AN ANALOGUE OF COBHAM’S THEOREM FOR FRACTALS

by

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Abstract. — We introduce the notion of $k$-self-similarity for compact subsets of $\mathbb{R}^n$ and show that it is a natural analogue of the notion of $k$-automatic subsets of integers. We show that various well-known fractals such as the triadic Cantor set, the Sierpinski carpet or the Menger sponge, turn out to be $k$-self-similar for some integers $k$. We then prove an analogue of Cobham’s theorem for compact sets of $\mathbb{R}$ that are self-similar with respect to two multiplicatively independent bases $k$ and $\ell$; namely, we show that $X$ is both a $k$- and an $\ell$-self-similar compact subset of $\mathbb{R}$ if and only if it is a finite union of closed intervals with rational endpoints.

1. Introduction

The notion of self-similarity is fundamental in the study of fractals. It can be motivated by looking at the usual triadic Cantor set $C$, which is the closed subset of $[0,1]$ consisting of all numbers whose ternary expansion does not contain any 1’s. We note that

$$C = \frac{1}{3} C \cup \left( \frac{1}{3} C + \frac{2}{3} \right).$$

The fact that $C$ is a disjoint union of a finite number of images of itself under affine transformations tells us that it is self-similar.

With this in mind, we define the notion of $k$-kernel for subsets of $[0,1]^d$. The $k$-kernel essentially looks at the possible sets one can obtain by taking the intersection of $X$ with certain cubes in $[0,1]^d$ with side length $1/k^a$ for some positive integer $a$ and then scaling by a factor of $k^a$.

Definition 1.1. — Given a subset $X \subseteq [0,1]^d$, we define the $k$-kernel to be the collection of distinct subsets of the form

$$\left\{ (k^a x_1 - b_1, \ldots, k^a x_d - b_d) \in [0,1]^d : (x_1, \ldots, x_d) \in X \cap \prod_{j=1}^d [b_j/k^a, (b_j + 1)/k^a] \right\},$$

where $a \geq 0$ and $0 \leq b_1, \ldots, b_d < k^a$ are integers.

We then define the notion of $k$-self-similarity in terms of $k$-kernels.

Definition 1.2. — A compact set $X \subseteq [0,1]^d$ is said to be $k$-self-similar if it has finite $k$-kernel.

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As we will see in Section 2, many famous examples of fractals are in fact $k$-self-similar sets for some $k$. For instance, the Cantor set is a 3-self-similar subset of $\mathbb{R}$, the Sierpinski carpet and the Reverend Back’s abbey floor are 3-self-similar subsets of $\mathbb{R}^2$, Pascal’s triangle modulo 2 is a 2-self-similar subset of $\mathbb{R}^2$, and the Menger sponge is a 3-self-similar subset of $\mathbb{R}^3$.

Our definition of self-similar compact subsets of $\mathbb{R}^n$ is given in terms of kernels. Actually, there are famous subsets of integers that can be defined in a similar way: *automatic or recognizable sets of integers*. We now briefly describe this analogy. We refer to the monograph [5] for a more formal and complete introduction to this topic.

Let $k \geq 2$ be a natural number. A set $N \subset \mathbb{N}$ is said to be $k$-automatic if there is a finite state machine that accepts as input the expansion of $n$ in base $k$ and outputs 1 if $n \in N$ and 0 otherwise. For example, the set of Thue–Morse integers $1, 2, 4, 7, 8, 11, 13, \ldots$ formed by the integers whose sum of binary digits is odd is 2-automatic. The associated automaton is given in Figure 1 below. It has two states. This automaton reads successively the binary digits of $n$ (starting say from the most significant digit and at the initial state $q_0$) and thus ends the reading either in state $q_0$ or in state $q_1$. The initial state $q_0$ gives the output 0, while $q_1$ gives the output 1.

![Figure 1. The finite state automaton recognizing the set of Thue–Morse integers.](image)

An equivalent formulation of $k$-automatic sets of integers is given in terms of $k$-kernels. Given a set $N \subseteq \mathbb{N}$, the $k$-kernel of $N$ is defined as the collection of distinct sets of the form

$$\{k^a n + b \mid n \in N, a \geq 0, 0 \leq b < k^a\}.$$ 

Then, we have the following characterization (see [5]): a set $N \subseteq \mathbb{N}$ is $k$-automatic if and only if its $k$-kernel is finite.

Another typical 2-automatic set of integers is given by the powers of 2: $1, 2, 4, 8, 16, \ldots$. Though these integers have very simple expansion in base 2, one can observe that this is not the case when writing them in base 3. One of the most important results in the theory of automatic sets formalizes this idea. Recall that two integers $k$ and $l$ larger than 1 are multiplicatively independent if $\log(k)/\log(l) \notin \mathbb{Q}$. Then, Cobham’s theorem says that only trivial sets of integers can be automatic with respect to two multiplicatively independent numbers [8]. More exactly, Cobham proved in 1969 the following result.

**Theorem 1.3.** — (Cobham) Let $k$ and $l$ be two multiplicatively independent integers. Then, a set $N \subseteq \mathbb{N}$ is both $k$- and $l$-automatic if and only if it is the union of a finite set and a finite number of arithmetic progressions.

Our main result is an analogue of Cobham’s theorem for $k$-self-similar compact subsets of $\mathbb{R}$.

**Theorem 1.4.** — Let $k$ and $\ell$ be two multiplicatively independent natural numbers. Then, a compact set $X \subseteq [0,1]$ is both $k$- and $\ell$-self-similar if and only if it is a finite union of closed intervals with rational endpoints.

Thus, the triadic Cantor set cannot be 2- or 7-self-similar. Theorem 1.4 is only concerned with one-dimensional sets, but we expect a similar picture in every dimension. More precisely, we suggest the following multidimensional version of Theorem 1.4.
Conjecture 1.5. — Let \( k \) and \( \ell \) be two multiplicatively independent natural numbers. Then, a compact set \( X \subseteq [0, 1]^d \) is both \( k \)- and \( \ell \)-self-similar if and only if it is a finite union of polyhedra whose vertices have rational coordinates.

Notice that Cobham’s theorem has been generalized to subsets of \( \mathbb{N}^d \) by Semenov [21]. Conjecture 1.5 can thus be thought of as an analogue of Semenov’s result.

To end this introduction, we mention that similar “independence principles” with respect to two multiplicatively independent integers are also expected in other contexts. This is a source of difficult questions arising from various fields. As an illustration, we quote below three famous open problems that rest on such principle. A long-standing question in dynamical systems is the so-called \( \times 2 \times 3 \) problem addressed by Furstenberg [12]: given \( k, l \geq 2 \) two multiplicatively independent integers, prove that the only Borel measures on \([0, 1]\) that are simultaneously ergodic for \( T_k(x) = kx \pmod{1} \) and \( T_l(x) = lx \pmod{1} \) are the Lebesgue measure and measures supported by those orbits that are periodic for both actions \( T_k \) and \( T_l \). With a number theoretic flavour, we recall another problem attributed to Mahler and Mendès France (see for instance [1]): given a binary sequence \( (a_n)_{n \geq 0} \in \{0, 1\}^\mathbb{N} \), prove that \[
\sum_{n \geq 0} \frac{a_n}{2^n} \quad \text{and} \quad \sum_{n \geq 0} \frac{a_n}{3^n}
\]
are both algebraic numbers only if both are rational numbers. The last problem we mention appeared in some work of Ramanujan (see [23]): prove that there is no non-zero real number \( x \) such that both \( 2^x \) and \( 3^x \) are integers. This corresponds to a particular instance of the four exponential conjecture, a famous open problem in transcendence theory.

The outline of this paper is as follows. In Section 2, we show how one can simply associate a fractal set with a finite automaton. Our approach is inspired by a recent work of Kedlaya concerning an extension of Christol’s theorem to Hahn’s power series [16]. These fractal sets are termed automatic fractals. We give several examples of famous fractals that turn out to be automatic and prove that automatic fractals all are self-similar sets. We then prove Theorem 1.4 in Section 3. We note that our proof of Theorem 1.4 does not rely on Cobham’s theorem. In Section 4, we investigate the link between the Hausdorff dimension of automatic fractals and the entropy of languages naturally associated with automatic fractals. We remark that some results of Mauldin and Williams [17] should allow one to compute the Hausdorff dimension of automatic fractals. We also observe that Hatmanis and Stearns [15] and Barbé and von Haeseler [6] previously considered similar fractals in the framework of automata.

2. Automatic fractals

Finite automata are devices that accept finite words as input. This can be naturally used to recognize sets of integers since those numbers correspond to finite words when representing them in an integer base. In contrast, most real numbers have infinite expansions in integer bases and are thus related to infinite words. Hence, it is not clear how to define a notion of sets of real numbers recognized by finite automata. In this section, we show how one can simply associate fractal sets with finite automata, called automatic fractals. We give several examples of famous automatic fractals and prove that a \( k \)-automatic fractal is always a \( k \)-self-similar set.

2.1. Automatic sets revisited. — Kedlaya [16] introduced the notion of automaticity for subsets of \( k \)-adic rationals numbers, by adding a radix point to the input set for a finite state automaton. The key point is that \( k \)-adic rational are exactly those real numbers with a finite expansion in base \( k \). Such real numbers are represented by finite words and can thus be read by a finite \( k \)-automaton. This is the way we will extend beyond the integers.
Let \( k > 1 \) be a positive integer and \( \Sigma_k := \{0, 1, \ldots, k-1\} \). Given a natural number \( n \) and a positive integer \( k \geq 2 \), we let \([n]_k\) denote the base-\( k \)-expansion of \( n \) and we let \( S_k \) denote the set of nonnegative \( k \)-adic rationals; i.e.,

\[
S_k = \{ a/k^b \mid a, b \in \mathbb{Z}, \ a \geq 0 \}.
\]

Then, a \( k \)-adic rational number has a finite base-\( k \) expansion of the form \([n]_k \cdot [m]_k\), where \( \cdot \) is the radix point.

We set

\[
\Sigma_k' = \{0, 1, \ldots, k-1, \cdot\}
\]

and we denote by \( \mathcal{L}(k) \) the language on the alphabet \( \Sigma_k' \) consisting of all words on \( \Sigma_k' \) with exactly one occurrence of the letter \( \cdot \) (the radix point) and whose first and last letters are not equal to 0. This is a regular language \([16, \text{Lemma 2.3.3}]\). We note that there is a bijection \([ \cdot ]_k: \mathcal{L}(k) \rightarrow S_k \) given by

\[
s_1 \cdots s_{i-1} \cdot s_{i+1} \cdots s_n \in \mathcal{L}(k) \mapsto \sum_{j=1}^{i-1} s_j k^{i-j} + \sum_{j=i+1}^n s_j k^{i-j},
\]

where \( s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n \in \{0, 1, \ldots, k-1\} \).

So, for example, we have \([110,32]_4 = [20,875]_{10} = 167/8 \). We also note that the fact that we exclude strings whose initial and terminal letters are 0 means we have the awkward looking expression \([\cdot]_k = 0 \).

We now recall the definition of a \( k \)-automatic subset of \( S_k \).

**Definition 2.1.** — We say that a subset \( S \) of \( S_k \) is \( k \)-**automatic** if there is a finite state automaton which takes words on \( \Sigma_k' \) as input and has the property that a word \( W \in \mathcal{L}_k \) is accepted by the automaton if and only if \([W]_k \in S \).

More generally, we can define automatic subsets of \( S_k^d \) as follows. We expand the alphabet \( \Sigma_k' \) to an alphabet \( \Sigma_k'' = \Sigma_k' \cup \{\cdot\} \). Then, an element of \( S_k^d \) is simply a \( d \)-tuple of elements of \( S_k \). With this in mind, we construct a regular language \( \mathcal{L}_d(k) \subseteq (\Sigma_k'')^* \) which is simply the collection of words of the form

\[
a_1, a_2, \ldots, a_d
\]

where \( a_1, a_2, \ldots, a_d \in \mathcal{L}(k) \subseteq \Sigma'' \). Then, there is a obvious way to extend the map \([\cdot]_k\) to a bijection \([ \cdot ]_k: \mathcal{L}_d(k) \rightarrow S_k^d \); namely,

\[
[a_1, \ldots, a_d]_k := ([a_1]_k, \ldots, [a_d]_k).
\]

**Definition 2.2.** — We say that a subset \( S \) of \( S_k^d \) is \( k \)-**automatic** if there is a finite state automaton which takes words on \( \mathcal{L}_d(k) \) as input and has the property that a word \( W \in \mathcal{L}_d(k) \) is accepted by the automaton if and only if \([W]_k \in S \).

We will also use the notion of a \( k \)-automatic function from \( S_k^d \) to a finite set.

**Definition 2.3.** — Let \( \Delta \) be a finite set. We say that a function \( f: S_k^d \rightarrow \Delta \) is \( k \)-**automatic** if there is a finite state automaton which takes words on \( \mathcal{L}_d(k) \) as input and has the property that reading a word \( W \in \mathcal{L}_d(k) \), the automaton outputs \( f([W]_k) \).

We make the following remark, which is a translation in our context of Theorem 6.6.2 of [5].

**Remark 2.4.** — If a subset \( S \) of \( S_k^d \) is \( k \)-automatic, then there are only finitely many distinct subsets of \( S_k^d \) of the form

\[
\left\{(x_1, \ldots, x_d) \in [0,1]^d \mid (x_1/k^a + b_1/k^a, \ldots, x_d/k^a + b_d/k^a) \in S\right\},
\]

where \( a \geq 0 \) and \( 0 \leq b_1, \ldots, b_d < k^a \).
The set $S^d_k$ is countable and hence automatic sets of $S^d_k$ are not very interesting in the framework of fractals. We can actually overcome this deficiency by considering the closure of automatic sets. As an example, we look at the Cantor set $C$. Since it can be described as the set of all $x \in [0, 1]$ which have a ternary expansion that does not have any 1’s, the set $C \cap S_3$ is just the set of 3-adic rationals in $[0, 1]$ whose ternary expansion does not contain any 1’s. We note that this set is a 3-automatic subset of $S_3$, as it can be described as being all numbers in $S_3$ that begin with a radix point and do not contain a 1. We also observe that $C \cap S_3$ is dense in $C$ and hence the Cantor set can be realized as the closure of a 3-automatic subset of $S_3$. This brings us to our definition of $k$-automatic fractals.

**Definition 2.5.** — We say that a compact subset $X \subseteq \mathbb{R}^d$ is a $k$-automatic fractal if it is the closure of a $k$-automatic subset of $S^d_k$.

**2.2. Examples of famous automatic fractals.** — To illustrate the relevance of the notion of automatic fractals, we give now several examples of classical fractals that turn out to be automatic.

**Example 2.6.** — The Cantor set is a 3-automatic fractal. It is the closure of the following 3-automatic set of $S_3$:

$$\{(\bullet[n_1n_2\cdots n_k]_3) | n_i \neq 1 \forall i \in [1,k]\}.$$

It is associated with the following automaton (see Figure 2).

**Figure 2.** The finite automaton associated with the triadic Cantor set.

**Figure 3.** The first steps in the construction of the triadic Cantor set.

**Example 2.7.** — The Sierpinski carpet is a 3-automatic fractal of $\mathbb{R}^2$.

It can be defined as the set of pairs of real numbers $(x, y)$ in $[0, 1]^2$ such that for every positive integer $n$, the $n$-th digit of the ternary expansion of $x$ and of $y$ are not both equal to 1. This set is thus the closure of the 3-automatic set of $S^2_3$ defined by:

$$\{(\bullet[n_1n_2\cdots n_k]_3, [m_1m_2\cdots m_l]_3) | (n_i, m_i) \neq (1, 1) \forall i \in [1, \min(k, l)]\}.$$
Example 2.8. — The Menger sponge is defined as a three-dimensional analogue of the Sierpinski carpet. It is a 3-automatic fractal of $\mathbb{R}^3$.

![The Menger sponge](image1)

Figure 5. The Menger sponge.

Example 2.9. — Pascal’s triangle modulo 2 is the subset of $[0,1]^2$ formed by taking the closure of

$$\left\{ (\cdot [n]_2, \cdot [m]_2) \mid \left(\frac{m}{n}\right) \equiv 1 \mod 2 \right\}.$$ 

We infer from [5] that this set is a 2-automatic fractal of $\mathbb{R}^2$.

![Pascal’s triangle modulo 2](image2)

Figure 6. Pascal’s triangle modulo 2.

Example 2.10. — We quote below the following example of 5-automatic fractal of $\mathbb{R}^2$.

It is obtained as the closure of the set

$$\left\{ (\cdot [n_1n_2 \ldots n_k]_5, \cdot [m_1m_2 \ldots m_l]_5) \mid (n_i, m_i) \not\in \{ (1, 2), (2, 1), (2, 3), (3, 2) \}, \forall i \in [1, \min(k, l)] \right\}.$$ 

The above examples of automatic fractals are all related to quite simple automata and should only be considered as basic examples. There is actually a large variety of finite automata, and more involved ones can be used to construct fractals with a much more complex structure (see for instance Example 2.11). On the other hand, some non-trivial automata can give rise to trivial fractals. For instance, we can use a small variation of the Thue–Morse automaton represented in Figure 1 to define a finite 2-automaton that recognizes exactly all elements of $S_2 \cap [0,1]$ whose binary expansion contains an odd number of 1’s. Though this set is not trivial, its closure is just the whole interval $[0, 1]$. 

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Example 2.11. — Reverend Back’s abbey floor is a 3-automatic fractal of \( \mathbb{R}^2 \) (see [5, page 410] for a definition).

2.3. Automatic fractals and self-similarity. — The following proposition gives the relationship between \( k \)-automatic fractals and \( k \)-self-similar sets.

**Proposition 2.12.** — If a compact subset \( X \) of \( [0, 1]^d \) is a \( k \)-automatic fractal, then it is \( k \)-self-similar.

**Proof.** — Suppose that \( X \subseteq [0, 1]^d \) is a \( k \)-automatic fractal. Then there is a \( k \)-automatic subset \( C \) of \( S_k^d \) whose closure is \( X \). Then by Remark 2.4, there are only finitely many distinct sets of the form

\[
\{(x_1, \ldots, x_d) \in S_k^d \cap [0, 1]^d \mid (x_1/k^a + b_1/k^a, \ldots, x_d/k^a + b_d/k^a) \in C\}
\]

with \( a \geq 0 \) and \( b_1, b_2, \ldots, b_d \in [1, k^a] \). Let \( C_1, \ldots, C_m \) denote these sets and let \( X_i \subseteq [0, 1]^d \) denote the closure of \( C_i \).

We claim that \( X \) has \( k \)-kernel \( \{X_1, \ldots, X_m\} \) and thus is \( k \)-self-similar. To see this, let us fix a nonnegative integer \( a \) and some integers \( b_1, b_2, \ldots, b_d \in [0, k^a] \). Thus, there exists an integer \( i, \ 1 \leq i \leq m \), such that

\[
\{(x_1, \ldots, x_d) \in S_k^d \cap [0, 1]^d \mid (x_1/k^a + b_1/k^a, \ldots, x_d/k^a + b_d/k^a) \in C\} = C_i.
\]

It then follows from the definition of \( X_i \) that

\[
\{(x_1, \ldots, x_d) \in S_k^d \cap [0, 1]^d \mid (x_1/k^a + b_1/k^a, \ldots, x_d/k^a + b_d/k^a) \in C\} = \overline{C_i} = X_i,
\]
which is equivalent to
\[ \{(x_1, \ldots, x_d) \in [0,1]^d \mid (x_1/k^a + b_1/k^a, \ldots, x_d/k^a + b_d/k^a) \in X\} = X_i. \]
Consequently
\[ \left\{ (k^a x_1 - b_1, \ldots, k^a x_d - b_d) \in [0,1]^d : (x_1, \ldots, x_d) \in X \cap \prod_{j=1}^{d} [b_j/k^a, (b_j + 1)/k^a] \right\} = X_i. \]
This concludes the proof since there are only a finite number of possible choices for \( \alpha \).

We note that the converse of Proposition 2.12 does not hold in general. This is not difficult to see: since there are only countably many finite state automata with input alphabet \( \Sigma_k \), there are only countably many one-dimensional \( k \)-automatic fractals. On the other hand, for each \( \alpha \in (1/2, 1] \), we have the set \( X_\alpha := \{\alpha/k^m \mid m \geq 1\} \cup \{0\} \) is \( k \)-self-similar. Hence, there are uncountably many \( k \)-self-similar subsets of \([0,1]\). We note that for \( \alpha \notin S_k \), \( X_\alpha \) does not even intersect the set \( S_k \).

3. Proof of Theorem 1.4

In this section we prove Theorem 1.4. The main idea we use is that if \( X \subset [0,1] \) is compact and is not a finite union of closed intervals, then its complement is a countable union of disjoint open intervals and the endpoints of these intervals have a limit point in \([0,1]\).

We first prove the easier part of Theorem 1.4, that is, a finite union of closed intervals with rational endpoints is always \( k \)-self-similar for every integer \( k \geq 2 \). This result follows from Propositions 3.1 and 3.2 below.

**Proposition 3.1.** — Suppose \( X = \bigcup_{i=1}^{m} [a_i, b_i] \subseteq [0,1] \) is \( k \)-self-similar and the intervals
\[ [a_1, b_1], \ldots, [a_m, b_m] \]
are disjoint. Then, \( a_i, b_i \in \mathbb{Q} \) for \( 1 \leq i \leq m \).

**Proof.** — Let \( X = X_1, \ldots, X_d \) be the \( k \)-kernel of \( X \). Then each \( X_i \) is a finite disjoint union of closed intervals. Let \( S \) denote the set of endpoints of the closed intervals that make up these sets. Then \( S \) is a finite set. Suppose that \( x \in S \) and let \( \bullet a_1 a_2 a_3 \cdots \) denote the base-\( k \) expansion of \( x \). By definition of the \( k \)-kernel of \( X \), we have that \( \bullet a_i a_{i+1} \cdots \in S \) for every positive integer \( i \). Since \( S \) is finite, we see that
\[ \bullet a_i a_{i+1} a_{i+2} \cdots = \bullet a_j a_{j+1} a_{j+2} \cdots \]
for some positive integers \( i \) and \( j \) with \( j > i \). It follows that \( x \) is a rational number, concluding the proof.

**Proposition 3.2.** — Suppose \( X \subseteq [0,1] \) is a finite union of closed intervals with rational endpoints. Then \( X \) is \( k \)-self-similar for every integer \( k \geq 2 \).

**Proof.** — Let \( k \geq 2 \) be an integer. Let us assume that \( X = [a_1, b_1] \cup \cdots \cup [a_d, b_d] \) with \( b_i < a_{i+1} \) and \((a_i, b_i) \in \mathbb{Q}^2 \) for \( 1 \leq i < d \). Then there exists a positive integer \( n \) such that \( a_i k^n (k^n - 1), b_i k^n (k^n - 1) \in \mathbb{N} \) for \( 1 \leq i \leq d \). Let \( S = \{a/k^n (k^n - 1) \mid 0 \leq a \leq k^n (k^n - 1)\} \), and let \( T \) denote the collection of subsets of \([0,1] \) that can be written as a finite union of closed intervals with endpoints all in \( S \). Then, \( T \) is finite. Notice that \( X \in T \) and
\[ \{k^i x - j \mid x \in [j/k^i, (j+1)/k^i] \cap X\} \in T \]
for every positive integer \( i \) and every nonnegative integer \( j \in [0,k^i) \). Thus the \( k \)-kernel of \( X \) is finite and so \( X \) is \( k \)-self-similar.
We now introduce the notion of wild point that will be often used in the sequel.

**Definition 3.3.** — Let \( X \subseteq [0,1] \). We say that a point \( \beta \in X \) is a right wild point of \( X \) if \( \beta \) is a limit point of both \( X \cap (\beta,1) \) and \( X^c \cap (\beta,1) \); we say \( \beta \) is a left wild point of \( X \) if it is a limit point of both \( X \cap [0,\beta) \) and \( X^c \cap [0,\beta) \). We simply say \( \beta \) is a wild point of \( X \) if it is either a left or right wild point of \( X \).

For example, if \( C \subseteq [0,1] \) denotes the triadic Cantor set, then \( 1/3 \) is a left wild point of \( C \) but is not a right wild point.

**Lemma 3.4.** — Suppose that \( X \subseteq [0,1] \) is closed and \( T : [0,1] \to [0,1] \) is a continuous injective map satisfying \( T(X) \subseteq X \) and \( T(X^c) \subseteq X^c \).

1. If \( \beta \in X \) is a wild point of \( X \) then \( T(\beta) \) is a wild point of \( X \).
2. If \( \beta_n \) are wild points of \( X \) and \( \beta_n \to \beta \) then \( \beta \) is a wild point of \( X \).

**Proof.** — The proof is a straightforward consequence of the definition of wild point.

For our next lemma we need to introduce some notation. Given a nonzero element \( \alpha \in S_k \), let \( v_k(\alpha) \) denote the \( k \)-adic valuation of \( \alpha \), that is, the largest integer \( n \) such that \( k^{-n}\alpha \in \mathbb{Z} \). When, \( \alpha = 0 \), our convention is that \( v_k(\alpha) = 0 \). Thus, \( v_k \) does not fit exactly with the usual definition of the \( k \)-adic valuation for which we should have \( v_k(0) = +\infty \). Lemma 3.5 shows a connection between \( k \)-self-similarity and \( k \)-automatic function defined over \( S_k \).

**Lemma 3.5.** — Let \( k \geq 2 \) be an integer, \( \alpha \) be an element of \( S_k \) and set \( v(\alpha) := v_k(\alpha) \). Suppose that \( Y \subseteq [0,1] \) is \( k \)-self-similar. Then, there are distinct sets \( Y = Y_1, \ldots, Y_m \) and a \( k \)-automatic function \( f : S_k \cap [0,1] \to \{1,2,\ldots,m\} \) such that for \( \alpha \in S_k \cap [0,1] \) we have

\[
Y_f(\alpha) = \{k^{-v(\alpha)}(x-\alpha) \mid x \in [\alpha, \alpha + k^{v(\alpha)}] \cap Y\}.
\]

In particular, if \( \gamma \in \mathbb{Q} \cap (0,1) \) and

\[
a_n = [k^n\gamma],
\]

then the sequence of sets

\[
T_n := \{k^n x - a_n \mid x \in [a_n/k^n, (a_n + 1)/k^n] \cap Y\}
\]

is eventually periodic.

**Proof.** — Since \( Y \) is a \( k \)-self-similar set, there are only a finite number of distinct sets of the form

\[
\{k^{-v(\alpha)}(x-\alpha) \mid x \in [\alpha, \alpha + k^{v(\alpha)}] \cap Y\},
\]

with \( \alpha \in S_k \cap [0,1] \). Let us denote by \( Y_1 = Y, Y_2, \ldots, Y_m \) these sets (the set \( Y \) corresponds to the case \( \alpha = 0 \) since by convention \( v_k(0) = 0 \)). Since these sets are distinct, we first note that one can define a map \( f : S_k \cap [0,1] \to \{1,2,\ldots,m\} \) such that for every \( \alpha \in S_k \cap [0,1] \), we have \( f(\alpha) = j \) if \( \{k^{-v(\alpha)}(x-\alpha) \mid x \in [\alpha, \alpha + k^{v(\alpha)}] \cap Y\} = Y_j \).

Recall that the \( k \)-kernel of a function whose domain is \( S_k \) is defined in an analogous manner to how \( k \)-kernels of functions whose domain is \( \mathbb{N} \) is. Namely, if \( f : S_k \to \Delta \) is a map, the \( k \)-kernel of \( f \) is the collection of all functions of the form \( g(x) = f((x+c)/k^a) \) with \( a \geq 0 \) and \( 0 \leq c < k^a \). It is easily checked from what is known about automatic functions whose domain is \( \mathbb{N} \) (see [5]), that having a finite \( k \)-kernel is the same as being \( k \)-automatic for such a function \( f \). To prove the first part of Lemma 3.5, it is thus sufficient to show that the \( k \)-kernel of the function \( f \) is finite.

We note that for \( \alpha \in S_k \cap (0,1) \), the set

\[
\{k^{-v(\alpha)}(x-\alpha) \mid x \in [\alpha, \alpha + k^{v(\alpha)}] \cap Y_j\}
\]

must be one of \( Y_1, \ldots, Y_m \). The reason for this is that there is some \( \beta \in S_k \cap [0,1] \) such that

\[
Y_j = \{k^{-v(\beta)}(x-\beta) \mid x \in [\beta, \beta + k^{v(\beta)}] \cap Y\}.
\]
Then, it follows that
\[ \{k^{-v(a)}(x - \alpha) \mid x \in [\alpha, \alpha + k^{v(a)}] \cap Y_j \} = \{k^{-v(a)-v(\beta)}(x - \beta') \mid x \in [\beta', \beta' + k^{v(\alpha)+v(\beta)}] \cap Y_j \}, \]
where \( \beta' = \beta + k^{v(\beta)}\alpha \). For every positive integer \( j \in [1, m] \), let \( f_j \) denote the function such that for \( \alpha \in S_k \cap [0, 1) \) we have
\[ Y_{f_j(\alpha)} = \{k^{-v(a)}(x - \alpha) \mid x \in [\alpha, \alpha + k^{v(a)}] \cap Y_j \}. \]
Let \( \alpha \in S_k \cap [0, 1) \). Let \( i \in \{0, 1, 2, \ldots, k - 1\} \) and set \( \ell = f_j(i/k) \).
Then
\[ Y_{f_j((i+\alpha)/k)} = \{k^{1-v(a)}(x - i/k - \alpha/k) \mid x \in [i/k + \alpha/k, i/k + \alpha/k + k^{v(a)-1}] \cap Y_j \} = \{k^{-v(a)}(x - \alpha) \mid x \in [\alpha, \alpha + k^{v(a)}] \cap Y_{\ell} \} = Y_{f_j(\alpha)}. \]
Since \( f = f_1 \), it follows that the \( k \)-kernel of \( f \) is contained in \( \{f_1, \ldots, f_m\} \) and hence \( f \) is \( k \)-automatic.

For the second part of Lemma 3.5, note that \( T_n \) is simply \( Y_{f(a_n/k^n)} \). We thus have to prove that the sequence \( (f(a_n/k^n))_{n \geq 1} \) is eventually periodic. Since \( \gamma \) is rational, its expansion in base \( k \) is eventually periodic. There thus exist two finite word \( U = a_1a_2 \ldots a_r \) (possibly empty) and \( V = a_{r+1}a_{r+2} \ldots a_{r+s} \) such that
\[ \gamma = [0_kUV^\infty] = [0_kUVVV \ldots ]_k. \]
Then, \( a_n = [k^n\gamma] = [a_1a_2 \ldots a_n]_k \) and thus
\[ a_n/k^n = [0_ka_1a_2 \ldots a_n]_k. \]
On the other hand, an easy adaptation of Theorem 5.5.2 in [5] gives the following result: if \( h \) is a \( k \)-automatic function from \( S_k \) into a finite set \( \Delta \) and if \( A \) and \( B \) are two finite words, then the sequence \( (h([0_kAB^n]_k))_{n \geq 0} \) is eventually periodic. This implies that all sequences \( (f(a_{ns+j}/k^{ns+j}))_{n \geq 0} \), with \( r \leq j < r + s \), are eventually periodic. Consequently, the sequence \( (f(a_n/k^n))_{n \geq 1} \) is eventually periodic, concluding the proof. \( \square \)

**Lemma 3.6.** — Let \( Y \subseteq X \subseteq [0, 1] \) be closed sets, \( a \) and \( b \) be nonnegative integers, \( c \) and \( d \) be nonpositive integers, and \( k \) and \( \ell \) be two integers larger than 1. Let \( T : [0, 1] \to \mathbb{R} \) and \( S : [0, 1] \to \mathbb{R} \) be given by \( T(x) = kx + c \) and \( S(x) = \ell x + d \). Suppose that the following conditions are satisfied:

(i) \( k \) and \( \ell \) are multiplicatively independent;
(ii) \( X \cap [b_1, b_2] = Y \) for some \( b_1, b_2 \in [0, 1] \);
(iii) \( X = T(X \cap [a/k, a/k + 1/k]) \);
(iv) \( Y = S(Y \cap [b/\ell, (b + 1)/\ell]) \);
(v) \( T(\beta) = S(\beta) = \beta \) for some \( \beta \in Y \);
(vi) \( \beta \) is not the rightmost point of \( Y \) and \( Y \) has at least two points in \( (b_1, b_2) \).

Then, \( X \cap [\beta, 1] = [\beta, 1] \).

**Proof.** — Suppose that \( Y^c \) contains an open interval \((x, y)\) with \( x, y \in Y \). By (vi), we may assume that \( b_1 < x < y < b_2 \). By hypothesis (i), for every positive real number \( \varepsilon \), the set
\[ \mathcal{N} = \{ (m, n) \in \mathbb{N}^2 \mid k^m < \ell^n < (1 + \varepsilon)k^m \}. \]
is infinite.

From now on, we fix a positive \( \varepsilon \) such that
\[ \varepsilon < (y - x)/2, \quad x > b_1(1 + \varepsilon) \quad \text{and} \quad b_2 > y(1 + \varepsilon). \]
Let \((m, n) \in \mathcal{N}\). We consider the set \(T^m(S^{-n}(Y))\). By (iv), we have \(S^{-n}(Y) \subseteq Y \subseteq X\), while (iii) implies that \(T^m(X) \cap [0, 1] = X\). Hence, we infer from (ii) that
\[
T^m(S^{-n}(Y)) \cap [b_1, b_2] \subseteq Y.
\]
On the other hand, (ii) and (v) give that \(\beta \in [b_1, b_2]\).

Then, we have
\[
T^m(S^{-n}(z)) = k^m \ell^{-n} z + \beta(1 - k^m \ell^{-n}) \in Y,
\]
whenever
\[
b_1 \leq k^m \ell^{-n} z + \beta(1 - k^m \ell^{-n}) \leq b_2.
\]
Notice that since \((m, n) \in \mathcal{N}\), (2) and (3) give
\[
T^m(S^{-n}(x)) = k^m \ell^{-n} x + \beta(1 - k^m \ell^{-n}) < x + \varepsilon < b < b_2
\]
and
\[
T^m(S^{-n}(x)) = k^m \ell^{-n} x + \beta(1 - k^m \ell^{-n}) > x/(1 + \varepsilon) > b_1.
\]
Thus
\[
T^m(S^{-n}(x)) \in Y, \quad \text{and} \quad T^m(S^{-n}(x)) < y.
\]
Since \((x, y) \subseteq Y^c\), we see that \(T^m(S^{-n}(x)) \leq x\). Similarly, \(T^m(S^{-n}(y)) \in Y\) and
\[
T^m(S^{-n}(y)) = k^m \ell^{-n} y + \beta(1 - k^m \ell^{-n}) > x.
\]
And again, since \((x, y) \subseteq Y^c\), we obtain \(T^m(S^{-n}(y)) > y\). These two inequalities give
\[
T^m(S^{-n}(y)) - T^m(S^{-n}(x)) = k^m \ell^{-n}(y - x) \geq y - x.
\]
This provides a contradiction, since (2) ensures that \(k^m \ell^{-n} < 1\).

It follows that \(Y^c\) cannot contain an open interval \((x, y)\) with \(x, y \in Y\). Thus \(Y\) is either empty or \(Y\) consists of a single closed interval. Since by (vi), \(\beta \in Y\) is not the rightmost point in \(Y\), \(Y\) is a closed interval containing \([\beta, \beta + \delta]\) for some positive \(\delta\).

We claim that \([\beta, 1] \subseteq X\). Indeed, if \(z \in (\beta, 1)\) is not in \(X\), then \((T^{-n}(z))_{n \geq 1}\) is a decreasing sequence converging to \(\beta\) and thus \(T^{-n}(z) \in [\beta, \beta + \delta]\) for every integer \(n\) large enough. But, we just obtained that \([\beta, \beta + \delta] \subseteq Y \subseteq X\). This would provide a contradiction since (iii), \(T^{-1}(X^c) \cap [0, 1] \subseteq X^c\). Thus \((\beta, 1) \subseteq X\) and since by assumption \(X\) is a closed set, we obtain that \([\beta, 1] \subseteq X\). This ends the proof.

Lemma 3.7. — Let \(k\) and \(\ell\) be two multiplicatively independent natural numbers and suppose that \(X \subseteq [0, 1]\) is a compact \(k\)- and \(\ell\)-self-similar set. If \(X\) is not a finite union of closed intervals with rational endpoints, then there exists a compact set \(\tilde{X} \subseteq [0, 1]\) which is both \(k\)- and \(\ell\)-self-similar, and which has a rational wild point with purely periodic base-\(k\) and base-\(\ell\) expansions.

Proof. — Since \(X^c\) is open, it is a countable disjoint union of open intervals. If the number of open intervals is finite we obtain that \(X\) is a finite union of closed intervals. By Proposition 3.1, this would provide a contradiction with our assumption. So, we may assume that we have a countably infinite set of closed intervals. Let
\[
\{(\alpha_i, \beta_i) \mid i \in \mathbb{N}\}
\]
be an enumeration of these intervals. We have that \(\beta_i \in [0, 1]\) for infinitely many \(i\) and so by the Bolzano-Weierstrass theorem there exists some number \(\beta \in [0, 1]\) that is a limit point of the set of \(\beta_i\). Write
\[
\beta = \sum_{i \geq 1} b_i/k^i,
\]
with \(0 \leq b_i < k\). Then \(\beta\) is a wild point of \(X\). For every positive integer \(n\), we set
\[
c_n := b_1k^{n-1} + b_2k^{n-2} + \cdots + b_n.
\]
Then
\[
c_n/k^n \leq \beta \leq c_n/k^n + 1/k^n.
\]
By assumption, the set $X$ is $k$-self-similar. Thus, there are only finitely many distinct sets of the form $$\{ k^n x - c_n \mid x \in [c_n / k^n, (c_n + 1) / k^n] \cap X \},$$ and hence there exist distinct positive integers $i$ and $j$ with $j > i$ such that

$$X_0 := \{ k^j x - c_i \mid x \in [c_i / k^j, (c_i + 1) / k^j] \cap X \} = \{ k^j x - c_j \mid x \in [c_j / k^j, (c_j + 1) / k^j] \cap X \}.$$

Note that the definition of $X_0$ implies that $X_0$ is also both $k$- and $\ell$-self-similar, and that $\beta' := k^j \beta - c_i$ is a wild point of $X_0$.

Let $m = j - i$. Then

$$X_0 = \{ k^m y - c_j + k^m c_i \mid y \in [c_j / k^m - c_i, c_j / k^m - c_i + 1 / k^m] \cap (k^j X - c_i) \}.$$

By definition, $X_0 = (k^i X - c_i) \cap [0,1]$, and so

$$X_0 = \{ k^m x + k^m c_i - c_j \mid x \in [c_j / k^m - c_i, c_j / k^m - c_i + 1 / k^m] \cap X_0 \}.$$

We thus have a map $U : [0,1] \to [0,1]$ given by

$$U(x) = (x + c_j - k^m c_i) / k^m,$$

which satisfies $U(X_0) \subseteq X_0$ and $U(X_0^c) \subseteq X_0^c$. Let $c = c_j k^{-m} - c_i$. By Lemma 3.4, $U(\beta'), U^2(\beta'), \ldots$ are all wild points of $X_0$. Moreover, this sequence is bounded and monotonic and hence it converges to the unique fixed point of $U$; namely, the point

$$\gamma := k^m c / (k^m - 1).$$

We thus infer from Lemma 3.4 that $\gamma = k^m c / (k^m - 1)$ is also a wild point of $X_0$.

Since $\gamma$ is rational, we can write

$$\gamma = a/\ell^p + a'/\ell^p(\ell^p - 1)$$

for some natural numbers $a$ and $a'$, and some positive integer $p$ such that $a' < \ell^p - 1$. For every positive integer $n$, set

$$a_n := \lfloor \ell^p \gamma \rfloor.$$

We are now going to use the fact that the set $X_0$ is also $\ell$-self-similar. Hence, there are only finitely many distinct sets of the form

$$\{ \ell^{np} x - a_n \mid x \in [a_n / \ell^{np}, (a_n + 1) / \ell^{np}] \cap X_0 \}.$$

Let $Y_1, \ldots, Y_\ell$ denote the distinct sets of this form. By Lemma 3.5, if $f(n)$ denotes the index such that

$$Y_{f(n)} = \{ \ell^{np} x - a_n \mid x \in [a_n / \ell^{np}, (a_n + 1) / \ell^{np}] \cap X_0 \},$$

then the sequence $(f(n))_{n \geq 1}$ is eventually periodic. Hence, there exist some index $i$ and some positive integer $d$ such that

$$Y_i = \{ \ell^{ndp} x - a_{ndp} \mid x \in [a_{ndp} / \ell^{ndp}, (a_{ndp} + 1) / \ell^{ndp}] \cap X_0 \}$$

for all $n \geq 1$. Since $X_0$ is both $k$- and $\ell$-self-similar, we obtain that $Y_i$ is also both $k$- and $\ell$-self-similar. Moreover, since $\gamma$ is a wild point of $X_0$, we also get that $\gamma' := \ell^p \gamma - a_{dp}$ is a wild point of $Y_i$. Furthermore, we have

$$\gamma' = a'/(\ell^p - 1).$$

To finish the proof, note that

$$\gamma = a/\ell^p + a'/\ell^p(\ell^p - 1)$$

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has the property that \((k^m - 1)\gamma \in \mathbb{Z}\). Thus \((k^m - 1)\ell^p \gamma = (k^m - 1)a + (k^m - 1)a'/(\ell^p - 1)\) is an integer. Consequently, \((k^m - 1)a'/(\ell^p - 1)\) is an integer. But this means that \(\gamma'\) has both a purely periodic base-\(k\) and base-\(\ell\) expansion, concluding the proof. \(\square\)

We are now ready to prove our main result.

**Proof of Theorem 1.4.** — Let \(k\) and \(l\) be two multiplicatively independent integers, and let \(X\) be a closed subset of \([0, 1]\) which is both \(k\)- and \(l\)-self-similar. We are going to argue by contradiction.

Let us assume that \(X\) is not a finite union of closed intervals with rational endpoints. By Lemma 3.7, there exists a compact set \(\tilde{X} \subseteq [0, 1]\) which is both \(k\)- and \(l\)-self-similar and which has a wild point, say \(\beta\), whose base-\(k\) and base-\(\ell\) expansions are purely periodic. By replacing \(k\) and \(\ell\) with an appropriate power, we may assume that this period is 1 for each base. It is also no loss of generality to assume that \(\beta\) is a right wild point of \(\tilde{X}\). Indeed, if \(\tilde{X}\) is \(k\)-self-similar then the set \(1 - \tilde{X} := \{x \in [0, 1] | 1 - x \in \tilde{X}\}\) is also \(k\)-self-similar, so that we could if necessary replace \(\tilde{X}\) by \(1 - \tilde{X}\).

For every integer \(n \geq 1\), there exist nonnegative integers \(a_n\) and \(b_n\) such that
\[
\frac{a_n}{k^n} \leq \beta \leq \frac{a_n}{k^n} + \frac{1}{k^n}
\]
and
\[
\frac{b_n}{\ell^n} \leq \beta \leq \frac{b_n}{\ell^n} + \frac{1}{\ell^n}.
\]

We first use the \(k\)-self-similarity of \(\tilde{X}\). Since \(\tilde{X}\) is \(k\)-self-similar, there are only finitely many distinct sets of the form
\[
\{k^n x - a_n | x \in [a_n/k^n, (a_n + 1)/k^n] \cap \tilde{X}\},
\]
and hence there exist distinct positive integers \(i\) and \(j\), with \(j > i\), and such that
\[
X_0 := \{k^j x - a_i | x \in [a_i/k^j, (a_i + 1)/k^j] \cap \tilde{X}\}
= \{k^j x - a_j | x \in [a_j/k^j, (a_j + 1)/k^j] \cap \tilde{X}\}.
\]

It thus follows from this definition, that \(X_0\) is both \(k\)- and \(\ell\)-self-similar. Furthermore, since \(\beta\) has a purely periodic base-\(k\) expansion, we also have that \(\beta = k^i\beta - a_i\) is a wild point of \(X_0\).

Set \(m := j - i\). Then
\[
X_0 = \{k^j x - a_j | x \in [a_j/k^j, (a_j + 1)/k^j] \cap \tilde{X}\}
= \{k^m (k^j x - a_i) - a_j + k^m a_i | x \in [a_j/k^j, (a_j + 1)/k^j] \cap \tilde{X}\}
= \{k^m y - a_j + k^m c_i | y \in [a_j/k^m - c_i, a_j/k^m - a_i + 1/k^m] \cap (k^iX - a_i)\}.
\]

By assumption, \(X_0 = (k^j \tilde{X} - a_i) \cap [0, 1]\), and so
\[
X_0 = \{k^m x + k^m a_i - a_j | x \in [a_j/k^m - a_i, a_j/k^m - a_i + 1/k^m] \cap X_0\}.
\]

Set \(c := a_j/k^m - a_i\) and let \(T : [0, 1] \to \mathbb{R}\) be the map defined by
\[
T(x) = k^m x - k^m c.
\]

Then, we infer from Equality (8) that
\[
X_0 = T([c, c + 1/k^m] \cap X_0).
\]

Furthermore, since \(\beta\) has a purely periodic expansion of period 1 in base \(k\), we infer from (6) that
\[
T(\beta) = \beta.
\]

We are now going to use the \(\ell\)-self-similarity of \(X_0\). Since \(X_0\) is \(\ell\)-self-similar, there are only finitely many distinct sets of the form
\[
\{\ell^n x - b_n | x \in [b_n/\ell^n, (b_n + 1)/\ell^n] \cap X_0\}.
\]
Let \(Y_1, \ldots, Y_n\) denote the distinct sets of this form. Let \(f(n)\) denote the index such that
\[
Y_{f(n)} = \{\ell^n x - b_n \mid x \in [b_n/\ell^n, (b_n + 1)/\ell^n] \cap X_0\}.
\]
Then, it follows from Lemma 3.5 that the sequence \((f(n))_{n \geq 1}\) is eventually periodic. Hence, there exist some index \(i\) and some positive integer \(d\) such that
\[
Y_i = \{\ell^{dn} x - b_{dn} \mid x \in [b_{dn}/\ell^{dn}, (b_{dn} + 1)/\ell^{dn}] \cap X_0\}
\]
for all \(n \geq 1\). To complete the proof, we define a subset \(Y\) of \(X_0\) by
\[
Y := X_0 \cap [b_d/\ell^d, (b_d + 1)/\ell^d].
\]
Let \(S : [0, 1] \to \mathbb{R}\) be the map defined by
\[
S(x) = \ell^d x - b_d.
\]
Since \(\beta\) has a purely periodic base-\(\ell\) expansion of period 1, we see that
\[
S(\beta) = \beta.
\]
We also claim that
\[
S(Y \cap [b_{2d}/\ell^{2d}, (b_{2d} + 1)/\ell^{2d}]) = Y.
\]
To see this, note that by (11) we have \(Y_i = S(X_0 \cap [b_d/\ell^d, (b_d + 1)/\ell^d]) = S(Y)\) and also
\[
Y_i = S^2(X_0 \cap [b_{2d}/\ell^{2d}, (b_{2d} + 1)/\ell^{2d}]) = S^2(Y \cap [b_{2d}/\ell^{2d}, (b_{2d} + 1)/\ell^{2d}]),
\]
since \([b_{2d}/\ell^{2d}, (b_{2d} + 1)/\ell^{2d}] \subset [b_d/\ell^d, (b_d + 1)/\ell^d]\). Thus,
\[
S(Y) = S^2(Y \cap [b_{2d}/\ell^{2d}, (b_{2d} + 1)/\ell^{2d}]),
\]
which implies Equality (14) since \(S\) is an injective map.

Since \(\beta\) is a right wild point of \(X_0\), we see that it is not the rightmost point in \(Y\). We are now ready to apply Lemma 3.6. Indeed, we infer from Equalities (9), (10), (12), (13), (14) and from Lemma 3.6 that \(X_0 \cap [\beta, 1] = [\beta, 1]\). This provides a contradiction since \(\beta\) is a right wild point of \(X_0\).

We thus have proved that \(X\) is a finite union of closed intervals with rational endpoints. In view of Propositions 3.1 and 3.2, this ends the proof of Theorem 1.4. \(\square\)

4. Entropy and Hausdorff dimension

In this section, we discuss two notions that can be naturally attached to automatic fractals.

An important notion in the study of fractals is the Hausdorff dimension. We do not recall here the definition of the Hausdorff definition and refer the reader to [11] or [19] for an introduction to this topic. One fact about most fractal objects is that their Hausdorff dimension (or the Hausdorff dimension of their boundary) is not a natural number. For instance, the Hausdorff dimension of the triadic Cantor set is equal to \(\log 2/\log 3\), and we respectively get the values \(\log 8/\log 3\), \(\log 3/\log 2\) and \(\log 20/\log 3\) for the Hausdorff dimension of the Sierpinski carpet, Pascal’s triangle modulo 2, and the Menger sponge.

In [17], Mauldin and Williams studied a large family of fractal sets of \(\mathbb{R}^n\) that they termed geometric graph directed constructions. These fractals are constructed by means of a directed labelled graph. Among other results, Mauldin and Williams showed that the Hausdorff dimension of such a set can be computed by working on the so-called weighted incidence matrix associated with the graph.

To a finite automaton is also attached a directed labelled graph. This could be used to show that every automatic fractal can be obtained as a geometric graph directed construction. For an automatic fractal \(X\), it should thus be possible to use the approach of Mauldin and Williams to compute \(\mathcal{H}(X)\), the Hausdorff dimension of \(X\).
Two fundamental notions in the theory of formal languages are the subword complexity and the entropy of factorial languages. Given a finite set $\mathcal{A}$, a language $\mathcal{L}$ over $\mathcal{A}$ is just a subset of $\mathcal{A}^*$. The language $\mathcal{L}$ is said factorial if for every word $W$ in $\mathcal{L}$, every subword of $W$ also belongs to $\mathcal{L}$. Then, the complexity function of a factorial language $\mathcal{L}$ is defined as the function that maps every integer $n$ to the integer

$$p(\mathcal{L}, n) := \#(\mathcal{L} \cap \mathcal{A}^n),$$

and the entropy of the language $\mathcal{L}$ is defined by

$$h(\mathcal{L}) := \lim_{n \to \infty} \frac{1}{n} \log p(\mathcal{L}, n).$$

The latter notion is well-defined; indeed, for a factorial language the limit above always exists because of the trivial inequality $p(\mathcal{L}, n + m) \leq p(\mathcal{L}, n) \times p(\mathcal{L}, m)$. With this definition, we always have

$$0 \leq h(\mathcal{L}) \leq \log(\#\mathcal{A}).$$

We now explain how to associate a factorial language with an automatic fractal. Let $X$ be a $k$-automatic fractal of $\mathbb{R}^d$. Let $x := (x_1, \ldots, x_d)$ be an element of $X \cap S_k^d$. Then, expanding each coordinate in base $k$, we obtain the existence of finite words $W_1, \ldots, W_d \in \Sigma_k^*$ such that

$$x_i = [W_i]_k,$$

for every integer $1 \leq j \leq d$. Set

$$W_i^0\infty := w^{(i)}_1 w^{(i)}_2 \ldots$$

Thus,

$$x = ([W_1^0\infty]_k, [W_2^0\infty]_k, \ldots, [W_d^0\infty]_k),$$

and we can associate with $x$ an element $w(x)$ of $(\Sigma_k^d)^*$ defined by

$$w(x) := \begin{pmatrix} w_1^{(1)} \\ \vdots \\ w_1^{(d)} \\ \vdots \\ w_2^{(1)} \\ \vdots \\ w_2^{(d)} \\ \vdots \\ w_n^{(1)} \\ \vdots \\ w_n^{(d)} \\ \vdots \end{pmatrix}.$$ 

We then consider the language $\mathcal{L}(X)$ formed by all finite words in $(\Sigma_k^d)^*$ having at least one occurrence in some $w(x)$, with $x \in X \cap S_k^d$. By definition, $\mathcal{L}(X)$ is a factorial language. We thus define the entropy of the automatic fractal $X$ to be

$$h(X) := \frac{1}{\log k} h(\mathcal{L}(X)).$$

Recall that a $k$-automatic fractal is also $k^n$-automatic for every positive integer $n$. With the normalization above, the entropy of a $k$-automatic fractal remains unchanged when viewing it as a $k^n$-automatic fractal.

Since $X$ is a $k$-automatic fractal, it can be shown that the language $\mathcal{L}(X)$ is recognized by a finite automaton (see [20]). Then, Kleene’s theorem implies that $\mathcal{L}(X)$ is a rational language, and by a result of Schützenberger, we get that the formal power series

$$\sum_{n \geq 1} p(\mathcal{L}(X), n) X^n$$

is a rational function, that is, it belongs to $\mathbb{Q}(X)$ (see [10]). In that case, the sequence $(p(\mathcal{L}(X), n))_{n \geq 1}$ satisfies a linear recurrence and this could be used to compute the entropy of $\mathcal{L}(X)$.

The Hausdorff dimension of an automatic fractal and its entropy appear to be strongly connected. Let us give a few examples.

With the previous definition, we easily obtain that the language associated with the triadic Cantor set is

$$\mathcal{L}(C) = \{a_1 a_2 \ldots a_k \mid a_i \in \{0, 2\}, \; 1 \leq i \leq k\}.$$
Consequently, \( p(\mathcal{L}(C), n) = 2^n \) and thus
\[
h(C) = \log 2 / \log 3 = \mathcal{H}(C).
\]

Let us denote by \( S \) the Sierpinski carpet defined in Example 2.7. We easily obtain that
\[
\mathcal{L}(S) = \left\{ \left( \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{array} \right) \mid \left( \begin{array}{c} a_i \\ b_i \end{array} \right) \neq \left( \begin{array}{c} 1 \\ 1 \end{array} \right), \ 1 \leq i \leq n \right\}.
\]
Thus, we get that \( p(\mathcal{L}(S), n) = 8^n \) for every positive integer \( n \), and consequently
\[
h(S) = \log 8 / \log 3 = \mathcal{H}(S).
\]
A similar computation with the Menger sponge \( M \) leads to equalities:
\[
h(M) = \log 20 / \log 3 = \mathcal{H}(M).
\]

It would be interesting to determine the exact link between the entropy and the Hausdorff dimension of automatic fractals. In particular, one may ask whether the equality
\[
h(X) = \mathcal{H}(X)
\]
holds for every automatic fractal \( X \).

### 5. Comments

The interplay between fractal sets and finite automata has quite a long history. It is not our purpose to give a survey of such studies, and so we will be content with few references.

Our definition of a \( k \)-automatic fractal uses two steps. First, we made the choice to consider finite state automata as devices that only take finite words as input; in this way, we associate a subset of the nonnegative \( k \)-adic rationals with a finite automaton \( A \) that takes words over the alphabet \( \Sigma_k' \) as input. Then, we obtain a \( k \)-automatic fractal \( X \) by taking the closure of this set. Another equivalent formulation can actually be given in terms of Büchi automata. More precisely, we would obtain the same automatic fractal \( X \) by considering the set of real numbers in \([0,1]\) whose base-\( k \) expansion has the property that either it is finite and accepted by \( A \) or it is infinite and infinitely many of its prefixes are accepted by \( A \).

Hartmanis and Stearns [15] were probably the first to use finite automata to describe some fractal subsets of \([0,1]\). Their approach, though different from ours, is in the same spirit as the one described just above in terms of Büchi automata. Note also that these authors were only concerned with one-dimensional sets. With a finite \( k \)-automaton \( A \), they associate the set of real numbers such that all prefixes in their base-\( k \) expansion are recognized by \( A \). In many cases, the set they get by this way is the same as the one obtained by using the process described in Section 2. However, this is not always the case. For instance, starting with the Thue–Morse automaton, they get the set \{1\}, while our construction gives the whole interval \([0,1]\).

In contrast, most contructions of fractals involving finite automata follow a rather different route. Starting with a classical automatic function \( f \) of \( \mathbb{N}^d \), a sequence of arrays (or matrices) corresponding to compact sets of \( \mathbb{R}^d \) is naturally associated with \( f \). After some kind of renormalization, this leads to a sequence of compact sets of \([0,1]^d\). Then, fractals are obtained as those sets that are a limit point of this sequence of compact sets with respect to the Hausdorff metric. Many authors already used this principle on specific examples such as the Sierpinski carpet or Pascal’s triangle (see for instance [22, 4, 3] and Chapter 14 of [5]). More recent and general accounts were given by von Haeseler et al. [13, 14] and by Barbé and von Haeseler [6]. In particular, [6] includes a systematic study of automatic fractals. By the process just described, these authors obtain a family of fractals which is essentially the same as the family of automatic fractals we defined in the present paper. In particular, our results would also apply in their framework.

Note that another geometric context in which a Cobham-type phenomenon occurs is given by stretching factors of self-similar tilings, as recently described by Cortez and Durand [9].
After writing a first draft of this article, we learned about a recent work of Boigelot and Brusten [7]. Although they are not motivated in any way by fractals and their results are written in terms of logic, there is no doubt an intimate connection with the present work.

We end this paper with two questions.

**Problem 5.1.** — Let $X$ be an automatic fractal of $\mathbb{R}^d$ whose Hausdorff dimension is $d$. Does the set $X$ always contain a nonempty open set?

The following problem is motivated by classical questions in number theory concerning the expansion of algebraic irrational numbers in integer bases.

**Problem 5.2.** — Is it true that an automatic fractal of $\mathbb{R}$ contains an algebraic irrational number only if it contains an open interval? Note that it would already be interesting to find a one-dimensional automatic fractal $X$ such that $0 < \mathcal{H}(X) < 1$, and for which one can prove that it contains no irrational algebraic number?

In the case where $X$ is the triadic Cantor set, the latter question corresponds to a famous problem addressed by Mahler (see [18]). Roughly, we could say that if an automatic set $X$ of $\mathbb{R}$ has Hausdorff dimension less than one, then all elements in $X$ should have serious restrictions regarding patterns that can occur in their base-$k$ expansion. On the other hand, it is expected that algebraic irrational numbers contain every possible finite sequence of digits in their base-$k$ expansion (see for instance [2]).

References


