Entrance and sojourn times for Markov chains

Application to (L, R)–random walks

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Stochastic processes under constraints
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The problem

My favorite and recurrent interests:

- **Hitting times, entrance times**
- **Exit times**
- **Overshooting times**
- **Sojourn times...**

For various stochastic processes:

- **Brownian motion, Ornstein-Uhlenbeck process**
- **Diffusion processes, Gaussian processes**
- **Lévy processes, stable processes**
- **Integrated Brownian motion and other integral functionals...**

The problem

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In this talk: **Random walks and more general Markov chains**

Main goal: **To provide a methodology for deriving the probability distribution of certain sojourn times...**

Motivation: high-order heat equation

- *Continuous-time (N integer > 1):*

$$\partial_t u(t,x) = (-1)^{N-1}\,\Delta_x^N u(t,x), \quad t > 0,\ x \in \mathbb{R}$$

where

$$\partial_t u(t,x) = \frac{\partial u}{\partial t}(t,x)$$

$$\Delta_x^N u(t,x) = \frac{\partial^{2N} u}{\partial x^{2N}}(t,x)$$

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$$\Delta_x^N u(t,x) = \frac{\partial^{2N} u}{\partial x^{2N}}(t,x)$$

$\longrightarrow$ *Pseudo-Brownian motion* $(B_t)_{t\geq 0}$ with pseudo-transition densities

$$\mathbb{P}_x\{B_t\in dy\}/dy = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(x-y)u - tu^{2N}}\,du$$

Warning: $\mathbb{P}_x$ *is a signed measure!*

Motivation: high-order heat equation

- ***Discrete-time (N integer > 1):***

$$\partial_n u(n,x) = (-1)^{N-1}\Delta_x^N u(n,x), \quad n \in \mathbb{N},\ x \in \mathbb{Z}$$

where
$$\partial_n u(n,x) = u(n+1,x) - u(n,x)$$
$$\Delta_x^N u(n,x) = \sum_{k=-N}^{N} (-1)^k \binom{2N}{k+N} u(n,x+k)$$

Motivation: high-order heat equation

- *Discrete-time (N integer > 1):*

$$\boxed{\partial_n u(n,x) = (-1)^{N-1}\Delta_x^N u(n,x), \quad n \in \mathbb{N},\ x \in \mathbb{Z}}$$

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$\longrightarrow$ *Pseudo-random walk* $(X_m)_{m\geq 0}$: $X_m = X_0 + \sum_{i=1}^m U_i$
where $(U_i)_{i\geq 1}$ is a sequence of i.i.d. pseudo-r.v. with pseudo-distribution

$$\begin{cases} \mathbb{P}\{U_1 = i\} = (-1)^{i+N-1}\binom{2N}{i+N} & \text{if } \begin{cases} -N \leq i \leq N \\ i \neq 0 \end{cases} \\ \mathbb{P}\{U_1 = 0\} = 1 + (-1)^N \binom{2N}{N} \end{cases}$$

Motivation: high-order heat equation

- *Discrete-time (N integer > 1):*

$$\partial_n u(n,x) = (-1)^{N-1}\Delta_x^N u(n,x), \quad n \in \mathbb{N},\ x \in \mathbb{Z}$$

where $\partial_n u(n,x) = u(n+1,x) - u(n,x)$

$$\Delta_x^N u(n,x) = \sum_{k=-N}^{N} (-1)^k \binom{2N}{k+N} u(n,x+k)$$

$\longrightarrow$ ***Pseudo-random walk*** $(X_m)_{m\geq 0}$: $X_m = X_0 + \sum_{i=1}^{m} U_i$

This is a $(-N,N)$***–random walk with signed distribution***

Plan

1. **General random walk on $\mathbb{R}$**

Plan

1. **General random walk on $\mathbb{R}$**
2. **Case of nearest neighbour random walk**

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1. **General random walk on $\mathbb{R}$**
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3. **Case of random walk with stagnation**

Plan

1. **General random walk on $\mathbb{R}$**
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4. **Case of (L, R)–random walk**

Plan

1. **General random walk on $\mathbb{R}$**
2. **Case of nearest neighbour random walk**
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4. **Case of (L, R)–random walk**
5. **General framework: Markov chains**

Plan

1. General random walk on $\mathbb{R}$
2. Case of nearest neighbour random walk
3. Case of random walk with stagnation
4. Case of (L, R)–random walk
5. General framework: Markov chains
6. Case of (L, R)–random walk (continued)

Plan

1. **General random walk on $\mathbb{R}$**
2. **Case of nearest neighbour random walk**
3. **Case of random walk with stagnation**
4. **Case of (L, R)–random walk**
5. **General framework: Markov chains**
6. **Case of (L, R)–random walk (continued)**
7. **Case of symmetric $(2, 2)$–random walk**

1. General random walk on $\mathbb{R}$

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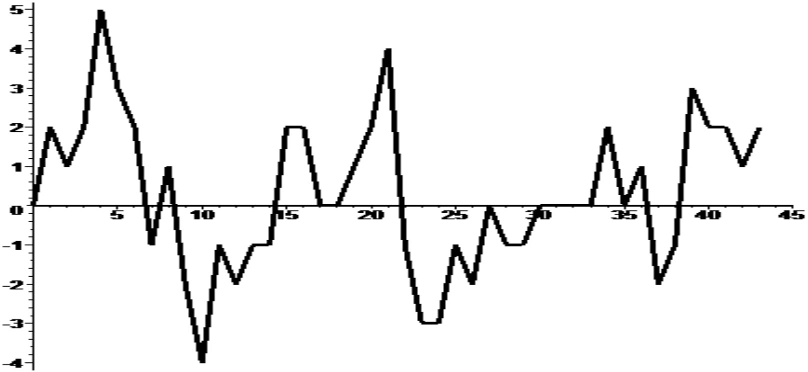
Definition – **Let $(U_i)_{i\geq 1}$ be a sequence of i.i.d. real-valued r.v.'s and $(X_m)_{m\geq 0}$ be the random walk defined on $\mathbb{R}$ by $X_0 = 0$ and**

$$X_m = \sum_{i=1}^{m} U_i \quad \text{for } m \geq 1$$

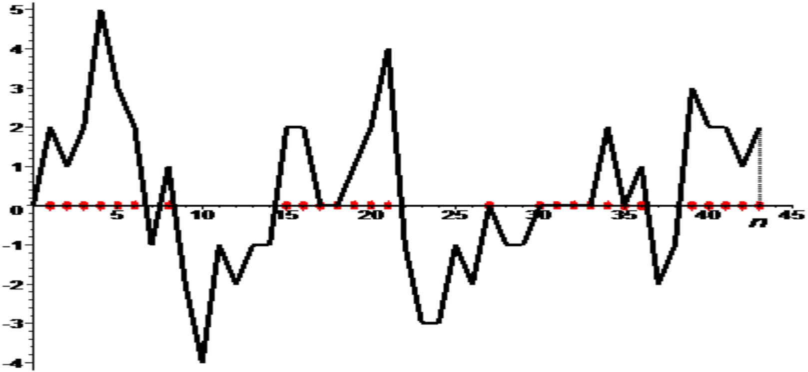


1. General random walk on $\mathbb{R}$

Definition – **Sojourn time of the walk $(X_m)_{m\geq 0}$ in $\mathbb{R}^\dagger = [0, +\infty)$ up to a fixed step $n \geq 1$**

$$T_n = \#\{m \in \{1, \ldots, n\} \colon X_m \geq 0\} = \sum_{m=1}^{n} \mathbb{1}_{\mathbb{R}^\dagger}(X_m)$$

(Convention: $T_0 = 0$)



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$\longrightarrow$ ***Problem: Probability distribution of T_n?***

1. General random walk on $\mathbb{R}$

Theorem A **(Sparre Andersen)**

[On the fluctuations of sums of random variables, 1953]

$$\mathbb{P}\{T_n = m\} = \mathbb{P}\{T_m = m\}\,\mathbb{P}\{T_{n-m} = 0\} \quad \text{for } 0 \le m \le n$$

with

$$\{T_m = m\} = \left\{\min_{1\le k\le m} X_k \ge 0\right\} = \{\tau^- > m\}$$

$$\{T_{n-m} = 0\} = \left\{\max_{1\le k\le n-m} X_k < 0\right\} = \{\tau^\dagger > n-m\}$$

where

$$\tau^- = \min\{k \ge 1 \colon X_k < 0\}$$

$$\tau^\dagger = \min\{k \ge 1 \colon X_k \ge 0\}$$

(Convention: $\min(\varnothing) = +\infty$)

1. General random walk on $\mathbb{R}$

Definition – **Generating function of the T_n, $n \geq 0$**

$$K(x,y) = \sum_{m,n\geq 0: m\leq n} \mathbb{P}\{T_n = m\}\, x^m y^{n-m} = \sum_{n=0}^{\infty} \mathbb{E}\left(x^{T_n} y^{n-T_n}\right)$$

Corollary – **The function K satisfies**

$$K(x,y) = K(x,0)K(0,y)$$

where

$$K(x,0) = \sum_{n=0}^{\infty} \mathbb{P}\{T_n = n\}\, x^n = \frac{1 - \mathbb{E}(x^{\tau^-} \mathbb{1}_{\{\tau^- < \infty\}})}{1-x}$$

$$K(0,y) = \sum_{n=0}^{\infty} \mathbb{P}\{T_n = 0\}\, y^n = \frac{1 - \mathbb{E}(y^{\tau^\dagger} \mathbb{1}_{\{\tau^\dagger < \infty\}})}{1-y}$$

1. General random walk on $\mathbb{R}$

Theorem B **(Spitzer)**

[A combinatorial lemma and its application to probability theory, 1955]

$$K(x,0) = \sum_{n=0}^{\infty} \mathbb{P}\left\{\min_{1\le k\le n} X_k \ge 0\right\} x^n = \exp\left(\sum_{k=1}^{\infty} \mathbb{P}\{X_k \ge 0\} \frac{x^k}{k}\right)$$

$$K(0,y) = \sum_{n=0}^{\infty} \mathbb{P}\left\{\max_{1\le k\le n} X_k < 0\right\} y^n = \exp\left(\sum_{k=1}^{\infty} \mathbb{P}\{X_k < 0\} \frac{y^k}{k}\right)$$

The probabilities $\mathbb{P}\{\max_{1\le k\le n} X_k < 0\}$ and $\mathbb{P}\{\min_{1\le k\le n} X_k \ge 0\}$ are implicitly known through their generating functions...

$\longrightarrow$ ***Next step: To invert these generating functions...***

2. Case of nearest neighbour random walk

2. Nearest neighbour random walk

Definition – **Let $(U_i)_{i\geq 1}$ be a sequence of i.i.d. Bernoulli r.v.'s with parameters** $\begin{cases} p = \mathbb{P}\{U_1 = +1\} \\ q = \mathbb{P}\{U_1 = -1\} \end{cases}$ **with $p + q = 1$ and $(X_m)_{m\geq 0}$ be the random walk defined on $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ by $X_0 = 0$ and**

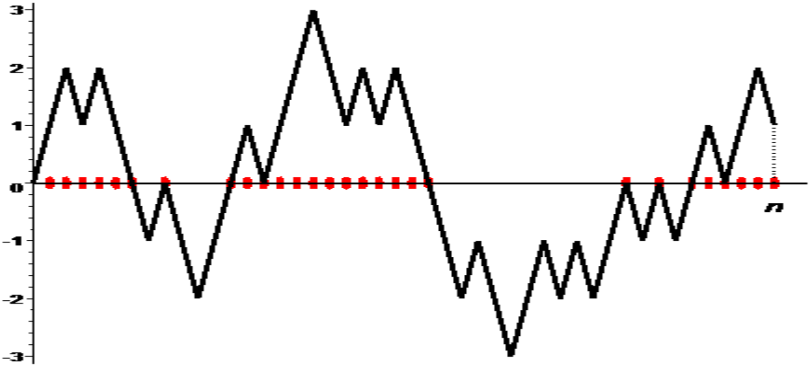
$$X_m = \sum_{i=1}^{m} U_i \quad \text{for } m \geq 1$$

2. Nearest neighbour random walk

Definition – **Sojourn time of the walk $(X_m)_{m\geq 0}$ in $\mathbb{Z}^\dagger = \mathbb{Z}\cap[0,+\infty)$ up to a fixed step $n \geq 1$**

$$T_n = \#\{m \in \{1,\ldots,n\} : X_m \geq 0\} = \sum_{m=1}^{n} \mathbb{1}_{\mathbb{Z}^\dagger}(X_m)$$

(Convention: $T_0 = 0$)



2. Nearest neighbour random walk

By Sparre Andersen and Spitzer:

$$\mathbb{P}\{T_n = m\} = \mathbb{P}\{\tau^- > m\}\,\mathbb{P}\{\tau^\dagger > n - m\}$$

Theorem – **Probability distribution of T_n**

$$\mathbb{P}\{T_n = m\} = \left(1 - q \sum_{0 \le i \le \frac{m-1}{2}} C_i (pq)^i\right)\left(p\delta_{mn} + q - \sum_{1 \le i \le \frac{n-m}{2}} C_{i-1} (pq)^i\right)$$

where $C_i = \frac{1}{i+1}\binom{2i}{i}$ (Catalan numbers !)

Quote – **As written by Spitzer:**

> ***"There is no doubt what causes the slight but ugly asymmetry in the distribution of T_n. It is slight but unpleasant difference between positive and non-negative partial sums"***

2. Nearest neighbour random walk

An alternative sojourn time by Chung & Feller

[On fluctuations in coin-tossings, 1949]

$$\widetilde{T}_n = \sum_{m=1}^{n} \delta_m \quad \text{with } \delta_m = \begin{cases} 1 & \text{if } (X_m > 0) \text{ or } (X_m = 0 \text{ and } X_{m-1} > 0) \\ 0 & \text{if } (X_m < 0) \text{ or } (X_m = 0 \text{ and } X_{m-1} < 0) \end{cases}$$

One counts each step m such that $X_m > 0$ and only those steps such that $X_m = 0$ which correspond to a downstep: $X_{m-1} = 1$

2. Nearest neighbour random walk

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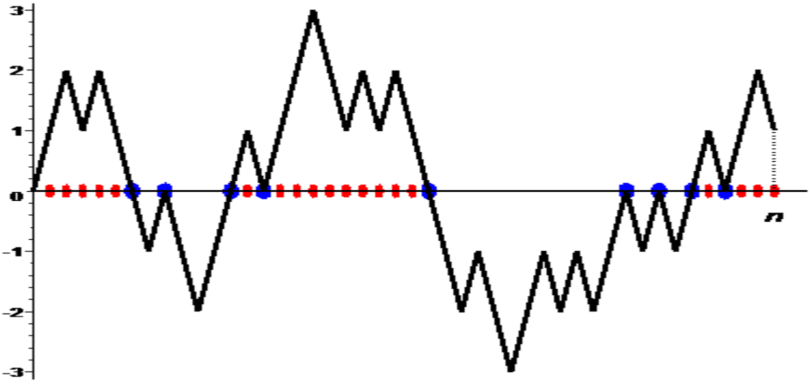

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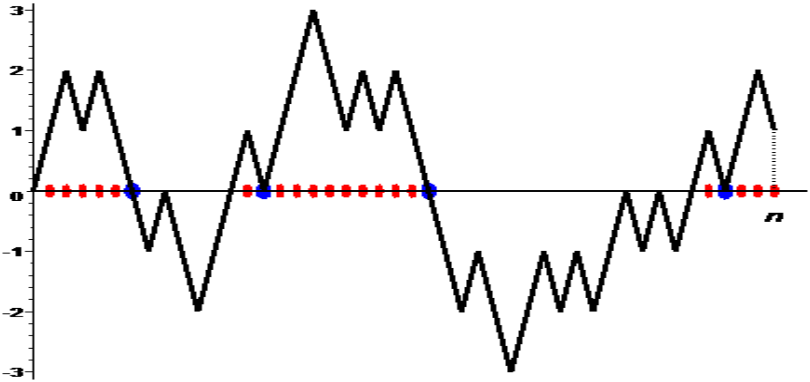


2. Nearest neighbour random walk

An alternative sojourn time by Chung & Feller

Symmetric case $p = q = 1/2$

Quote – **As written by Chung & Feller:**

> ***"The elegance of the results to be announced depends on this convention [definition of δ_m]"***

$\longrightarrow$ **It produces a remarkable result!**

Theorem – **Probability distribution of $\widetilde{T}_n$**

[Chung & Feller: On fluctuations in coin-tossings, 1949]

$$\mathbb{P}\left\{\widetilde{T}_n = m\right\} = \mathbb{P}\left\{\widetilde{T}_m = m\right\} \mathbb{P}\left\{\widetilde{T}_{n-m} = 0\right\} = \frac{1}{2^n} \binom{m}{m/2} \binom{(n-m)}{(n-m)/2}$$

2. Nearest neighbour random walk

An alternative sojourn time by Chung & Feller

Symmetric case $p = q = 1/2$

***Quote* – As written by Chung & Feller:**

> ***"The elegance of the results to be announced depends on this convention [definition of δ_m]"***

$\longrightarrow$ It produces a remarkable result!

Problem – What about the case of an arbitrary p? Well-known? Surprisingly, no precise reference in the literature...

2. Nearest neighbour random walk

An alternative sojourn time by Chung & Feller

Definition – **Generating function of the $\widetilde{T}_n$, $n \geq 0$**

$$\widetilde{K}(x,y) = \sum_{m,n\geq 0: m\leq n} \mathbb{P}\left\{\widetilde{T}_n = m\right\} x^m y^{n-m}$$

Theorem – **The function $\widetilde{K}$ is given by**

[AL: Sojourn time in $\mathbb{Z}^+$ for the Bernoulli random walk on $\mathbb{Z}$ (ESAIM: P&S, 2012)]

$$\widetilde{K}(x,y) = \frac{(p-q)(x-y) + (1-y)\sqrt{\Delta(x)} + (1-x)\sqrt{\Delta(y)}}{(1-x)(1-y)\left(\sqrt{\Delta(x)} + \sqrt{\Delta(y)}\right)}$$

where $\Delta(u) = 1 - 4pqu^2$

An alternative sojourn time by Chung & Feller

Corollary – Probability distribution of $\widetilde{T}_n$

$$\mathbb{P}\left\{\widetilde{T}_n = m\right\}$$
$$= \mathbb{1}_{\{n-m\text{ even}\}}\left(p\sum_{\frac{n-m}{2}\le i\le\frac{n}{2}} C_i(pq)^i - \sum_{\frac{n-m}{2}\le i\le\frac{n}{2}-1}\left(\sum_{0\le j\le i-\frac{n-m}{2}} C_jC_{i-j}\right)(pq)^{i+1}\right)$$
$$+ \mathbb{1}_{\{m\text{ even}\}}\left(q\sum_{\frac{m}{2}\le i\le\frac{n}{2}} C_i(pq)^i - \sum_{\frac{m}{2}\le i\le\frac{n}{2}-2}\left(\sum_{0\le j\le i-\frac{m}{2}} C_jC_{i-j}\right)(pq)^{i+1}\right)$$

where $C_i = \frac{1}{i+1}\binom{2i}{i}$ (Catalan numbers)

2. Nearest neighbour random walk

An alternative sojourn time by Chung & Feller

Remark – **Similar results hold for pinned random walk/bridge of random walk** (joint distribution of $\left(\widetilde{T}_n \,|\, X_n\right)$)...

Theorem – **Probability distribution of** $\left(\widetilde{T}_n \,|\, X_n = 0\right)$ **(even n)**

$$\mathbb{P}\left\{\widetilde{T}_n = m \,|\, X_n = 0\right\} = \begin{cases} \dfrac{2}{n+2} & \text{if } m \text{ is even} \leq n \\ 0 & \text{else} \end{cases}$$

$\longrightarrow$ **The r.v.** $\left(\widetilde{T}_n \,|\, X_n = 0\right)$ **is uniformly distributed on the set** $\{0, 2, 4, \ldots, n-2, n\}$

2. Nearest neighbour random walk

An alternative sojourn time by Chung & Feller

Proof – **Two possible methods:**

- **Strong Markov property related to the first hitting time of 0**

  ⟶ Splitting the paths into two parts : the first excursion and the refreshed walk (yielding recurrence relations)

- **Theory of excursions away from 0**

  ⟶ Splitting the paths into excursions

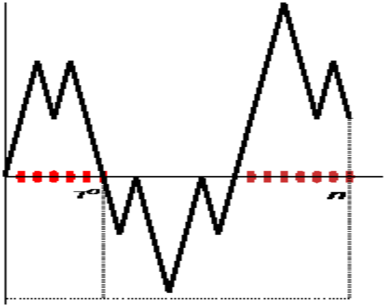
2. Nearest neighbour random walk

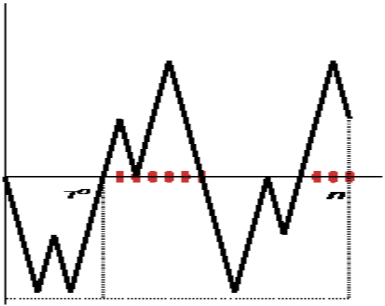
An alternative sojourn time by Chung & Feller

If $1 \le \widetilde{T}_n \le n-1$, then $1 \le \tau^o \le n-1$ and

$$\widetilde{T}_n = \begin{cases} \tau^o + \widetilde{T}_{\tau^o,n} & \text{if } X_1 > 0 \\ \widetilde{T}_{\tau^o,n} & \text{if } X_1 < 0 \end{cases}$$

where $\tau^o = \min\{m \ge 1 \colon X_m = 0\}$ and $\widetilde{T}_n = \sum_{m=\tau^o+1}^{n} \delta_m$





2. Nearest neighbour random walk

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If $1 \le \widetilde{T}_n \le n-1$, then $1 \le \tau^o \le n-1$ and

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where $\tau^o = \min\{m \ge 1 \colon X_m = 0\}$ **and** $\widetilde{T}_n = \sum_{m=\tau^o+1}^{n} \delta_m$

By the strong Markov property, if $1 \le m \le n-1$:

$$\begin{aligned} \mathbb{P}\Big\{\widetilde{T}_n = m\Big\} = \mathbb{P}\Big\{\widetilde{T}_n = m, 1 \le \tau^o \le n-1\Big\} &= \sum_{j=1}^{n-1} \mathbb{P}\Big\{\widetilde{T}_n, \tau^o = j\Big\} \\ &= \sum_{j=1}^{n-1} \mathbb{P}\{\tau^o = j, X_1 > 0\}\, \mathbb{P}\Big\{\widetilde{T}_{n-j} = m-j\Big\} \\ &\quad + \sum_{j=1}^{n-m} \mathbb{P}\{\tau^o = j, X_1 < 0\}\, \mathbb{P}\Big\{\widetilde{T}_{n-j} = m\Big\} \text{ etc.} \end{aligned}$$

2. Nearest neighbour random walk

An alternative sojourn time by Chung & Feller

Remark – **The generating function $\widetilde{K}$ does not satisfy**

$$\widetilde{K}(x,y) = \widetilde{K}(x,0)\widetilde{K}(0,y)$$

But the partial generating function of the $\widetilde{T}_n$, $n \geq 0$

$$\widetilde{K}'(x,y) = \sum_{\substack{m,n\geq 0: m\leq n \\ \text{even } m,n}} \mathbb{P}\left\{\widetilde{T}_n = m\right\} x^m y^{n-m}$$

does satisfy

$$\widetilde{K}'(x,y) = \widetilde{K}'(x,0)\widetilde{K}'(0,y)...$$

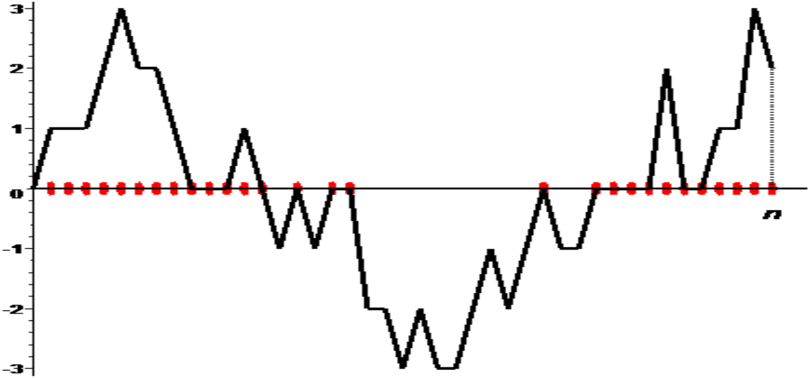
3. Case of random walk with stagnation

3. Random walk with stagnation

Definition – **Let $(U_i)_{i\geq 1}$ be a sequence of i.i.d. r.v.'s with distribution**

$$\begin{cases} p = \mathbb{P}\{U_1 = +1\} \\ q = \mathbb{P}\{U_1 = -1\} \\ r = \mathbb{P}\{U_1 = 0\} \end{cases} \quad \text{with } p+q+r=1$$

and $(X_m)_{m\geq 0}$ be the corresponding random walk



3. Random walk with stagnation

By Sparre Andersen and Spitzer:

$$\mathbb{P}\{T_n = m\} = \mathbb{P}\{\tau^- > m\}\,\mathbb{P}\{\tau^\dagger > n - m\}$$

Theorem – **Probability distribution of T_n**

$$\mathbb{P}\{T_n = m\} = \left(1 + \frac{1}{p}\sum_{i=2}^{m+1} A_i\right)\left(p\delta_{mn} + q + \sum_{i=2}^{n-m} A_i\right)$$

where

$$A_i = \frac{2}{4^i}\sum_{j=0}^{i} C_{j-1} C_{i-j-1}\left(r - 2\sqrt{pq}\right)^j \left(r + 2\sqrt{pq}\right)^{i-j}$$

$C_i = \frac{1}{i+1}\binom{2i}{i}$ and $C_{-1} = -\frac{1}{2}$ (Catalan numbers)

3. Random walk with stagnation

An alternative sojourn time by AL and V. Cammarota

$$\widetilde{T}_n = \sum_{m=1}^{n} \delta_m$$

$\delta_m = 1$ if	$\delta_m = 0$ if
$\begin{cases} X_m > 0 \text{ or} \\ X_m = 0, X_{m-1} > 0 \text{ or} \\ X_m = X_{m-1} = 0, X_{m-2} > 0 \text{ or} \\ \vdots \\ X_m = X_{m-1} = \cdots = X_2 = 0, X_1 > 0 \end{cases}$	$\begin{cases} X_m < 0 \text{ or} \\ X_m = 0, X_{m-1} < 0 \text{ or} \\ X_m = X_{m-1} = 0, X_{m-2} < 0 \text{ or} \\ \vdots \\ X_m = X_{m-1} = \cdots = X_2 = 0, X_1 \leq 0 \end{cases}$

(Convention: $\widetilde{T}_0 = 0$)

One counts each step m such that $X_m > 0$ and only those steps such that $X_m = 0$ which correspond to a previous descent: $X_{m-1} = 1$ or ($X_{m-1} = 0$ and $X_{m-2} = 1$), etc.

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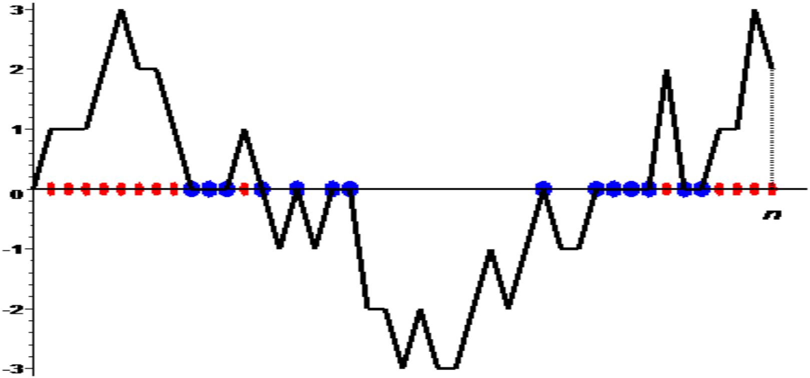

3. Random walk with stagnation

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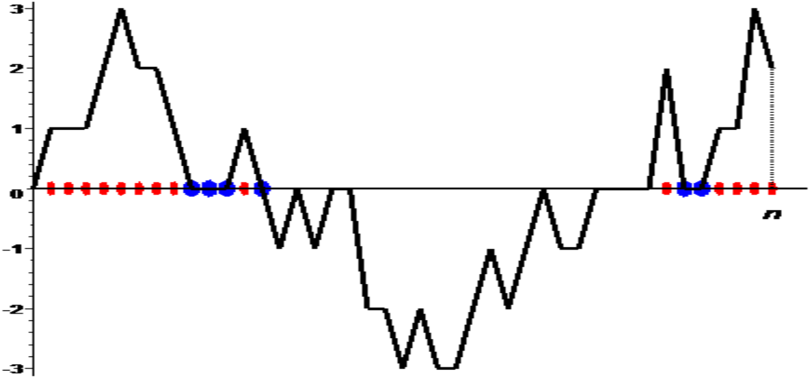

3. Random walk with stagnation

Theorem – **Generating function of the $\widetilde{T}_n$, $n \geq 0$ (with $X_1 \neq 0$)**

[V. Cammarota & AL: Entrance and sojourn times for Markov chains.
Application to (L, R)–random walks (MPRF, 2015)]

[AL: Excursions for nearest neighbour random walk including stagnation.
Application to occupation times (Work in progress)]

$$\widetilde{K}(x,y) = \frac{a(x,y) + b(y)\sqrt{\Delta(x)} + b(x)\sqrt{\Delta(y)}}{(1-x)(1-y)\Big((1-ry)\sqrt{\Delta(x)} + (1-rx)\sqrt{\Delta(y)}\Big)}$$

with

$$a(x,y) = (p-q)(x-y)(1-rx)(1-ry)$$
$$b(u) = (1-r)(1-u)(1-ru)$$
$$\Delta(u) = (1-ru)^2 - 4pqu^2$$

3. Random walk with stagnation

Theorem – **Generating function of the $\widetilde{T}_n$, $n \geq 0$ (with $X_1 \neq 0$)**

[V. Cammarota & AL: Entrance and sojourn times for Markov chains. Application to (L, R)–random walks (MPRF, 2015)]

[AL: Excursions for nearest neighbour random walk including stagnation. Application to occupation times (Work in progress)]

$$\widetilde{K}(x,y) = \frac{a(x,y) + b(y)\sqrt{\Delta(x)} + b(x)\sqrt{\Delta(y)}}{(1-x)(1-y)\Big((1-ry)\sqrt{\Delta(x)} + (1-rx)\sqrt{\Delta(y)}\Big)}$$

Proof – **Two possible methods:**

- **Strong Markov property related to the first hitting time of 0 (yielding recurrence relations)**
- **Theory of excursions away from 0 *(in progress)***

3. Random walk with stagnation

***Theorem* – Generating function of the $\widetilde{T}_n$, $n \geq 0$ (with $X_1 \neq 0$)**

*[V. Cammarota & AL: Entrance and sojourn times for Markov chains.
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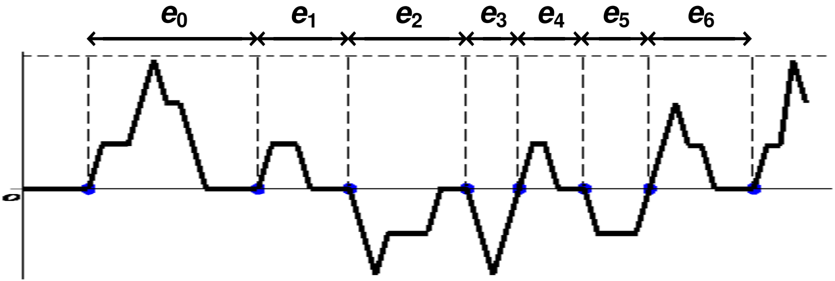

3. Random walk with stagnation

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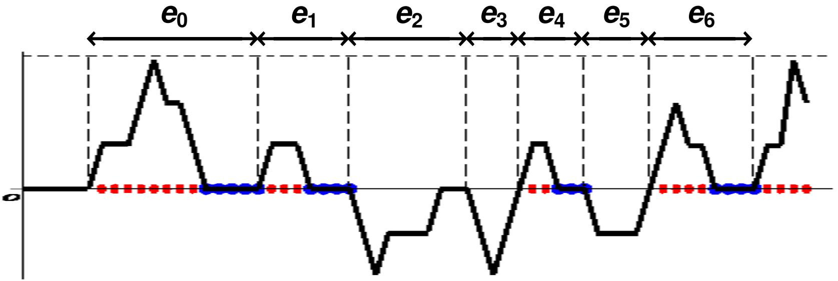

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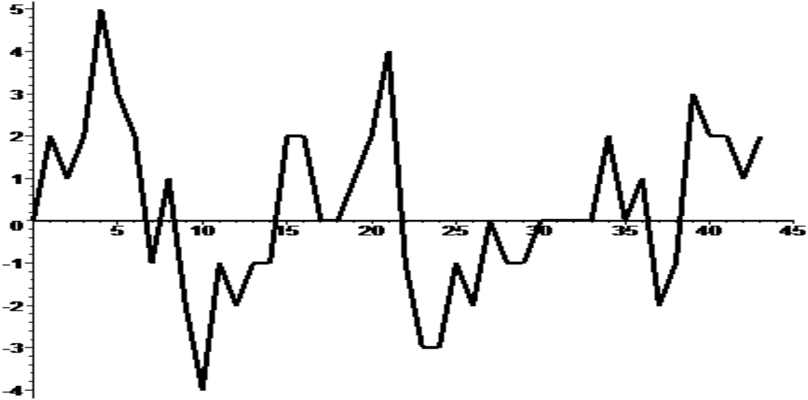
$\longrightarrow$ ***Next step: To invert this generating function** (in progress)...*

4. Case of (L,R)–random walk

4. (L, R)–random walk

Definition – **Let L, R be positive integers and let $(U_i)_{i\geq 1}$ be a sequence of i.i.d. r.v.'s with values in $\{-L, -L+1, \ldots, R-1, R\}$ and $(X_m)_{m\geq 0}$ be the corresponding random walk**



Example: $L = 5$, $R = 4$

4. (L, R)–random walk

Definition – **Let L, R be positive integers and let $(U_i)_{i\geq 1}$ be a sequence of i.i.d. r.v.'s with values in $\{-L, -L+1, \ldots, R-1, R\}$ and $(X_m)_{m\geq 0}$ be the corresponding random walk**

Facts

- **In the previous examples, $\{0\}$ was a natural boundary between $\mathbb{Z}^-$ and $\mathbb{Z}^\dagger$ such that $\{0\} \subset \mathbb{Z}^\dagger$**

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Facts

- **In the previous examples, $\{0\}$ was a natural boundary between $\mathbb{Z}^-$ and $\mathbb{Z}^\dagger$ such that $\{0\} \subset \mathbb{Z}^\dagger$**
- **Moving from $\mathbb{Z}^-$ to $\mathbb{Z}^\dagger$ induces an up-crossing jump of maximal size R:** $\begin{cases} \tau^\dagger = \min\{k \geq 1 \colon X_k \geq 0\} \\ X_{\tau^\dagger} \in \{0, 1, 2, \ldots, R-1\} \end{cases}$

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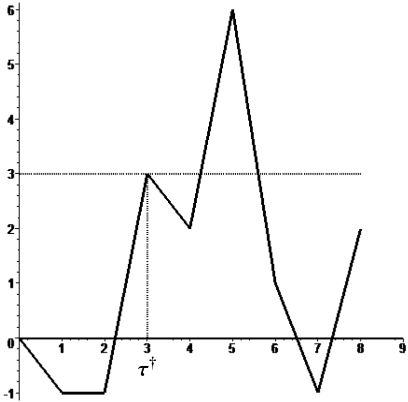

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- **In the previous examples, $\{0\}$ was a natural boundary between $\mathbb{Z}^-$ and $\mathbb{Z}^\dagger$ such that $\{0\} \subset \mathbb{Z}^\dagger$**
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- **Moving from $\mathbb{Z}^\dagger$ to $\mathbb{Z}^-$ induces a down-crossing jump of maximal size L:** $\begin{cases} \tau^- = \min\{k \geq 1 \colon X_k < 0\} \\ X_{\tau^- -1} \in \{0, 1, 2, \ldots, L-1\} \end{cases}$

4. (L, R)–random walk

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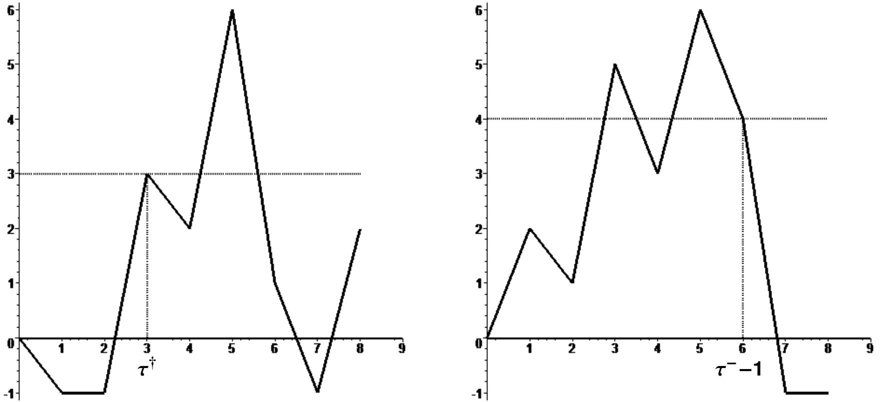

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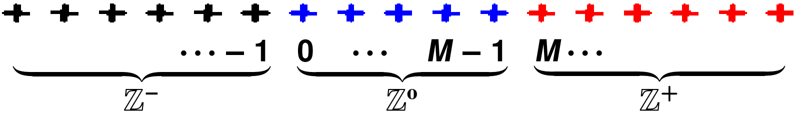
Facts

⟶ **Occurrence of a natural "boundary"**

$$\mathbb{Z}^{o} = \{0, 1, \dots, M-1\} \quad \text{where } M = \max(L, R)$$

yielding a partition

$$\mathbb{Z} = \mathbb{Z}^{-} \cup \mathbb{Z}^{o} \cup \mathbb{Z}^{+} \quad \text{where } \mathbb{Z}^{+} = \{M, M+1, M+2, \dots\}$$



5. General framework : Markov chains

5. Markov chains

Settings

- $(X_m)_{m\geq 0}$: homogeneous Markov chain on $\mathcal{E}$ (finite or denumerable)
- $E^\dagger$: subset of $\mathcal{E}$ and $E^- = \mathcal{E}\setminus E^\dagger$ ('non-negative' and 'negative' states)
  $\longrightarrow$ Partition $\boxed{\mathcal{E} = E^\dagger \cup E^-}$
- E^o: subset of $E^\dagger$ and $E^+ = E^\dagger \setminus E^o$ ('null' and 'positive' states)
  $\longrightarrow$ Partition $\boxed{\mathcal{E} = E^+ \cup E^o \cup E^-}$
- Conditional probabilities $\mathbb{P}_i\{\cdots\} = \mathbb{P}\{\cdots | X_0 = i\}$ and transition probabilities $p_{ij} = \mathbb{P}_i\{X_1 = j\}$ for $i, j \in \mathcal{E}$
- Sojourn time of $(X_m)_{m\geq 0}$ in $E^\dagger$ up to a fixed time $n \geq 1$

$$\boxed{T_n = \#\left\{m \in \{1, \ldots n\} : X_m \in E^\dagger\right\} = \sum_{m=1}^{n} \mathbb{1}_{E^\dagger}(X_m)}$$

(Convention: $T_0 = 0$)

5. Markov chains

Settings

- **First entrance times: $\tau^{o}, \tau^{\dagger}, \tau^{+}, \tau^{-}$ in $E^{o}, E^{\dagger}, E^{+}, E^{-}$**

$$\begin{aligned}
\tau^{o} &= \min\{m \geq 1 : X_m \in E^{o}\} \\
\tau^{\dagger} &= \min\{m \geq 1 : X_m \in E^{\dagger}\} \\
\tau^{+} &= \min\{m \geq 1 : X_m \in E^{+}\} \\
\tau^{-} &= \min\{m \geq 1 : X_m \in E^{-}\}
\end{aligned}$$

(Convention: $\min(\varnothing) = +\infty$)

Assumptions on $E^{\dagger}$ and E^{o}

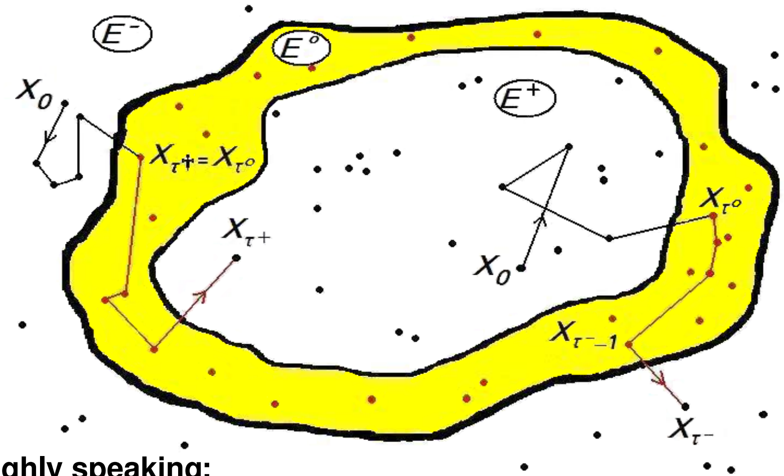
(A_1) **if $X_0 \in E^{-}$, then $\tau^{o} = \tau^{\dagger}$**

"The chain starting out of $E^{\dagger}$ enters $E^{\dagger}$ necessarily by passing through E^{o}"

(A_2) **if $X_0 \in E^{+}$, then $\tau^{o} \leq \tau^{-} - 1$**

"The chain starting in E^{+} exits $E^{\dagger}$ necessarily by passing through E^{o}"

Settings



Roughly speaking:

E^o acts as a kind of 'boundary' of $E^\dagger$
while E^+ acts as a kind of 'interior' of $E^\dagger$

Settings

- Generating functions: for $i, j \in \mathcal{E}$ and any real number x
  - Generating function of the numbers $\mathbb{P}_i\{X_m = j\}$, $m \geq 0$:

$$G_{ij}(x) = \sum_{m=0}^{\infty} \mathbb{P}_i\{X_m = j\}\, x^m$$

  - Generating function of (τ^o, X_{τ^o}):

$$H^o_{ij}(x) = \sum_{m=1}^{\infty} \mathbb{P}_i\{\tau^o = m, X_{\tau^o} = j\}\, x^m = \mathbb{E}_i\Big(x^{\tau^o} \mathbb{1}_{\{X_{\tau^o}=j, \tau^o<\infty\}}\Big)$$

$$H^{o\dagger}_{ij}(x) = \mathbb{E}_i\Big(x^{\tau^o} \mathbb{1}_{\{X_1 \in E^\dagger, X_{\tau^o}=j, \tau^o<\infty\}}\Big)$$

$$H^{o-}_{ij}(x) = \mathbb{E}_i\Big(x^{\tau^o} \mathbb{1}_{\{X_1 \in E^-, X_{\tau^o}=j, \tau^o<\infty\}}\Big)$$

5. Markov chains

Settings

- **Generating functions: linear systems of equations**

Chapman-Kolmogorov equation

$$G_{ij}(x) = \delta_{ij} + x \sum_{k \in \mathcal{E}} p_{ik} G_{kj}(x) \quad \text{for } i, j \in \mathcal{E}$$

$\longrightarrow$ **yields the $G_{ij}(x)$'s**

Strong Markov property

$$G_{ij}(x) = \delta_{ij} + \sum_{k \in E^o} H^o_{ik}(x) G_{kj}(x) \quad \text{for } i \in \mathcal{E}, j \in E^o$$

$\longrightarrow$ **yields the $H^o_{ij}(x)$'s**

Settings

- Generating functions: linear systems of equations

Markov property

$$H_{ij}^{o\dagger}(x) = x\left(p_{ij} + \sum_{k\in E^+} p_{ik} H_{kj}^{o}(x)\right) \quad \text{for } i \in \mathcal{E}, j \in E^{o}$$

$$H_{ij}^{o-}(x) = x \sum_{k\in E^-} p_{ik} H_{kj}^{o}(x) \quad \text{for } i \in \mathcal{E}, j \in E^{o}$$

5. Markov chains

Definition – **Generating function of the T_n, $n \geq 0$: for any $i \in \mathcal{E}$**

$$K_i(x,y) = \sum_{m,n\geq 0 : m \leq n} \mathbb{P}_i\{T_n = m\}\, x^m y^{n-m}$$

Theorem – **The K_i, $i \in \mathcal{E}$, satisfy the linear system of equations**

$$\begin{aligned} K_i(x,y) = {} & K_i(x,0) + K_i(0,y) - 1 \\ & + \sum_{j \in E^o} \left(H_{ij}^{o\dagger}(x) + \frac{x}{y} H_{ij}^{o-}(y) \right) K_j(x,y) \\ & - \sum_{j \in E^o} H_{ij}^{o\dagger}(x) K_j(x,0) \end{aligned}$$

where

$$K_i(x,0) = \frac{1 - \mathbb{E}_i\left(x^{\tau^-} \mathbb{1}_{\{\tau^- < \infty\}}\right)}{1-x} \quad \text{and} \quad K_i(0,y) = \frac{1 - \mathbb{E}_i\left(y^{\tau^\dagger} \mathbb{1}_{\{\tau^\dagger < \infty\}}\right)}{1-y}$$

Remarks

- It is enough to know $K_i(x, y)$ only for $i \in E^o$ to derive $K_i(x, y)$ for $i \in \mathcal{E} \backslash E^o$
- It provides a methodology for determining the $K_i(x, y)$'s, $i \in E^o$

Theorem – Matrix approach

$$\mathbb{K}(x, y) = \left(\mathbb{I} - \mathbb{H}^{o\dagger}(x) - \frac{x}{y}\,\mathbb{H}^{o-}(y)\right)^{-1}\left(\left(\mathbb{I} - \mathbb{H}^{o\dagger}(x)\right)\mathbb{K}(x, 0) + \mathbb{K}(0, y) - \mathbb{1}\right)$$

with the matrices

$$\mathbb{K}(x, y) = \left(K_i(x, y)\right)_{i \in E^o} \qquad \mathbb{I} = \left(\delta_{ij}\right)_{i,j \in E^o} \qquad \mathbb{1} = \left(1\right)_{i \in E^o}$$

$$\mathbb{H}^{o\dagger}(x) = \left(H_{ij}^{o\dagger}(x)\right)_{i,j \in E^o} \qquad \mathbb{H}^{o-}(y) = \left(H_{ij}^{o-}(y)\right)_{i,j \in E^o}$$

5. Markov chains

Particular case – **If $E^{o} = \{i_0\}$, then**

$$K_{i_0}(x,y) = \frac{\left(1 - H^{o\dagger}_{i_0 i_0}(x)\right)K_{i_0}(x,0) + K_{i_0}(0,y) - 1}{1 - H^{o\dagger}_{i_0 i_0}(x) - \frac{x}{y}\, H^{o-}_{i_0 i_0}(y)}$$

where

$$H^{o\dagger}_{i_0 i_0}(x) = x\left(\sum_{k\in E^{\dagger}} p_{i_0 k}\, G_{k i_0}(x)\right) / G_{i_0 i_0}(x)$$

$$H^{o-}_{i_0 i_0}(y) = y\left(\sum_{k\in E^{-}} p_{i_0 k}\, G_{k i_0}(y)\right) / G_{i_0 i_0}(y)$$

Definition **– An alternative sojourn time**

$$\widetilde{T}_n = \sum_{m=1}^{n} \delta_m$$

with

$\delta_m = 1$ if	$\delta_m = 0$ if
$\begin{cases} X_m \in E^+ \text{ or} \\ X_m \in E^o, X_{m-1} \in E^+ \text{ or} \\ X_m, X_{m-1} \in E^o, X_{m-2} \in E^+ \text{ or} \\ \vdots \\ X_m, X_{m-1}, \ldots, X_2 \in E^o, X_1 \in E^+ \end{cases}$	$\begin{cases} X_m \in E^- \text{ or} \\ X_m \in E^o, X_{m-1} \in E^- \text{ or} \\ X_m, X_{m-1} \in E^o, X_{m-2} \in E^- \text{ or} \\ \vdots \\ X_m, X_{m-1}, \ldots, X_2 \in E^o, X_1 \in E^- \cup E^o \end{cases}$

(Convention: $\widetilde{T}_0 = 0$)

Definition – **Generating function of the $\widetilde{T}_n$, $n \geq 0$: for any $i \in \mathcal{E}$**

$$\widetilde{K}_i(x, y) = \sum_{m,n \geq 0 : m \leq n} \mathbb{P}_i\left\{\widetilde{T}_n = m\right\} x^m y^{n-m}$$

Theorem – **The $\widetilde{K}_i$, $i \in \mathcal{E}$, satisfy an intricate system of equations...**

[V. Cammarota& AL: Entrance and sojourn times for Markov chains. Application to (L, R)–random walks (MPRF, 2015)]

$$\widetilde{K}_i(x, y) = \cdots$$

Matrix approach

$$\widetilde{\mathbb{K}}(x, y) = \cdots$$

where $\widetilde{\mathbb{K}}(x, y) = \left(\widetilde{K}_i(x, y)\right)_{i \in E^o}$

5. Markov chains

Definition – **Generating function of the $\widetilde{T}_n$, $n \geq 0$: for any $i \in \mathcal{E}$**

$$\widetilde{K}_i(x,y) = \sum_{m,n\geq 0 : m\leq n} \mathbb{P}_i\Big\{\widetilde{T}_n = m\Big\}\, x^m y^{n-m}$$

Particular case – **If $E^{\mathrm{o}} = \{i_0\}$ and $p_{i_0 i_0} = 0$ (no stagnation at i_0), then**

$$\widetilde{K}_{i_0}(x,y) = \frac{\Big(1 - H^{\mathrm{o}+}_{i_0 i_0}(x)\Big)\widetilde{K}_{i_0}(x,0) + \Big(1 - H^{\mathrm{o}-}_{i_0 i_0}(y)\Big)\widetilde{K}_{i_0}(0,y) - 1}{1 - H^{\mathrm{o}+}_{i_0 i_0}(x) - H^{\mathrm{o}-}_{i_0 i_0}(y)}$$

where

$$H^{\mathrm{o}+}_{i_0 i_0}(x) = \frac{x}{G_{i_0 i_0}(x)} \sum_{k\in E^+} p_{i_0 k}\, G_{k i_0}(x)$$

$$H^{\mathrm{o}-}_{i_0 i_0}(y) = \frac{y}{G_{i_0 i_0}(y)} \sum_{k\in E^-} p_{i_0 k}\, G_{k i_0}(y)$$

6. Case of (L,R)–random walk (continued)

6. (L, R)–random walk (continued)

Definition – **Let L, R be positive integers and let $(U_i)_{i\geq 1}$ be a sequence of i.i.d. r.v.'s with values in $\{-L, -L+1, \ldots, R-1, R\}$**
Set

$$\pi_i = \begin{cases} \mathbb{P}\{U_1 = i\} & \text{for } i \in \{-L, \ldots, R\} \\ 0 & \text{for } i \in \mathbb{Z} \backslash \{-L, \ldots, R\} \end{cases}$$

Let $(X_m)_{m\geq 0}$ be the random walk defined on $\mathbb{Z}$ by $X_0 = 0$ and

$$X_m = \sum_{i=1}^{m} U_i \quad \text{for } m \geq 1$$

6. (L, R)–random walk (continued)

Definition – **Generating function of the X_m, $m \geq 0$**

$$\Gamma_{j-i}(x) = \sum_{m=0}^{\infty} \mathbb{P}_i\{X_m = j\}\, x^m$$

Proposition – **The function Γ_{j-i} admits the representation**

$$\Gamma_{j-i}(x) = \begin{cases} \displaystyle\sum_{\ell \in \mathcal{L}^-} \frac{z_\ell(x)^{i-j+L-1}}{P_x'(z_\ell(x))} & \text{if } i > j \\ \displaystyle -\sum_{\ell \in \mathcal{L}^+} \frac{z_\ell(x)^{i-j+L-1}}{P_x'(z_\ell(x))} & \text{if } i \leq j \end{cases}$$

where the $z_\ell(x)$'s, $1 \leq \ell \leq L + R$, are the roots of the polynomial $P_x : z \mapsto z^L - x \sum_{j=0}^{L+R} \pi_{j-L} z^j$ and

$$\mathcal{L}^+ = \{\ell : |z_\ell(x)| > 1\} \qquad \mathcal{L}^- = \{\ell : |z_\ell(x)| < 1\}$$

6. (L, R)–random walk (continued)

***Settings* –** Set $M = \max(L, R)$

We choose here

$$E^{o} = \{0, 1, \dots, M-1\}$$
$$E^{\dagger} = \{0, 1, 2, \dots\}$$
$$E^{+} = \{M, M+1, M+2, \dots\}$$
$$E^{-} = \{\dots, -3, -2, -1\}$$

The settings can be rewritten in this context as

$$T_n = \#\{m \in \{1, \dots, n\} : X_m \geq 0\}$$
$$\tau^{o} = \min\{m \geq 1 : X_m \in \{0, 1, \dots, M-1\}\}$$
$$\tau^{\dagger} = \min\{m \geq 1 : X_m \geq 0\}$$
$$\tau^{+} = \min\{m \geq 1 : X_m \geq M\}$$
$$\tau^{-} = \min\{m \geq 1 : X_m \leq -1\}$$

Assumptions (A_1) and (A_2) are fulfilled

6. (L,R)–random walk (continued)

Theorem – **The functions K_i, $0 \le i \le M-1$, satisfy the linear system**

$$K_i(x,y) = x \sum_{j=0}^{M-1} \left(\pi_{j-i} + \sum_{k=M}^{2M-1} \pi_{k-i} H^{o}_{kj}(x) + \sum_{k=-M}^{-1} \pi_{k-i} H^{o}_{kj}(y) \right) K_j(x,y)$$
$$+ K_i(x,0) + K_i(0,y) - 1$$
$$- x \sum_{j=0}^{M-1} \left(\pi_{j-i} + \sum_{k=M}^{2M-1} \pi_{k-i} H^{o}_{kj}(x) \right) K_j(x,0), \quad 0 \le i \le M-1$$

where

$$K_i(x,0) = \frac{1 - \mathbb{E}_{i+M}\left(x^{\tau^o} \mathbb{1}_{\{\tau^o < \infty\}}\right)}{1-x}$$

$$K_i(0,y) = \frac{1}{1-y} \left(1 - y \sum_{j=0}^{2M-1} \pi_{j-i} - y \sum_{k=-M}^{-1} \pi_{k-i} \mathbb{E}_k\left(y^{\tau^o} \mathbb{1}_{\{\tau^o < \infty\}}\right) \right)$$

6. (L,R)–random walk (continued)

Theorem – **The functions K_i, $0 \le i \le M-1$, satisfy the linear system**

$$K_i(x,y) = x \sum_{j=0}^{M-1} \left(\pi_{j-i} + \sum_{k=M}^{2M-1} \pi_{k-i} H^{o}_{kj}(x) + \sum_{k=-M}^{-1} \pi_{k-i} H^{o}_{kj}(y) \right) K_j(x,y)$$
$$+ K_i(x,0) + K_i(0,y) - 1$$
$$- x \sum_{j=0}^{M-1} \left(\pi_{j-i} + \sum_{k=M}^{2M-1} \pi_{k-i} H^{o}_{kj}(x) \right) K_j(x,0), \quad 0 \le i \le M-1$$

and where the functions H^{o}_{ij} solve the systems

$$\sum_{k=0}^{M-1} H^{o}_{ik}(x) \Gamma_{j-k}(x) = \Gamma_{j-i}(x) \qquad M \le i \le 2M-1,\ 0 \le j \le M-1$$

$$\sum_{k=0}^{M-1} H^{o}_{ik}(y) \Gamma_{j-k}(y) = \Gamma_{j-i}(y) \qquad -M \le i \le -1,\ 0 \le j \le M-1$$

6. (L, R)–random walk (continued)

Symmetric random walk
$L = R = M$, steps lying in $\{-M, -M+1, \ldots, M-1, M\}$, such that $\pi_i = \pi_{-i}$ for all integer i

---

Example 1

$$\begin{cases} \pi_i = c\dbinom{2M}{i+M} & \text{for } i \in \{-M, \ldots, -1, 1, \ldots, M\} \\ \pi_0 = 1 - c\left[4^M - \dbinom{2M}{M}\right] \end{cases}$$

where $0 < c \leq 1/\left[4^M - \binom{2M}{M}\right]$

For $c = 1/4^M$, we have $\pi_i = \binom{2M}{i+M}/4^M$ for any i

For $c = 1/\left[4^M - \binom{2M}{M}\right]$, we have $\pi_0 = 0$

6. (L, R)–random walk (continued)

Symmetric random walk
$L = R = M$, steps lying in $\{-M, -M+1, \ldots, M-1, M\}$, such that $\pi_i = \pi_{-i}$ for all integer i

---

Example 1

$$\Gamma_j(x) = \frac{1}{M\,(1-(1-c\,4^M)\,x)} \sum_{\ell=1}^{M} \frac{1+z_\ell(x)}{1-z_\ell(x)}\, z_\ell(x)^{|j|}$$

where the z_ℓ, $1 \leq \ell \leq M$, are the roots of

$$(z+1)^2 - e^{i\frac{2\pi}{M}r}\sqrt{\frac{1-(1-c\,4^M)\,x}{c\,x}}\, z = 0, \quad 0 \leq r \leq M-1$$

6. (L, R)–random walk (continued)

Symmetric random walk
$L = R = M$, steps lying in $\{-M, -M+1, \ldots, M-1, M\}$, such that $\pi_i = \pi_{-i}$ for all integer i

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Example 2

$$\begin{cases} \pi_i = c\rho^{|i|}\binom{M}{|i|} & \text{for } i \in \{-M, \ldots, -1, 1, \ldots, M\} \\ \pi_0 = 1 - 2c\left((\rho+1)^M - 1\right) \end{cases}$$

where $c \leq 1/\left(2(\rho+1)^M - 1\right)$

For $c = 1/\left(2(\rho+1)^M - 1\right)$, we have $\pi_0 = 0$

6. (L, R)–random walk (continued)

Symmetric random walk
$L = R = M$, steps lying in $\{-M, -M+1, \dots, M-1, M\}$, such that $\pi_i = \pi_{-i}$ for all integer i

Example 2

When $\rho = 1$:

$$\Gamma_j(x) = \frac{1}{M\,(1-(1-c\,2^{M+1})\,x)} \sum_{\ell=1}^{M} \frac{(1+z_\ell(x))(1+z_\ell(x)^M)}{1-z_\ell(x)^{M+1}}\, z_\ell(x)^{|j|}$$

where the z_ℓ, $1 \le \ell \le M$, are the roots of

$$\left(1-(1-c\,2^{M+1})x\right) z^M - cx(z^M+1)(z+1)^M = 0$$

6. (L, R)–random walk (continued)

Symmetric random walk
$L = R = M$, steps lying in $\{-M, -M+1, \ldots, M-1, M\}$, such that $\pi_i = \pi_{-i}$ for all integer i

Example 3

$$\begin{cases} \pi_i = c \quad \text{for } i \in \{-M, \ldots, -1, 1, \ldots, M\} \\ \pi_0 = 1 - 2Mc \end{cases}$$

where $0 < c \leq 1/(2M)$

For $c = 1/(2M)$, we have $\pi_0 = 0$
For $c = 1/(2M+1)$, the jumps are identically distributed

Symmetric random walk
$L = R = M$, steps lying in $\{-M, -M+1, \ldots, M-1, M\}$, such that $\pi_i = \pi_{-i}$ for all integer i

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Example 4 – Symmetric $(2,2)$-random walk ($L = R = M = 2$)

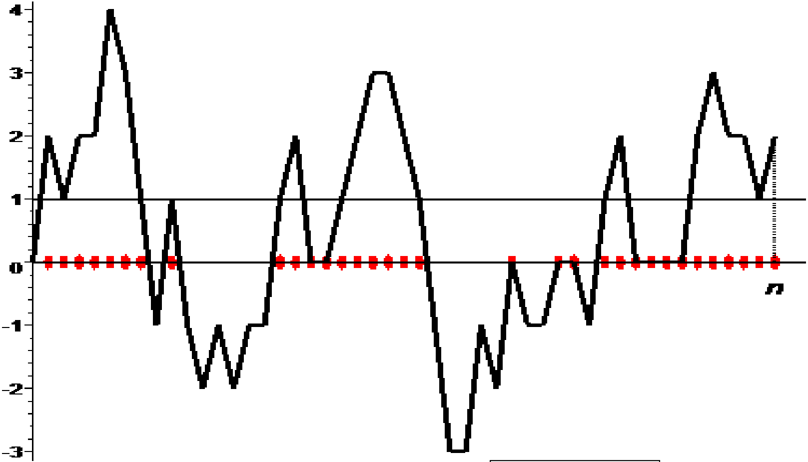
$$\begin{cases} \pi_0 = \mathbb{P}\{U_1 = 0\} \\ \pi_1 = \mathbb{P}\{U_1 = +1\} = \mathbb{P}\{U_1 = -1\} \\ \pi_2 = \mathbb{P}\{U_1 = +2\} = \mathbb{P}\{U_1 = -2\} \end{cases}$$

with $\pi_0 + 2\pi_1 + 2\pi_2 = 1$

7. Case of symmetric $(2, 2)$–random walk

7. Symmetric $(2,2)$–random walk

Definition – **Let $(U_i)_{i\geq 1}$ be a sequence of i.i.d. r.v.'s with values in $\{-2,-1,0,1,2\}$ and $(X_m)_{m\geq 0}$ be the corresponding random walk**



The natural boundary is $\boxed{\mathbb{Z}^{\mathbf{o}} = \{0,1\}}$

7. Symmetric $(2,2)$–random walk

Definition – **Let $(U_i)_{i\geq 1}$ be a sequence of i.i.d. r.v.'s with values in $\{-2,-1,0,1,2\}$ and $(X_m)_{m\geq 0}$ be the corresponding random walk**
Set

$$\begin{cases} \pi_0 = \mathbb{P}\{U_1 = 0\} \\ \pi_1 = \mathbb{P}\{U_1 = +1\} = \mathbb{P}\{U_1 = -1\} \\ \pi_2 = \mathbb{P}\{U_1 = +2\} = \mathbb{P}\{U_1 = -2\} \end{cases} \quad \text{with } \pi_0 + 2\pi_1 + 2\pi_2 = 1$$

7. Symmetric $(2,2)$–random walk

Definition – **Generating function of the X_m, $m \geq 0$**

$$\Gamma_{j-i}(x) = \sum_{m=0}^{\infty} \mathbb{P}_i\{X_m = j\}\, x^m$$

Proposition – **The function Γ_{j-i} is given by**

$$\Gamma_{j-i}(x) = \frac{1}{x\sqrt{\delta(x)}}\left(\frac{z_1(x)^{|i-j|+1}}{1-z_1(x)^2} - \frac{z_2(x)^{|i-j|+1}}{1-z_2(x)^2}\right)$$

where

$$\delta(x) = (\pi_1 + 4\pi_2)^2 + 4\pi_2(1/x - 1)$$

$$z_1(x) = -\frac{1}{4\pi_2}\left(\pi_1 - \sqrt{\delta(x)} + \sqrt{2}\sqrt{\pi_1^2 + 4\pi_1\pi_2 - 2\pi_2 + 2\pi_2/x - \pi_1\sqrt{\delta(x)}}\right)$$

$$z_2(x) = -\frac{1}{4\pi_2}\left(\pi_1 + \sqrt{\delta(x)} + \sqrt{2}\sqrt{\pi_1^2 + 4\pi_1\pi_2 - 2\pi_2 + 2\pi_2/x + \pi_1\sqrt{\delta(x)}}\right)$$

7. Symmetric $(2,2)$–random walk

Definition – **Generating matrices of T_n and $\widetilde{T}_n$**

$$\mathbb{K}(x,y) = \begin{pmatrix} K_0(x,y) \\ K_1(x,y) \end{pmatrix} \qquad \widetilde{\mathbb{K}}(x,y) = \begin{pmatrix} \widetilde{K}_0(x,y) \\ \widetilde{K}_1(x,y) \end{pmatrix}$$

Theorem – **The matrices $\mathbb{K}$ and $\widetilde{\mathbb{K}}$ admit the representations**

$$\mathbb{K}(x,y) = \mathbb{A}(x,y)\mathbb{B}(x,y) \qquad \widetilde{\mathbb{K}}(x,y) = \widetilde{\mathbb{A}}(x,y)\widetilde{\mathbb{B}}(x,y)$$

where $\mathrm{A}(x,y), \mathrm{B}(x,y), \widetilde{\mathrm{A}}(x,y), \widetilde{\mathrm{B}}(x,y)$ are explicit matrices given by... very complicated formulae !

7. Symmetric $(2,2)$–random walk

$$\mathbb{A}(x,y) = \frac{d(x)d(y)}{A(x,y)} \begin{pmatrix} A_{00}(x,y) & A_{01}(x,y) \\ A_{10}(x,y) & A_{11}(x,y) \end{pmatrix}$$

where

$$A_{00}(x,y) = (1-\pi_0 x)d(x)d(y) - xd(y)A'_{11}(x) - yd(x)A'_{00}(y)$$
$$A_{01}(x,y) = \pi_1 xd(x)d(y) + xd(y)A'_{01}(x) + yd(x)A'_{10}(y)$$
$$A_{10}(x,y) = \pi_1 xd(x)d(y) + xd(y)A'_{10}(x) + yd(x)A'_{01}(y)$$
$$A_{11}(x,y) = (1-\pi_0 x)d(x)d(y) - xd(y)A'_{00}(x) - yd(x)A'_{11}(y)$$

$$d(z) = \Gamma_0(z)^2 - \Gamma_1(z)^2$$
$$A(x,y) = A_{00}(x,y)A_{11}(x,y) - A_{01}(x,y)A_{10}(x,y)$$

7. Symmetric $(2,2)$–random walk

$$\mathbb{A}(x,y) = \frac{d(x)d(y)}{A(x,y)} \begin{pmatrix} A_{00}(x,y) & A_{01}(x,y) \\ A_{10}(x,y) & A_{11}(x,y) \end{pmatrix}$$

where

$$A'_{00}(z) = \pi_2\Big(\Gamma_0(z)\Gamma_2(z) - \Gamma_1(z)^2\Big)$$
$$A'_{01}(z) = \pi_2\Big(\Gamma_0(z)\Gamma_1(z) - \Gamma_1(z)\Gamma_2(z)\Big)$$
$$A'_{10}(z) = \pi_1\Big(\Gamma_0(z)\Gamma_2(z) - \Gamma_1(z)^2\Big) + \pi_2\Big(\Gamma_0(z)\Gamma_3(z) - \Gamma_1(z)\Gamma_2(z)\Big)$$
$$A'_{11}(z) = \pi_1\Big(\Gamma_0(z)\Gamma_1(z) - \Gamma_1(z)\Gamma_2(z)\Big) + \pi_2\Big(\Gamma_0(z)\Gamma_2(z) - \Gamma_1(z)\Gamma_3(z)\Big)$$

7. Symmetric (2, 2)–random walk

$$\mathbb{B}(x,y) = \begin{pmatrix} B_0(x,y) \\ B_1(x,y) \end{pmatrix}$$

where

$$B_0(x,y) = \frac{1}{(1-x)d(x)^2}\Big[\Big((1-\pi_0 x)d(x) - xA'_{00}(x)\Big)B_0^-(x) - x\Big(\pi_1 d(x) + A'_{01}(x)\Big)B_1^-(x)\Big] + \frac{1}{(1-y)d(y)}B_0^\dagger(y) - 1$$

$$B_1(x,y) = \frac{1}{(1-x)d(x)^2}\Big[\Big((1-\pi_0 x)d(x) - xA'_{11}(x)\Big)B_1^-(x) - x\Big(\pi_1 d(x) + A'_{10}(x)\Big)B_0^-(x)\Big] + \frac{1}{(1-y)d(y)}B_1^\dagger(y) - 1$$

7. Symmetric $(2,2)$–random walk

$$\mathbb{B}(x,y) = \begin{pmatrix} B_0(x,y) \\ B_1(x,y) \end{pmatrix}$$

where

$$B_0^-(x) = d(x) - \Big(\Gamma_0(x) - \Gamma_1(x)\Big)\Big(\Gamma_1(x) + \Gamma_2(x)\Big)$$

$$B_1^-(x) = d(x) - \Big(\Gamma_0(x) - \Gamma_1(x)\Big)\Big(\Gamma_2(x) + \Gamma_3(x)\Big)$$

$$B_0^\dagger(y) = \Big(1 - (1 - \pi_1 - \pi_2)y\Big)d(y) \\ - y\Big(\Gamma_0(y) - \Gamma_1(y)\Big)\Big(\pi_1\Gamma_1(y) + (\pi_1 + \pi_2)\Gamma_2(y) + \pi_2\Gamma_3(y)\Big)$$

$$B_1^\dagger(y) = \Big(1 - (1 - \pi_2)y\Big)d(y) \\ - \pi_2 y\Big(\Gamma_0(y) - \Gamma_1(y)\Big)\Big(\Gamma_1(y) + \Gamma_2(y)\Big)$$

Thank you for your attention!

http://math.univ-lyon1.fr/~alachal/exposes/slides_augsburg_2016.pdf