}} \mathbb{E}\left[(1-\alpha)\,(x-X)_+^2 + \alpha\,(X-x)_+^2\right].$$

In the elliptical case, the expectile of level α is the solution of:

$$\Psi_R(x) = \Phi_R(x) + \frac{1}{x}\int_x^{+\infty} y c_1 g_1(y^2)dy = \frac{\alpha}{2\alpha-1}.$$

Theoretical conditional expectiles :

$$e_\alpha(X_2|X_1 = x_1) = \mu_{2|1} + \sigma_{2|1}\Psi_{R^*}^{-1}\left(\frac{\alpha}{2\alpha-1}\right).$$

Expectile regression

Approximate the conditional expectile by:
([Newey and Powell, 1987])

$$\hat{e}_\alpha(X_2|X_1 = x_1) = {\beta^*}^T x_1 + \beta_0^*$$

where β^* and β_0^* are the solutions of the minimization problem :

$$(\beta^*, \beta_0^*) = \underset{\beta\in\mathbb{R}^N, \beta_0\in\mathbb{R}}{\arg\min}\ \mathbb{E}[s_\alpha(X_2 - \beta^T X_1 - \beta_0)]$$

with the loss function s_α:

$$s_\alpha(x) = (1-\alpha)x^2\mathbf{1}_{\{x<0\}} + \alpha x^2\mathbf{1}_{\{x>0\}}$$

Expectile Regression for elliptical distributions

Theorem

Let $X = (X_1, X_2)$ be an elliptical distribution, the optimal β^* is given by :

$$\begin{cases} \beta^* = & \Sigma_{11}^{-1}\Sigma_{12} \\ \beta_0^* = & \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1 + \sigma_{2|1}\Psi_R^{-1}\left(\frac{\alpha}{2\alpha-1}\right) \end{cases}$$

The Expectile Regression Predictor $\alpha \in [0,1]$ is:

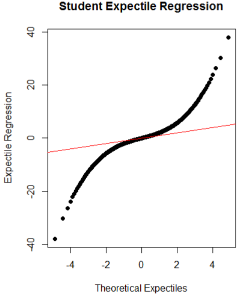
$$\hat{e}_\alpha(X_2|X_1 = x_1) = \mu_{2|1} + \sigma_{2|1}\Psi_R^{-1}\left(\frac{\alpha}{2\alpha - 1}\right).$$

Furthermore,

$$\hat{e}_\alpha(X_2|X_1) \sim \mathcal{E}_1\left(\mu_2 + \sigma_{2|1}\Psi_R^{-1}\left(\frac{\alpha}{2\alpha - 1}\right), \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}, R\right).$$

How good is the expectile regression?

Gaussian case The Quantile Regression Predictor is exactly the conditional quantile.
Student case Semi-explicit formula, Ψ_R^{-1} computed using MM algorithms. The error may be huge. Below $N = 5$.



Plan

Extreme approximations

In case $\alpha \sim 1$ or $\alpha \sim 0$, alternative methods have to be proposed.
In the case of quantile, we found an equivalent of $\Phi_{R^*}^{-1}(\alpha)$.
In the case of expectile, we found an equivalent of $\Psi_{R^*}^{-1}(\alpha)$.

Some asymptotic relationships

Theorem

Under some technical assumptions, their exist $0 < \ell < +\infty$ *and* $\eta \in \mathbb{R}$ *such that :*

$$\left[\Phi_R^{-1}\left(1 - \frac{1}{\frac{\ell}{1-\alpha} + 2(1-\ell)}\right)\right]^{\frac{1}{\eta}} \underset{\alpha \to 1}{\sim} \Phi_{R^*}^{-1}(\alpha)$$

Examples

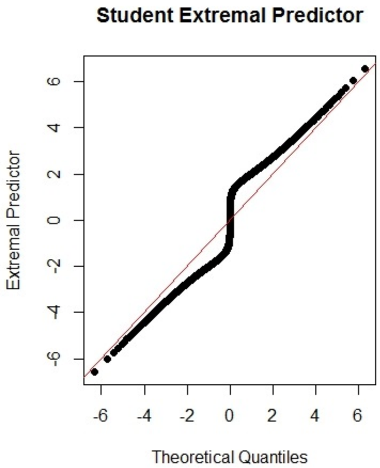
Property

The Gaussian, Student and Slash distributions satisfy the previous assumptions, with coefficients η and ℓ given in the table below.

Distribution	η	ℓ
Gaussian	1	1
Student, $\nu > 0$	$\frac{N}{\nu} + 1$	$\frac{\Gamma\left(\frac{\nu+N+1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+N}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)} \left(1 + \frac{q_1}{\nu}\right)^{\frac{N+\nu}{2}} \frac{\nu^{\frac{N}{2}+1}}{\nu+N}$
Slash, $a > 0$	$\frac{N}{a} + 1$	$\frac{\Gamma\left(\frac{N+1+a}{2}\right) q_1^{\frac{N+a}{2}}}{\Gamma\left(\frac{N+a}{2}\right)(N+a)\chi^2_{N+a}(q_1) 2^{\frac{a}{2}-1}\Gamma\left(\frac{1+a}{2}\right)}$

Examples

Extremal correction in the Student case



Some asymptotic relationships

Under some technical assumptions, their exist $0 < \ell < +\infty$ and $\gamma \in \mathbb{R}$ such that:

$$\begin{cases} \hat{\hat{e}}_{\alpha\uparrow}(X_2|X_1 = x_1) & \underset{\alpha\to 1}{\sim} e_\alpha(X_2|X_1 = x_1) \\ \hat{\hat{e}}_{\alpha\downarrow}(X_2|X_1 = x_1) & \underset{\alpha\to 0}{\sim} e_\alpha(X_2|X_1 = x_1) \end{cases}$$

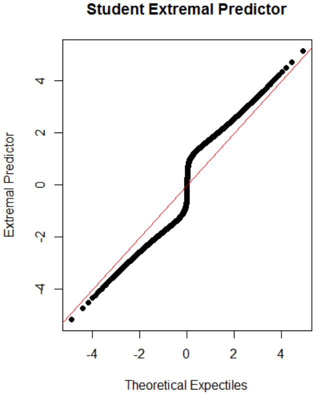
with

$$\begin{cases} \hat{\hat{e}}_{\alpha\uparrow}(X_2|X_1 = x_1) & = \mu_{2|1} + \sigma_{2|1}\left[\Psi_R^{-1}\left(1 - \frac{\alpha-1}{(2\alpha-1)\ell}\right)\right]^{\frac{1}{\gamma}} \\ \hat{\hat{e}}_{\alpha\downarrow}(X_2|X_1 = x_1) & = \mu_{2|1} - \sigma_{2|1}\left[\Psi_R^{-1}\left(1 - \frac{\alpha}{(2\alpha-1)\ell}\right)\right]^{\frac{1}{\gamma}} \end{cases}$$

Hypothesis satisfied for Gaussian, Student, Slash distributions.

Example

Extremal correction in the Student case

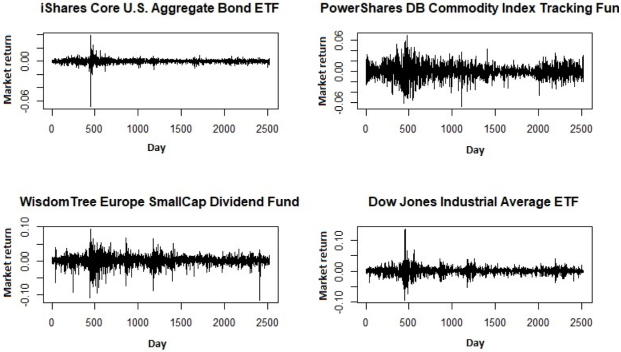


Estimations

Under additional assumptions (heavy tail + order two condition, estimations of the parameters ℓ, η, γ + asymptotic normality of the estimators ([Usseglio-Carleve, 2017]).

Plan

Financial example



[Usseglio-Carleve, 2017]. These four values are the first available every day ⇒ anticipate the behaviour of the return of WisdomTree Japan Hedged Equity Fund X_2.

Financial example

The sample size is 2520. The first 2519 days (from January 3, 2007 to December 5, 2016) = learning sample, and we focus on the 2520th day: $x_1 = (-0.0185\%, -0.4464\%, 0.9614\%, 0.1405\%)$.

Estimate quantiles / expectiles / TVaR of $X_2|X_1 = x_1$.

Financial example

The sample size is 2520. The first 2519 days (from January 3, 2007 to December 5, 2016) = learning sample, and we focus on the 2520th day: $x_1 = (-0.0185\%, -0.4464\%, 0.9614\%, 0.1405\%)$.

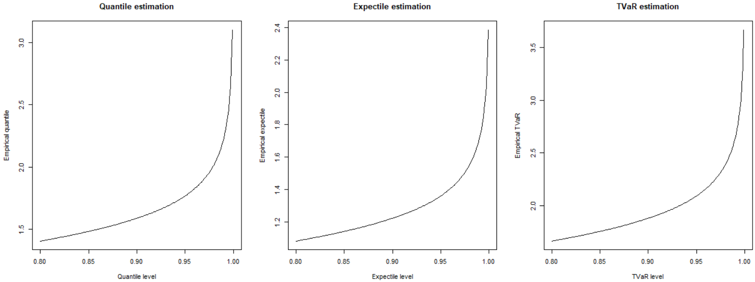
Estimate quantiles / expectiles / TVaR of $X_2 | X_1 = x_1$.

Data exploration:

- the daily returns can be considered as independent.
- the marginals seem symmetrical.
- the measured tail index is approximately the same for the marginals.

Could be assumed to be elliptical.

Financial example



E.g., for $\alpha = 0.999$, the estimated VaR is 3.1%, the observed value is 0.7141%.

Plan

Conclusion / perspectives

- Regression methods are not satisfactory for non gaussian distributions.
- Framework adapted to a large class of risk measures (L^p quantile, Haezendonck-Goovaerts risk measures).
- New technics needed in the high dimension case (N large), see Antoine Usseglio-Carleve (2019).
- More details in [Maume-Deschamps et al., 2017a, Maume-Deschamps et al., 2017b, Usseglio-Carleve, 2017].
- Mixed approaches for non central but non extreme risk levels?
- Non symetric distributions?

References I

Artzner, P., Delbaen, F., Eber, J., and Heath, D. (1999).
Coherent measures of risk.
Mathematical Finance, 9(3):203–228.

Bellini, F. and Bignozzi (2015).
On elicitable risk measures.
Quantitative Finance, 15(5):725â733.

Bellini, F. and Di Bernardino, E. (2017).
Risk management with expectiles.
The European Journal of Finance, 23(6):487â506.

Cambanis, S., Huang, S., and Simons, G. (1981).
On the theory of elliptically contoured distributions.
Journal of Multivariate Analysis, 11:368–385.

References II

Gneiting, T. (2011).
Making and evaluating point forecasts .
Journal of the American Statistical Association, 106(494):746â762.

Kano, Y. (1994).
Consistency property of elliptical probability density functions.
Journal of Multivariate Analysis, 51:139–147.

Koenker, R. and Bassett, G. J. (1978).
Regression quantiles.
Econometrica, 46(1):33–50.

References III

Krige, D. (1951).
A statistical approach to some basic mine valuation problems on the witwatersrand.
Journal of the Chemical, Metallurgical and Mining Society, 52:119–139.

Matheron, G. (1963).
Traité de géostatistique appliquée.
Bureau de recherches géologiques et minières (France).

Maume-Deschamps, V., Rullière, D., and Usseglio-Carleve, A. (2017a).
Quantile predictions for elliptical random fields.
Journal of Multivariate Analysis, 159:1 – 17.

References IV

Maume-Deschamps, V., Rullière, D., and Usseglio-Carleve, A. (2017b).
Spatial expectile predictions for elliptical random fields.
Methodology and Computing in Applied Probability, 20(2):643–671.

Newey, W. and Powell, J. (1987).
Asymmetric least squares estimation and testing.
Econometrica, (55):819–847.

Usseglio-Carleve, A. (2017).
Estimation of conditional extreme risk measures from heavy-tailed elliptical random vectors.
working paper.

References V

Ziegel, J. (2016).
Coherence and elicitability.
Mathematical Finance, 26(4):901–918.

Thank you

Obrigada pela vossa atenção.